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THE USE OF MATHEMATICAL MODELLING AS A TOOL FOR LEARNING MATHEMATICS

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ABSTRACT

In this paper we discuss the use of mathematical modelling as a tool for learning mathematics in contrast with other views giving more emphasis to other factors (schemas, automation of rules et c) .
We sketch the "flow - diagram" of the modelling process in the classroom when the teacher gives such problems for solution to the students and we present methods to recognize the attainment levels of students at defined stages of the mathematical modelilng process and to measure the mathematical model building abilities of them.

Key words and phrases: Mathematical Modelling and Applications, Problem Solving, Ergodic Markov Chains.

1. Introduction

It is well known that, the reformation attempted during the 60's in mathematics education, with the introduction of the "modern mathematics" in the school curricula was proved to be a failure , e.g. see Niss (1989: section 2). One of the most negative consequences was that, the attempt to teach the fundamental generalizations before presenting the material that can be generalized, had as a result the despoliation of the curricula from examples and applications connecting mathematics with the real situations of our daily life as well as with the other sciences, that use them as a tool , and therefore they give birth to very many mathematical problems and theories.
Thus, after the rather vague "wave" of the "back to the basics", considerable emphasis has been placed during the 80's on the use of the problem as a tool and motive to teach and understand better mathematics, with two coordinates: "Problem- solving", where attention is given to the use of the proper heuristic

strategies for solving pure (mainly) mathematical problems, e.g. see Polya (1945 and 1963), Schoenfeld (1983), Voskoglou – Perdikaris (1991: section 1) etc, and “Mathematical modelling and Applications”, a process of solving a particular type of problems generated by corresponding situations of the real world, e.g. see Pollak (1979), Niss (1987), Voskoglou (1995: sections 1 and 2) etc.
Although views appeared later disputing the effectiveness of using the problem-solving as a learning device in mathematics and giving emphasis to the acquisition of the appropriate schemas of knowledge and the automation of rules, e.g. Owen and Sweller (1989), it is more or less acceptable nowadays that through the problem solving processes we can give to the students a balanced view of mathematics and we can face effectively the false opposition between “learning mathematics” and “learning to apply mathematics”, e.g. Niss (1987), Voss (1987), Lawson (1990) etc.
Even Marshall (1995), the introducer of the current schema theory, present schemas as the vehicles for problem solving, that can simplify and reconstruct a problem in order to make it more accessible to the solver.

2. The application – orientated teaching of mathematics.

To focus on mathematical modelling, the transformation from a situation of the real word to a mathematical problem is achieved through the use of a mathematical model, which, briefly speaking, is an idealized (simplified) representation of the basic characteristics of the real situation through the use of a suitable set of mathematical symbols, relations and functions.
One of the first persons who described the process of modelling in such a way that it could be used for teaching mathematics was Pollak (1979). He represented the interaction between mathematics and the real world with the scheme shown in figure 1, which is known as the “circle of modelling” .

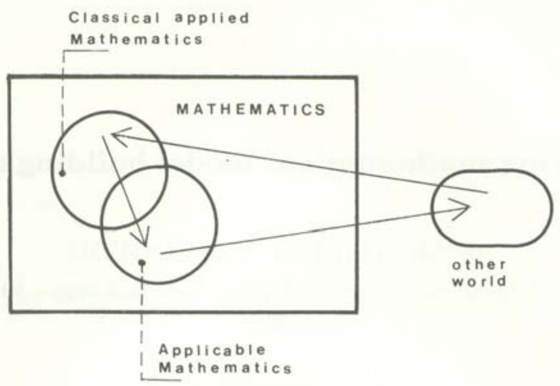

Figure 1.

