

Tools for analyzing learning processes in mathematics

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Premise

My talk continues the lectures I gave at previous YESS presenting some fresh developments got in these last years by the Turin research team that I coordinate¹. Since most participants to YESS 5 are new-comers the lecture is self-contained. In the first part I shall sketchily illustrate the main problems that we study through some examples, and shall introduce a theoretical frame suitable for describing didactical phenomena in the classroom; in the second part I shall use such a frame to develop a fine analysis of the argumentative productions of students who are solving elementary calculus problems. Also this part will be structured through examples.

In both parts the YESS students will be asked to solve some problems and to debate some questions: in particular they will be asked to answer some questions like those at the end of this Summary. To do that they are asked to see the videoclip in the web and to do some preliminary work of analysis before coming to YESS (see the Questions at the end of this Summary).

This Summary and its Appendices contain the material that students should read before coming to YESS.

PART I

The focus of our researches concerns the *embodied* and *multimodal* way in which the learning processes happen and develop in the mathematics classroom. In fact, in their mathematical activities students and teachers use a variety of resources: words (orally or in written form); extra-linguistic modes of expression (gestures, glances, ...); different types of inscriptions (drawings, sketches, graphs, ...); different instruments (from the pencil to the most sophisticated ICT devices), and so on. This issue puts forward four types of intertwined problems:

1. (*What*-problem) What is necessary to observe in the classroom?
2. (*Why*-problem) Which theoretical frames are suitable to answer the *What*-problem?
3. (*How*-problem)
 - (i) How to observe all that is necessary?
 - (ii) How to interpret the observed data according to the assumed frame?
4. (*Goal*-problem) How to improve consequent didactical practices in the classroom?

The presentation will first address Problem 1 presenting some concrete examples (one is in the video-clip that you find at the following address:

<http://www2.dm.unito.it/paginepersonali/arzarello/index.htm>, click on YESS Texts and Videoclips).

The different resources used by the students and by the teacher illustrate the embodied (and multimodal) paradigm and allow introducing a theoretical frame: the *Space of Action, Production and Communication* (APC-space)⁽²⁾. The APC-space gives reason of the didactical phenomena within the embodied paradigm and represents my solution to Problem 2. It studies learning processes using a semiotic lens within a Vygotskian perspective; as such it is suitable also for studying the *semiotic mediation*⁽³⁾ promoted by the teacher.

To concretely focus the learning processes within the APC-space – hence answering Problem 3(i) – I shall introduce a semiotic analysis tool, namely the *Semiotic Bundle* ⁽²⁾.

¹ It is made by the colleagues Luciana Bazzini and Ornella Robutti, by some young researchers, like Francesca Ferrara and Cristina Sabena, and by many teachers (from the elementary to the higher school level) that participate actively to our researches, like Riccardo Barbero, Silvia Beltramino, Silvia Ghirardi, Marina Gilardi, Patrizia Laiolo, Donatella Merlo, Domingo Paola, Ketty Savioli, Bruna Villa and others.

² See Appendix 1

³ See Appendix 2.

Successively I shall come back to the classroom episodes introduced at the beginning. I shall read them through the theoretical tools previously defined, namely the APC-space and the Semiotic Bundle. I will so be able to answer Problem 3(ii).

The content of Part I is summarized in sections 1, 2, 3: first I shall discuss the main focus of our research (§1), then I shall sketch the main points of the embodied and multimodal paradigm (§2) and finally I shall make some comments about the APC-space and the Semiotic Bundle tool (§3).

Part II sketches how the semiotic lens can be used to analyse students' argumentative processes while solving elementary calculus problems.

