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The importance of supplementary variables in a case of an educational research

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Résumé. Le but de ce papier est ce de faire ressortir l'importance des variables supplémentaires pour une recherche didactique sur la manière avec laquelle les élèves peuvent argumenter et conjecturer sur la fameuse conjecture de Goldbach . In particulier, puisque l'analyse des données fut exécutée par le logiciel "Chic", l'usage des variables supplémentaires fut efficace pour déduire des résultats significatifs en ce qui concerne l'attitude des élèves en présence du problème de la théorie des nombres.

1 The framework of the research

The theoretical framework of this educational research is the theory of didactical situations in mathematics by Guy Brousseau[1].
It is known that this theory is based on the conception of the didactic situations, and in particular, this work concerns an a-didactic situation, namely that part of a didactic situation which teacher's intention respect to pupils is not clear into. An a-didactic situation is really the moment of the didactic situation in which the teacher does not declare the task to be reached but he gets the pupil to think about the proposed task which is chosen in order to allow him to acquire a new knowledge and that it is to be looked for within the same logic of the problem.
So, an a-didactic situation allows a pupil to appropriate and to manage the staking dinamycs, to get him to be a protagonist of the process, to get him to perceive the responsibility of it as a knowledge and not as a guilt of the saught result.

2 The historical context of the research

The research concerns some conceptions of pupils facing a conjecture, and in particular a famous historical conjecture like Goldbach's one. Goldbach's conjecture was chosen because it has a long historical background allowing an efficient a-priori analysis, which is an important phase for the experimentation in order to foresee the possible pupils's answers and behaviours in front of the conjecture. Moreover, it has a fascinating formulation allowing pupils to mix many numerical examples, and to discuss fruitfully about its validity and some possible attempts of a demonstration.
So, the historical context is important because it suggestes an interplay between the history of mathematics and the mathematics education.

[1] See G. Brousseau, *Théorie des situations didactiques, Didactique des mathématiques* 1970-1990, Textes rassamblés, La pensée sauvage, Grenoble 1998.

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3 Using implicative analysis

This research was carried out by a quantitative analysis along with a qualitative analysis. The statistical survey for the quantitative analysis was made by two phases: in the first experiment, which was realized with a sample of pupils attending the third and fourth year of study (16-17 years) of secondary school, the method of individual and matched activity was used; the second experiment was carried out in three levels: pupils from the first school (6-10 years), pupils from primary school (11-15 years) and pupils from secondary school.
The quantitative analysis of the data drawn from pupils's protocols was made by the software of inferential statistics[2] CHIC 2000 (*Classification Hiérarchique Implicative et Cohésitive*) and the factorial statistical survey S.P.S.S. (*Statistical Package for Social Sciences).*
The research pointed out some important misconception by pupils and some knots in the passage from an argumentative phase to a demonstrative one of their activity which need to be deeped.

4 The experimentation

The research was realized on different levels by two experiments. The first experiment, was realized with pupils attending the third and fourth year of study (16-17 years) of secondary school, the method of individual and matched activity was used. Pupils working individually were expected, within two hours, to answer the following question:

a) Using the enclosed table of primes, the following even numbers can be written as a sum of two primes (in an alone or in a manner more)? 248; 356; 1278; 3896.
b) If you have answered the previous question, are you able to prove that it occurs for every even number?
The pupils working in couples were expected, within an hour, to answer this question (in a written form and only if they have agreed):

Is it always true that every even natural number greater than 2 is a sum of two prime numbers? Let argue about the demonstrative processes motivating them.
In both cases the procedure was acoustically recorded and the transcript of those records with comments was made.
The second experiment was carried out in three levels: pupils from the primary school (6-10 years), pupils from middle school (11-15 years) and pupils from higher secondary school. The experiment was carried out on the lowest level in two phases: In the first phase the pupils could answer this question:

How can you obtain the first 30 even numbers by putting together prime numbers of the table you have just made?