In the universe of mathematics according to Pollak “classical applied mathematics” and “applicable mathematics” are two intersected but not equal sets. In fact, they are topics from classical mathematics with great theoretical interest but without any visible, for the moment, applications (although it is possible to find such applications in future), while at the same time they are branches of mathematics with many practical applications, which are not characterized by many people as classical (e.g. probability and statistics, linear programming etc).
But the most important feature of Pollak’s scheme is the direction of the arrows, representing a looping between the other world (including all the other sciences and the human activities of everyday life) and the “universe” of mathematics, and that is the substance of what we call mathematical modelling. That is, starting from a real situation or a real problem, we transfer to the other part of the scheme, where we use or develop suitable mathematics, and then we go back to the other world interpreting the mathematical results and even more, if these results are not satisfactory, we make the circle from the beginning again.
From the time that Pollak presented the above scheme in ICME-3 (Karlsruhe, 1976) until nowadays much effort has been placed to analyze in detail the process of mathematical modelling (e.g. see recent works of Berry and Davies1996, Edwards and Hauson 1996, etc)
Summarizing all the existing ideas one could say that the main stages of the process are the following:
s_1 = analysis of the problem (understanding the statement and recognizing the restrictions and requirements of the real system).
s_2 = mathematising , which involves the formulation of the real situation in such a way that it will be ready for mathematical treatment and the construction of the model.
The formulation of the problem, which for many researchers must be considered as an independent stage, involves a deep abstracting process, in order to transfer from the real system to the, so called, “assumed real system”, where emphasis is given to certain, dominating for the system’s performance, variables.
s_3 = solution of the model, which is achieved by proper mathematical manipulation.
s_4 = validation (control) of the model, which is usually achieved by reproducing, through the model, the behaviour of the real system under the conditions existing before the solution of the model (empirical results, special cases etc).
s_5 = interpretation of the final mathematical results and implementation of them to the real system , in order to give the “answer” to our problem.
Mathematical modelling appears today as a dynamic tool for the teaching of mathematics, because it connects mathematics with our everyday life and gives to the students the possibility to understand the usefulness of them in practice; it has also the potential to enhance the performance in mathematics of students generally (Matos, 1998).
A special didactic methodology was developed across these lines by De Lang in Netherlands, called by G. Kaiser (Hamburg) as the “Application – orientated teaching of mathematics”. But we must be careful! The process of modelling could

not be considered as a general, and therefore applicable in all cases, method to teach mathematics. In fact, such a consideration could lead to far-fetched situations, where more emphasis is given to the search of the proper application rather, than to the consolidation of the new mathematical knowledge!

3. The “flow – diagram” of the mathematical modelling process in the classroom.

Let us now try to analyze the “flow-diagram” of the mathematical modelling process in the classroom, when the teacher is giving such problems for solution to the students.

The solver of a problem involving mathematical modelling from the initial state, which is always s_1, proceeds via s_2 to s_3 .

From this state, if the mathematical relations obtained are not suitable to allow an analytic solution of the model, the solver should return to s_2, in order to make the proper simplifications – modifications to the model. Then he (she) returns to s_3, to continue the process

After the solution of the problem within the model the solver should return to the real system, in order to check the validity of the model (state s_4).

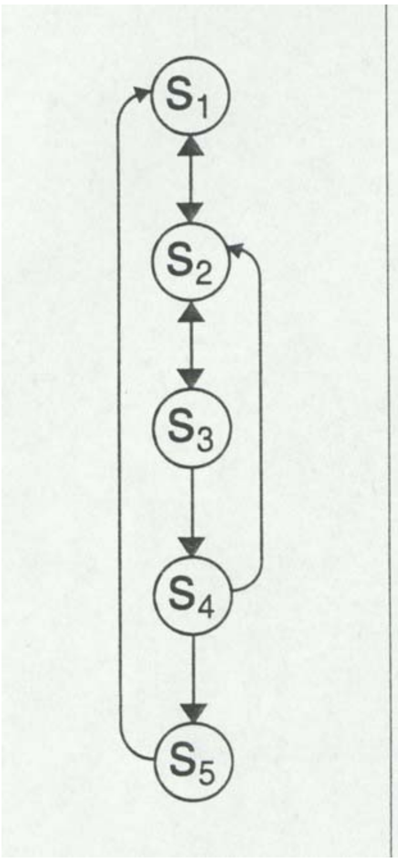

Figure 2

If the model does not give a reliable prediction of the system’s performance, (e.g. if the solution obtained is not satisfying the natural restrictions resulting from the real system, or if it is not verified by known special cases etc), the solver returns from s_4 to s_2 , in order to correct the model. From there he (she) will return, via s_3, to s_4 to continue the process.

After ensuring that the model is valid, the solver from s_4 reaches the state s_5, where he (she) interprets the final mathematical results and applies the conclusions to the real system. i.e. he (she) gives the "answer” to the enquiries of the problem.

When the process of modelling is completed in state s_5, it is assumed that the teacher gives to the students a new problem for solution and therefore the process starts again from s_1.

Notice also that, a solver, who finally fails to construct a solvable mathematical model giving a reliable prediction of the real system’s performance and being unable to make any

other "movement" for the solution of the problem during the time given by the teacher, returns from the state s_2 to s_1 waiting for a new problem, to be given for solution.
According to the above description the "flow - diagram" of the mathematical modelling process is that shown in figure 2.