1. The main problem

The major focus of didactics of mathematics should concern learning processes, as pointed out by H. Freudenthal in all his work:

“...the use of and the emphasis on *processes* is a *didactic principle*. Indeed, didactics itself is concerned with processes. Most educational research, however, and almost all of it that is based on or related to empirical evidence, focuses on states (or time sequences of states when education is to be viewed as development). States are *products* of previous *processes*. As a matter of fact, *products* of learning are more easily accessible to observation and analysis than are learning *processes* which, on the one hand, explains why researchers prefer to deal with states (or sequences of states), and on the other hand why much of this educational research is didactically pointless.” (Freudenthal, 1991, p. 87, emphasis in the original)

If one looks carefully to the phenomenology of learning processes in the class of mathematics, one sees a variety of actions and productions activated by the students and by the teacher using different semiotic resources: for some commented examples, see [A], [B] in Appendix 1. As shown there, such resources are used with great flexibility: generally the same subject exploits simultaneously many of them (e.g. speech and gesture). Sometimes they are shared by the students (and possibly by the teacher) and used as communication tools, other times they reveal as crucial thinking tools. As illustrated in [A] and [B] all such resources, with the actions and productions they support, appear important in the building of mathematical ideas. To use the slogan from Arzarello (2008), they are useful to bridge the gap between the time-less and context-less sentences of formal mathematics and the worldly experience that allows people to grasp the meaning of mathematical concepts.

These general observations show that in order to describe scientifically the learning processes that happen in the classroom according to the claim of Freudenthal (*What* Problem 1), it is necessary to consider all such resources, the activities that they allow and also how they develop. The broad answer to the first Problem requires to approach suitably also Problems 2 (*Why*) and 3i (*How* observing): first, one must develop tools, which allow observing *all* the resources that are acted on or are produced in the classroom (Problem 3i); second, one must reconsider the epistemological and cognitive paradigms according to which mathematics learning is described (Problem 2).

In the following section I shall summarise the fresh general paradigm, from which our research has critically developed, namely the paradigm of embodiment.

2. Multimodality

The new paradigm of *embodiment* is developing in these last years (see Wilson [2002] for an incisive summary). It is a movement afoot in cognitive science that grants the body a central role in shaping the mind. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment.

It concerns different disciplines, in particular cognitive science and neuroscience, interested with how the body is involved in thinking and learning (e.g. about mathematics).

In fact, a major result of neuroscience is that

“conceptual knowledge is embodied, that is, it is mapped within the sensory-motor system...The sensory-motor system not only provides structure to conceptual content, but also characterises the semantic content of concepts in terms of the way in which we function with our bodies in the world” (Gallese & Lakoff, 2005, p.456).

The embodiment stance points out that the boundaries between perception and action on the one side, and cognition on the other side, are *porous* (the terminology is from Seitz, 2000): for an example, see the protocols of Eleanore in Arzarello (in print).

This idea is a strong challenge to those common positions, according to which “all concepts – even concepts about action and perception– are symbolic and abstract and therefore must be implemented outside the brain’s sensory-motor system” (Gallese & Lakoff, 2005, p. 455). The new frame states that we must analyse concepts not on the basis of “formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body’s functioning in the world” (*ibid.*), but considering the *multimodality* of our cognitive performances.

In fact, the sensory-motor system of the brain is multimodal rather than modular; this means that for example

“an action like grasping...(1) is neurally enacted using neural substrates used for both action and perception, and (2) the modalities of action and perception are integrated at the level of the sensory-motor system itself and not via higher association areas.” (*ibid.*, p. 459).

“Accordingly, language is inherently multimodal in this sense, that is, it uses many modalities linked together—sight, hearing, touch, motor actions, and so on. Language exploits the pre-existing multimodal character of the sensory-motor system.” (*ibid.*, p. 456).

Our frame considers the consequences of this multimodal approach for learning processes in mathematics.

The stance of multimodality implies that “the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context.” (Nemirovsky, 2003; p. 108). For a concrete example see [B], § 3. Conceptualisation processes are inherently multimodal in the sense that they use many different modalities linked together: sight, hearing, touch, motor actions, and so on. Verbal language itself (e.g. metaphorical productions) is part of these cognitive multimodal activities.

A major consequence of this approach is that the construction of mathematical concepts must be re-elaborated according to the new paradigm (Answer to Problem 1 [*What?*] and partially to Problem 2 [*Why?*]).