In the second phase, the pupils created small groups and tried to answer the following question:
Can you derive the even numbers obtained by summing always and only two primes? If it is so, can you state this is always the case for an even number?
The pupils from lower secondary school solved the following problem within 100 minutes:

Is the following statement always true? "Can an even number be resolved into a sum of prime numbers?" Argue your claims.

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[2] See R. Gras, *Les fondements de l'analyse statistique implicative*, Quaderni di Ricerca in Didattica del G.R.I.M., Dipartimento di Matematica dell'Università di Palermo, n. 9, 2000, pp. 187-208.

The procedure had four phases:
c) discussion about the task in couples (10 min.)
d) individual written description of a chosen solving strategy (30 min.)
e) dividing of the class into two groups discussing the task (30 min.)
f) proof of a strategic processing given by the competitive groups (30 min.)
Pupils from higher secondary school solved the same problem like the pupils from middle school in the same way and within the same time limit.
Individual works were analyzed (a-priori analysis), the identification of parameters was carried out and those were subsequently used as a basis for the characteristics of pupils' answers. It enabled to do a quantitative analysis of the answers, to establish an implicative graph (graph functionality), hierarchical diagram, diagram of similarities and also factor data analysis. The analyses, graphs and diagrams (or trees) were, together with conclusions, part of the evaluation of each experiment.

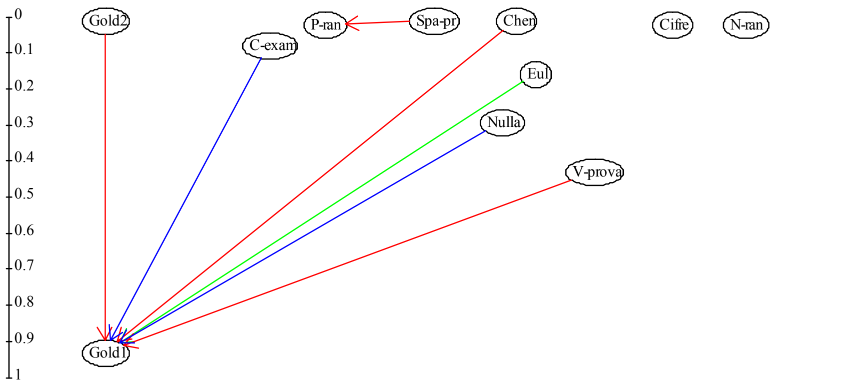
5 The first experiment and the analysis by CHIC

As I said, the first statistical survey was made by using a sample of 88 pupils attending the third and fourth year of study of secondary school in Palermo (Sicily). The students worked in pairs for the part relating to interviews and individually for the production of solution protocols related to the proposed conjecture. The variables used for the a-priori analysis were 15 and they were explained by the following steps:

1) He/she verifies the conjecture by natural number taken at random.(N-random)
2) He/she sums two prime numbers at random and checks if the result is an even number. (Pr-random)
3) He/she factorizes the even number and sums its factors, trying to obtain two primes. (Factor)
4) *Golbach's method 1.* He/she considers odd prime numbers lesser than an even number, summing each of them with successive primes. (Gold1)
5) *Golbach's method* 2 (*letter to Euler*). He/she writes an even number as a sum of more units, combining these in order to get two primes. (Gold2)
6) *Cantor's method.* Given the even number $2n$, by subtracting from it the prime numbers $x \leq 2n$ one by one, by a table of primes one tempts if the obtained difference $2n - x$ is a prime. If it is, then $2n$ is a sum of two primes. (Cant)
7) *The strategy for Cantor's method.* He/she considers the primes lower then the given number and calculates the difference between the given number and each of primes. (S-Cant)
8) *Euler.* He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)
9) *Chen Jing-run's method (1966).* He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)
l0) He/she subtracts a prime number from an any even number (lower then the given even number) and he/she ascertains if he/she obtains a prime, so the condition is verified. (Spa-pr)
11) He/she looks for a counter-example which invalidates the statement. (C-exam)
12) He/she considers the final digits of a prime to ascertain the truth of the statement. (Cifre)
13) He/she thinks that a verification of the statement by some numerical examples needs to prove the statement. (V-prova)
14) He/she does not argue anything for the second question. (Nulla)
15) He/she thinks the conjecture is a postulate. (Post)

Table 1- A-priori analysis.