4. Recognizing attainment levels of students within mathematical modelling.

A central object of the educational research taking place in the area of Mathematical Modelling and Applications is to recognize the attainment level of students at defined stages of the modelling process.
In Voskoglou (1995) we obtained a stochastic method for the description of the process of mathematical modelling in the classroom, when the teacher gives such kind of problems for solution to the students.
To do so we assumed that the above process has the “Markov property”, i.e. that the probability for it to be at one of its stages at a certain phase depends mainly from the stage occupied in the previous phase and not in older ones. This is a simplification (not faraway from the truth) made to the real situation in order to transfer from the real to the “assumed real system” (see section 2) and our final results, which are consistent with other recently reported research (see below), show that this simplification gives a reliable prediction of the real system’s performance. This simplification enables us to introduce a finite Markov chain having as states the main stages of the mathematical modelling process that we have described in section 2 and form its transition matrix in terms of the “flow-diagram” of the process shown in figure 2.
Since the modeling process starts again from $s_1$as soon as it is completed at state s_5 (as the teacher gives to the class a new problem for solution) the resulting chain is an ergodic one (i.e. it is possible to go between any two states not necessarily in one step) and therefore it reaches an equilibrium situation in the long run.
Applying standard results from the theory of ergodic Markov chains (e.g. see Kemeny and Snell 1976, Chapter 5) we expressed mathematically the “gravity” of each stage of the mathematical modelling process (where bigger “gravity” means more difficulties for the students in the corresponding stage) and we also obtained a measure for the student’s model building abilities in general.
The full presentation of this technique is out of the purposes of the present paper.
An improved version of our Markov model (Voskoglou 2005) has been accepted for presentation in the ICTMA 12 Conference (London , July 2005).
In this version among the others we introduce a 2-states submodel in order to underline the principles involved in our 5-states more complex model and to make it accessible more easily to those who have barely a passing acquaintance with Markov chains. The 2 states of our submodel are t_1=solution of the problem

(including states s_1, s_2 and s_3) and t_2=answer to the problem (including states s_4 and s_5).
An application in the classroom presented in Voskoglou (1995) is also reconsidered in the above paper in order to illustrate the improvements of our model. Two are the main results obtained from this application:
(i) There is a comparison between two groups of students with indications that the teaching of one group might be more effective than that of the other one , and
(ii) The analysis shows that students of both groups found the step of mathematisation more difficult than the other steps of the mathematical modelling process.
The second result was logically expected, since the formulation of the problem involves, as we have already justified in section 2, a deep abstracting process, which is not always an easy thing to do for a non expert
Inspecting the attempts of our students towards the solution of the problems I found that in the smashing majority of cases the successful formulation of the problem was followed by a successful construction of the corresponding mathematical model.
This important founding could also be illustrated mathematically through our model, if we had considered the formulation of the problem as an independent state of our Markov chain. Such a consideration however should make our model technically more complicated, since we should have to deal with 6x6 matrices and determinants.
Mathematics does not explain the natural behaviour of an object, it simply describes it. This description however is so much effective so that an elementary mathematical equation can describe simply and clearly a relation, that in order to be expressed with words could need entire pages. We believe that this is exactly the main advantage of our model compared with other qualitative methodologies used by other researchers for similar purposes, such as the analyses of questionnaire's collected answers by using respond maps (Stillman and Galbraith 1998), or multiple choice tests (Crouch and Haines 2001) etc, and the related discussion activities
Galbraith and Haines (2000) describe a hierarchy of procedural and conceptual skills in mathematical modelling, where the relative degree of success is

mechanical > interpretive > constructive

and demonstrate the validity of this taxonomy.
Crouch and Haines (2004, section 1) report that it is, however, the interface between the real world problem and the mathematical model that presents difficulties to the students, i.e. the transition from the real word to the mathematical model (which is consistent with the conclusions of our application) and vice versa the transition from the solution of the model to the real world. On the contrary the results of our application ("gravities" of the states s_4 and s_5 respectively) do not indicate any particular difficulty of our students across these stages of the modelling process.

According to our results the solution of the model is the stage having the second (after mathematising) greater "gravity" for both groups of student. This is partially crossed by Stillman and Galbraith (1998) reporting on an intensive study of problem solving activity of female students at the senior secondary level, where they found that more time was spent in general on execution (and orientation) activities with little time being spent on organization and verification activities.
Conclusively, as Haines and Crouch (2001) observe, further research remains to be done on how experts model and the relations, if any, between the processes employed by the expert modeller and by the novice (see also Crouch and Haines 2004, section 2). This could provide insights on links among the several stages of the mathematical modelling process.

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