But the multimodal paradigm has important consequences also for problem 3i (*How* observing?). in fact, it puts forward the relevance of time grains within which the multimodal activities and productions must be observed. An important result of our research entails the necessity of observing carefully the mutual relationships among the different components active in the multimodal processes (e.g. how speech and gestures or gestures and drawings are related each-other). This implies the necessity of using an extremely fine-grain analysis: for example to grasp some of these relationships one must consider scales of time of few hundredths of second. This requires suitable tools for collecting data, typically videotaping what is happening in the classroom (possibly with more than one video-camera) and studying carefully the videos. That means studying: what one sees in each frame (in the most common records, one can get 24 frames per second); the relationships among the frames; the relationships among what one sees in the video and what one reads on the transcriptions of the oral part of the video; the relationships among the different inscriptions used/produced by the students and their words and actions seen in the video, and so on. Hence these observation tools integrate the most usual ones based on the analysis of protocols produced by students and by recording their speech. It is interesting to stress that the multimodality of processes can be observed and studied only using such digital tools; hence it is because such sophisticated technological tools exist that we can approach learning processes in a fresh way, which was not possible a few years ago.

But it is also necessary to consider the evolution of the data categorised through this fine grain analysis in a large-scale dimension; namely it is important to focus the relationships among the main components of the learning processes in middle and long periods of time (from some days to different years). Hence the fine grain analysis must be integrated with a long-term analysis: for this generally different tools are necessary. For a discussion on this see Arzarello et al. (2002). Time will not allow me to enter into this side of the problem during the Talk.

All this heavy work is necessary to realise as better as possible the invitation of Freudenthal to study learning processes and not only states.

Now we have a complete answer to Problems 1, 3i; it is time to come to Problem 2: the multimodality paradigm constitutes a first partial answer to it. But multimodality is a general approach. What we need is a specific theoretical frame to explain the learning processes in mathematics. This theoretical framework is the APC-space and the related notion of Semiotic Bundle (see Appendix 1).

3. APC space and Semiotic Bundles.

The paper [B] illustrates how the processes of abstraction can develop in mathematics because of the semiotic activities and productions of students (and teachers). Such activities and productions are centered on the use of *semiotic resources*, from the canonical symbols of the semiotic registers to the broader ones of the semiotic bundles (see [B], for the distinction between the two), which include gestures, drawings, artifacts, and so on.

The Semiotic Bundle is a tool that allows entering into the semiotic activities and productions that happen in the classroom. These processes develop because of the didactic situation designed by the teacher and consist in students' (and teacher's) activities, productions and communication acts (e.g. the interactions with their mates, with the teacher, with the artifacts or the inscriptions given/produced in the didactic situation). Such processes have both biological and cultural roots: in fact they develop according to the multimodal paradigm but also within a culturally carved environment. The last one can be analyzed at least from three different points of view:

1. considering students' semiotic resources as *cultural semiotic systems* (see Radford, 2003; or [2] for a summary);
2. analysing the artifacts as *communication and representation infrastructures* (see Kaput et. Al., 2002; Hegedus and Moreno-Armella (2009); or [B]);
3. considering the *semiotic mediation* (see Appendix 2) promoted by the teacher.

As to the last one, a first general observation is that the teacher is responsible for the construction of the mathematical knowledge in the classroom because of the didactic situation that she/he designs: the semiotic activities/productions of the students develop because of their response to the task given by the teacher. The teacher takes care to support the evolution of the personal senses that the students attach to the proposed situation towards the scientific and shared sense that the task requires in order to be solved. In doing that, she/he becomes a part of the APC space; in fact she/he acts and communicates with students supporting such an evolution with suitable actions. The students re-act to the teacher actions and produce new elements of the APC space (fresh inscriptions, gestures, sentences, metaphors, conjectures, validations, and so on). The semiotic lens allows us to focus the dynamics and the evolution of such processes, which I call of semiotic mediation.

The Semiotic Bundle is like a movie of all the semiotic resources activated in the classroom during a specific period of time; it allows studying the didactical phenomena that happen in the classroom as semiotic activities: in the Talk I shall illustrate a concrete tool (the timeline analysis) to do that. While the Semiotic Bundle is like a movie, the APC-space is the didactical environment where such bundles develop: it allows considering the didactical aspects that support the phenomena that we see in the Semiotic Bundle.