5.1 The implicative Graph



Graphe implicatif : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\goldbach1.csv 99 95 90 85

Fig. 1 – The implicative graph 90%.

The analysis of the implicative graph shows, with percentages of 90%, 95% and 99%, that pupils' choice of following some of the strategies is strictly linked to a relevant strategy, namely Gold 1, or the one according which the pupil considers odd prime numbers summing each of them with successive primes. Hence the basis of pupil behaviour is the sequential thinking.

5.2 The hierarchic tree

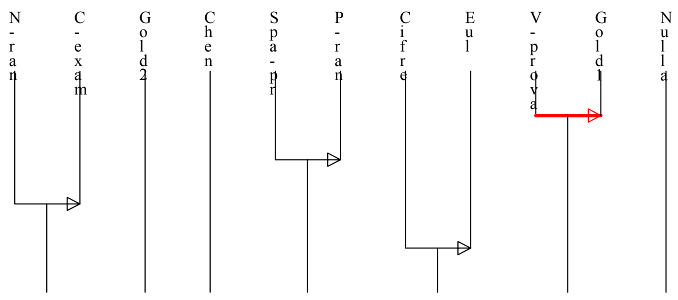


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Fig. 2 – The hierarchic tree.

The tree shows that the most hierachic link is between the variables V-prova and Gold1, as it appears also in the implicative graph.

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5.3 Similarity tree

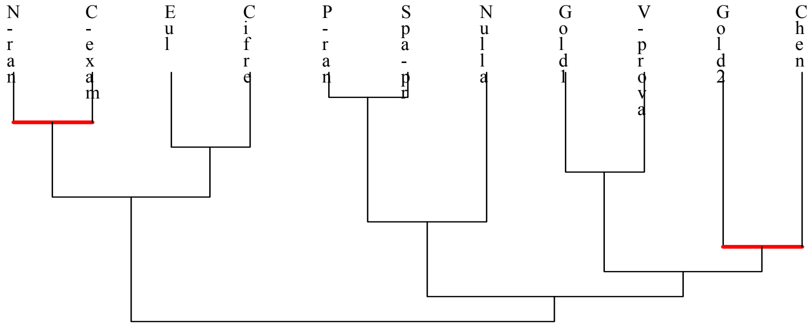

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Fig. 3 – Similarity tree.

We observe a similarity of the first order between five pairs of variables, and a similarity of the second order between three pairs of variables.

5.4 The factorial analysis by S.P.S.S.

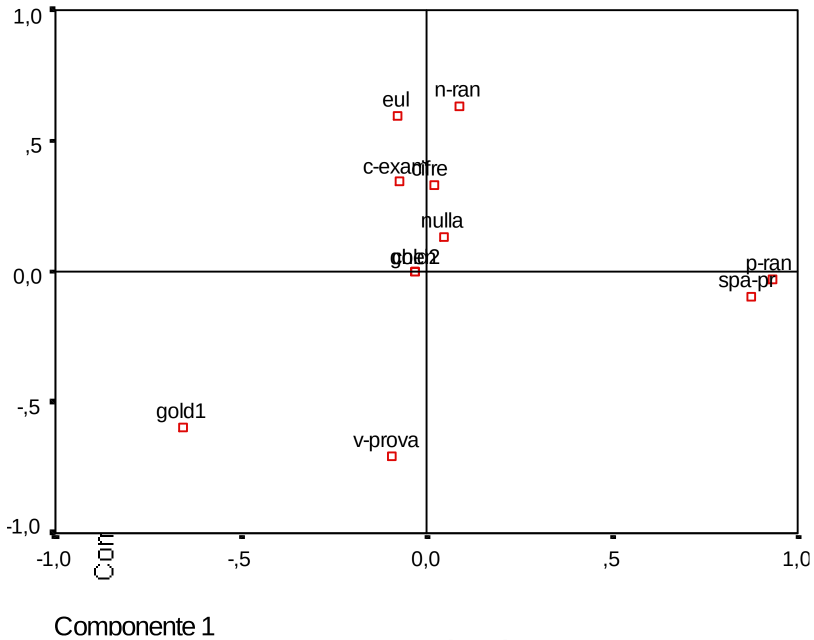

Fig. 4 – Factorial analysis.

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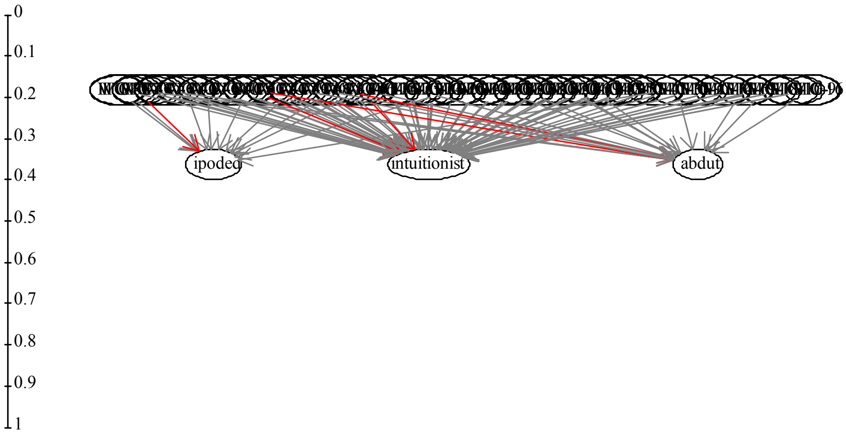
The graph shows that a part of pupils is inclined either to proceed by a sequential fashion or by preferring a method based on a random choice. On the other hand, the second component shows that the real strong characterization of most pupils is Gold1-Chen which is nearer to the intersection of the two components. So, this is the winning strategy among pupils to pass from an argumentation to a possible demonstration. This is a kind of a photo of the more frequent approachs to the conjecture by students.

5.5 Supplementary variables and pupils' profiles.

A further step was made by introducing three supplementary variables to get other informations about the obtained data. They were the following:
a) Abdut: this is the pupil proceeding by *abduction*, indicates (Peirce) the first moment of an inductive process, when a pupil chooses a hypothesis by which he may explain determined empirical facts. On the base of such a definition, the pupil named Abdut is who observes how Goldbach's conjecture to be verified in a large number of cases, therefore he supposes it is also valid for any very large even number, and that leads him to the final thesis, that is the conjecture to be valid for every even natural number.
b) Intuitionist: (at the present who proceeds by an inductive argumentation) is instead the pupil having the N-random and Euler strategies in common with Abdut, but thinking that the demonstration of the conjecture can be deduced by a simple numerical evidence, because he is convinced that what happens for the elements of a small finite set of values can be generalized to the infinite set which the small set belongs to; so he uses the V-prova strategy. In short, in an inductive argumentation used by the intuitionist the statement is deduced as a generic case after researching from specific cases.
d) Ipoded: is just the pupil using a deductive argumentation which can be directly transposed into a deductive demonstration.
With these new additional variables a transposed matrix (changing rows by columns) was made by Excel and interpreted by CHIC. The more interesting results were the following:

a) The implicative graph



Graphe implicatif : C:\WINDOWS\Desktop\tesiFP-Dott\aldo\Goldtrasp2.csv 99 95 90 85

Fig. 5 – The implicative graph with supplementary variables.