The APC space represents an alternative, complementary approach to the classical frame of the theory of didactic situations (see Brousseau, 1997), based on the classical phases of action, formulation, devolution, validation and institutionalisation and where the milieu is conceived as the antagonist environment where the activities of students develop and are regulated by a feedback process, which does not require the active intervention of the teacher (at least after the devolution and before the institutionalisation phase).

The APC space is particularly suitable to study the teaching of such ‘global’ concepts like that of function, of derivative, of integral and so on, where a-didactical situations (Brousseau, 1997) are not so easy to find because of the complexity of the subject and where the use of artifacts (e.g. specific software) puts forward the necessity of a semiotic mediation, which requires a continuous intervention of the teacher.

The APC space is built up according to a Vygotskian approach. It allows interpreting all the learning processes within a multimodal paradigm but considering also their cultural side. The two aspects are deeply intertwined, as pointed out by L. Radford:

“an account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual’s subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual’s actions.” (Radford et al. 2005).

PART II

4. Semiotic and theoretic control.

This part sketchily illustrates how the Semiotic Bundle lens allows to point out two intertwined modalities of control, through which students manage their argumentative activities while solving mathematical problems: the *semiotic* and the *theoretic control*.

We speak of *semiotic control* when the decisions concern mainly the selection and implementation of semiotic resources, namely when the decisions concern activities featured by the treatment of signs, according to a wide Peircean interpretation⁴ of what a sign is (from the symbols and graphs of mathematics to the drawings sketched by the subjects, to their gestures, etc.). On the other hand, the control is *theoretic* when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it, and is accomplished through a more or less organised cluster of properties, algorithms and possibly conceptions that subjects activate to elaborate an argument or a proof. For example, a semiotic control is necessary to choose a suitable semiotic representation for solving a task (e.g. an algebraic formula Vs a Cartesian graph), while a theoretic control intervenes when a subject decides to use a theorem of Calculus or of Euclidean Geometry for supporting an argument. Of course the two modalities are often intertwined in the concrete actions of subjects who solve a task and are here distinguished for the sake of analysis.

In many cases the semiotic analysis of students’ activities while arguing and proving distinguishes three typologies of actions:

- 1) Signs are perceived as markers of facts: phenomenological description;
- 2) Local relationships between facts are looked for; (economic, explanatory and testable) hypothesis are detected and made explicit: abductions;
- 3) Global relationships between facts are looked for; production of arguments that give reasons why the observed facts are in the way they are; explanations are given within a theory: deductions.

These three kinds of semiotic actions are deeply intertwined. Whereas there may be an evolution from the first to the third kind (through the second), the previous kind of activities do not disappear when passing to the next ones. The transition between the different semiotic actions involves also a shift of attention and is therefore related to what we call semiotic and theoretic control. In fact, when perceiving signs as facts, the attention is mainly concentrated in the signs themselves, i.e. in looking at meaningful facts: perception has a relevant role in this process. Hence, the semiotic control is particularly relevant. In the second kind of semiotic actions, the attention is focussing on the relationships between the different observed components. Here the logical control starts to be relevant and a dialectic interaction between logical and semiotic control is needed. Abductions have an important role at this point. Finally, in the third kind the relationships are explained within a mathematical theory. The arguments produced in the first two kinds of semiotic actions are mainly local and have an abductive nature (see [D]), whereas the third kind is marked by arguments that

⁴ See [D] in Appendix 1.

have also deductive, global and formal features. We do not intend that during argumentation and proving *processes* the semiotic actions of the third kind destroy the previous ones. Generally, they prevail in the *products* of such processes. The semiotic elements are always present on the scene, but they may be more in the foreground (actions 1) or more in the background (actions 3). According to our model, we find therefore different elements in the foreground, and the attention is mainly concentrating on them:

- 1) Facts and perceived signs are on the fore, and the verbal productions are mainly phenomenological descriptions;
- 2) Local relationships between are on the fore, and the verbal productions are mainly abductions;
- 3) Global relationships between facts are on the fore, and the verbal productions are mainly deductions within a theory.

In our classroom observations, we have noticed an evolution in students' processes, namely a shift of control from a phase where it is mainly semiotic, with a low theoretic control, towards a phase where argumentation becomes the centre of their activities and evolves from abductive to deductive and more formal structures. An analysis through the Toulmin model of arguments (see [E]) can be useful to focus the structure of their arguments (Pedemonte, 2007). According to the Toulmin terminology, there is a transition from *substantial* to *analytical* arguments (Toulmin, 1958/2003, p. 114 ff.), even if the distinction is never so sharp: also in experts the two aspects generally live together. We can find an explanation for such a transition in the teacher's didactical design of the activities and in his role fostering the interactions of the students.

In the second part of the talk students will be asked to comment some excerpts from students' activities using this interpretative frame.

APPENDIX 1⁵

A. The space of Action, Production and Communication (APC-space)

The discussion above puts forward two main streams, which, according to our view, are necessary to give a fresh and faithful description of the processes of mathematics learning: the embodied and the semiotic-cultural approach. These two streams can be integrated into a unitary frame, namely the cognitive *space of action, production and communication* (APC-space), introduced by one of the authors (see Arzarello, in press; and Arzarello & Olivero, 2005). As a result of both theoretical and empirical research, the APC-space is a model, which frames the processes that develop in the classroom among students (and the teacher) while working together. It allows analysing them considering their different components and by a variety of mutually dependent relationships among them. The components are: the body, the physical world, the cultural context, that means the students themselves along with the environment, where they are acting and learning. Our point is that when students learn mathematics the integration of these and other components (e.g. the emotional ones) take an active part in the learning process, interacting together. The interaction comes from the students' work, the teacher's mediation and the use of artefacts. The three letters A, P, C refer to the main dynamic relationships among its components, namely students' Actions and interactions (e.g. in a situation at stake, with their mates, with the teacher, with themselves, with tools), their Productions (e.g. answering a question, posing other questions, making a conjecture, introducing a new sign to represent a situation, and so on) and Communication aspects (e.g. when the discovered solution is

⁵ The excerpts A, B, C, D, E are taken from different papers of the author.

communicated to a mate or to the teacher orally or in written form, using suitable representations). In short, *an APC space is the unitary system of its components and of their mutual relationships, amalgamated in a dynamically evolving unit within a concrete learning situation in the classroom, because of the action and mediation of the teacher, who suitably orchestrates their integration.*

The APC space is a typical complex system, which cannot be described in a linear manner as resulting by the simple superposition of its ingredients (i.e. the components and their mutual relationships). In particular it models how the relationships among its components evolve in the classroom through the specific action of the teacher, who triggers their activation and coaches their integration. It is beyond the aims of this paper analysing the reasons why the integration process may possibly fail, even if it is an important question.

The APC-space properly frames the multimodal way of learning, discussed above. Namely it allows to consider how action and perception determine the processes of learning and to describe them so that doing, touching, moving and seeing appear as their important ingredients, which act together. The example that we shall discuss below [see the full paper] will illustrate how this happens also in learning processes in adult students concerning the most advanced topics; for an example involving younger students, see Arzarello et al. (2006) and for further examples see Arzarello (in press). Of course, a learning approach based on perceptuo-motor activities, requires suitable modalities of teaching, in which the students are actively involved in the construction of mathematical concepts, as in the example that we will discuss below. In this perspective, the artefacts that are introduced in the didactical practice can support and mediate the construction of the experiential base necessary for learning, according to the philosophy of the mathematical laboratory, sketched above.

From the research point of view, studying the APC-space means to enter in its ingredients and to analyse them: from students' sensory-motor experiences to the embodied templates that they activate, from their words to every kind of signs or representations they are using while working with their mates in a context with material and cultural features. An essential point to analyse is the role of the teacher in this systemic frame, which consists in favouring and supporting the production through the various interactions, as we shall discuss below (last but one section).