It is evident that the three profiles corresponding to the additional variables are significative as much as they *catalyze* the outlines of reasoning of the pupils. The supplementary variables play in this case the same role

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plaied by an equivalence relation when we put it into a set. In fact, what the relation does in this case? It orders the elements of the set, so the supplementary variables, in this case give much more order to the data. They get the interpretation of data more effective. Really, they begin attractors for pupils's behaviours.

b) Factorial Analysis

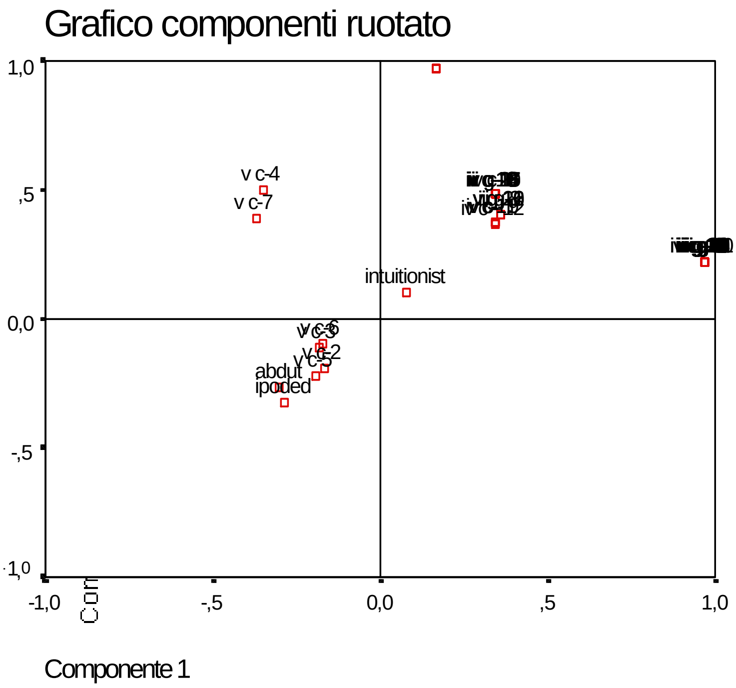

Fig. 6 – Factorial analysis with supplementary variables

From the viewpoint of the horizontal component the variable Intuitionist characterizes it weakly, while the variables Abdut and Ipoded with a lot of other variables characterize it much more. On the other hand, this is a paradigmatic situation which has its historical counterpart in the attemps made along centuries by different mathematicians facing the conjecture. So, Abdut and Ipoded profiles are winners, while it is less productive the intuitive method of approach. This characteristic situation is stationary also when one observes the graph from the viewpoint of the second component. This means that in any way Abdut and Ipoded methods are more interesting for pupils.

6 Some final observations

The experimentation about Goldbach's conjecture has pointed up that in general most pupils, while facing an unsolved historical conjecture (without knowing it is yet unsolved), start at once with an empirical verification of it which can support their intuition, but after they distinguish theirselves along three different solving tipologies:

- a congruos part of pupils bites off more than one can chew with the following conclusion: since the conjecture is true for all of these particular cases, then it has to be true anyway.

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These are pupils who have a strong faith in their convinctions, but who do not know clearly enough how to pass from an argumentation to a demonstration, by using the achieved data.

- a part of pupils proceeds at the same time by an empirical verification and by an attempt of argumentation and demonstration ending in a mental stalemate. They try to clear a following hurdle: how can I deduce anything general from the empirical evidence?

These are pupils who before making any generalization want to be sure of the made steps, therefore they tread carefully.

- few pupils, after a short empirical verification, look at once for a formalization of their argumentations, but if they are not able to do that, they are not diffident about claiming they are in front of something which is undemonstrable. These pupils have a high consideration for their mental processes therefore they think that if they are not able to demonstrate anything, then it has to be undemonstrable anyway.

By this experimentation we argue that the argumentation favoured by pupils facing a historical conjecture like Goldbach's is the abductive one. Some questions arise from the results which would be advanced by other experimentations:

- Is this result generalizable?
- To what extent it is generalizable?

But the fundamental kernel of this experimentation about the interplay between history of mathematics and mathematics education is that such results could not be pointed out if the a-priori analysis had not been made by the historical-epistemogical remarks which have inspired it.

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Summary

The aim of this work is to point out the importance of supplementary variables for an educational research about arguing and conjecturing by pupils on Goldbach's conjecture concerning prime numbers. In particular, since the analysis of data was carried out by the software CHIC, using supplemetary variables was effective to deduce significative results on pupils's behaviours.in front of that number theoretic problem