Generally a good teacher can do this almost in an unconscious manner. What we shall try to do here is to explicit this unconscious work. In order to do this, we need suitable tools to investigate the main ingredients of the APC-space, e.g.:

- *historical and social sciences* to analyse the cultural environment;
- *semiotics* to analyse signs, languages and cultural artefacts;
- *psychology* to analyse perceptions, actions and gestures.

Of course the same tool can be used to analyse different ingredients. E.g. gestures can be studied both in psychology and in semiotics; languages can be studied also with psychological and neurological instruments, and so on. The list above gives only an idea of the different complementary tools, which are necessary to carry out the multiple analysis of APC-space. In a sense, each lens gives a picture of the whole from a particular point of view. It is beyond the aim of this paper to develop all these analysis and we limit ourselves to unfold a semiotic analysis, which particularly fits with the approach we have settled in the previous chapters. In any case, analysing the activities in the classroom through a semiotic lens allows us to suitably describe most of the didactical phenomena, because of the systemic nature of the APC-space.

B. The semiotic bundle

The basis of the semiotic analysis of the APC-space is an analysis tool, called *semiotic bundle*, which enlarges the classical notion of semiotic system. According to Ernest, (2006, pp. 69-70), a *semiotic system* consists of three *components*:

1. A set of signs, the tokens of which might possibly be uttered, spoken, written, drawn or encoded electronically.
2. A set of rules of sign production and transformation, including the potential capacity for creativity in producing both atomic (single) and molecular (compound) signs.
3. A set of relationships between the signs and their meanings embodied in an underlying meaning structure.

An essential feature of a semiotic system has been pointed out by Duval (1999), who introduced the concept of *semiotic representations*: namely the signs, relationships and rules of production and transformation are semiotic representations insofar as they bear an intentional character.

Other important aspects of semiotic systems are their *semiotic functions*, which can be distinguished in *transformational* or *symbolic* (see Arzarello et al., 1994). The transformational function consists in the possibility of transforming signs within a fixed system or from a system to another, according to precise rules (algorithms). For example, one can transform the sign $x(x+1)$ into $(x^2 + x)$ within the algebraic system (register) or into the graph of a parabola from the Algebraic to the Cartesian system. Duval (2002, 2006) calls *treatment* the first type of transformations and *conversion* the second ones.

The *symbolic* function is the possibility of interpreting a sign within a register, possibly in different ways, but without any concrete treatment or conversion on it. E.g. if one asks if the number $n(n+1)$ is odd or even one must interpret n and $(n+1)$ with respect to their oddity and see that one of the two is always even. This is achieved without any transformation on the written signs, but interpreting differently the signs n , $(n+1)$ and their mutual relationships, the first time as odd-even numbers and then as even-odd numbers. The symbolic function of signs has been described by different authors with different words and from different perspectives: C.S. Peirce, C.K.Ogden & I.A. Richards (semiotics), G. Frege (logic), L. Vygotsky (psychology) and others: see Steinbring (2005, chapter 1) for an interesting summary focusing the problem from the point of view of mathematical education.

Semiotic systems allow us to study important components of the APC-space but unfortunately, they are not enough: in APC-space it is the same notion of sign and of operations upon them that needs to be considered within a wider perspective. In fact, in the classroom one observes phenomena that can be considered as signs that enter the semiotic activities of students^(s) but they are not inscriptions, nor are processed through specific algorithms as in the more standard definition. For example, observing students, who solve problems working in group, also gestures, gazes and generally their body language reveal as crucial semiotic tools. Namely, non-written signs and non-algorithmic procedures must be taken into consideration within a semiotic approach. Roughly speaking, it is the same notion of sign and of operations upon them that needs to be broadened. In fact, during the years many scholars have tried to widen the classical formal horizon of semiotic systems considering also less formal or not formal components.

This tendency is already in the complex evolution of the sign definition in Peirce (see this evolution considering the item 'sign' in the analytic Index of his Collected Works: Hartshorne & Weiss, 1933) and is also contained in some pioneering observations of Vygotsky concerning the relationships between gestures and written

signs, like the following: “*The gesture is the initial visual sign that contains the child’s future writing as an acorn contains a future oak. Gestures, it has been correctly said, are writing in air, and written signs frequently are simply gestures that have been fixed.*” (Vygotsky, 1978, p. 107; see also: Vygotsky, L. S. 1997, p. 133.).

But it is specifically in some recent researches of Radford (2006), Arzarello & Edwards (2005), Bosch & Chevallard (1999) and others that semiotics systems are studied within a wider approach. Using our theoretical frame we can say that we need to describe *all* the semiotic activities that happen in the classroom. To give reason of them, it is necessary to broaden the notion of semiotic system, so to encompass all the variety of phenomena of semiotic activity in the classroom. Moreover, the psychological processes of internalisation, described by Vygotsky and so important in describing the semiotic mediation of signs and tools, must find a natural place within the new framework.

As a consequence, the same idea of operation within or between different registers changes its meaning. It is not any longer a treatment or conversion (in the terminology of Duval) within or between semiotic representations according to algorithmic rules (e.g. the conversion from the geometric to the Cartesian register). On the contrary, the (within or between) operations must be widened to encompass also phenomena that are not algorithmic at all: for example practices with instruments, gestures and so on. Hence the above definition of Ernest can be widened and we get the notion of *semiotic bundle*: it has been introduced in Arzarello (2006), from which this part of the paper is widely taken.

To define it, we need first the notion of *semiotic set*, which is a widening of the notion of semiotic system.

A semiotic set consists of:

- a) The signs which may possibly be produced with different actions that have an intentional character, such as speaking, writing, drawing, gesticulating, handling an artefact.
- b) The modes for producing signs and possibly transforming them; such modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject.
- c) The relationships among these signs and their meanings embodied in an underlying meaning structure.

The three components above (signs, modes of production/transformation and relationships) may structure in different ways, which span from the compositional systems, usually studied in traditional semiotics (e.g. formal languages) to the sets of signs (e.g. sketches, drawings, gestures). The former are made of elementary constituents and their rules of production involve both atomic (single) and molecular (compound) signs. The latter have holistic features, cannot be split into atomic components, and the modes of production and transformation are often idiosyncratic to the subject who produces them (even if they embody deeply shared cultural aspects, according to the notion of *semiotic systems of cultural meanings* elaborated by Radford, quoted above). The word set must be interpreted in a very wide sense, e.g. as a variable collection. A semiotic bundle is:

- (i) A collection of semiotic sets.
- (ii) The relationships between the sets of the bundle.

Some of the relationships may be conversion modes between them.

A semiotic bundle is a *dynamic structure*, which can change in time because of the semiotic activities of the subject: for example the collection of semiotic sets that constitute it may change; as well as the relationships among its components may vary in time; sometimes the conversion rules have a genetic nature, namely one semiotic

set is generated by another one enlarging the bundle itself (we speak of *genetic conversions*).

Semiotic bundles are semiotic representations, provided one considers the intentionality as a relative feature.

An example of semiotic bundle is represented by the unity speech-gesture: “we should regard the gesture and the spoken utterance as different sides of a single underlying mental process” (McNeill, 1992, p.1), namely “gesture and language are one system” (ibid., p.2). In our terminology, gesture and language are a semiotic bundle, made of two deeply intertwined semiotic sets (only one, speech, is also a semiotic system). Research on gestures has discovered some important relationships between the two (e.g.: match and mismatch in Goldin-Meadow, 2003; redundant and non-redundant gestures with respect to speech, in Kita, 2000: see Notes 11, 12 below). A semiotic bundle must not be considered as a juxtaposition of semiotic sets; on the contrary, it is a unitary system and it is only for sake of analysis that we distinguish its components as semiotic sets.

C. Semiotic bundles and the Vygotskian frame

Semiotic bundles allow to deepen the Vygotskian notion of semiotic mediation sketched above. The dynamics in the process of internalisation according to Vygotsky is based on the semiotic activities with tools and signs, externally oriented, which produce new psychological tools, internally oriented, completely transformed but still maintaining some aspects of their origin. According to Vygotsky, a major (yet not unique) component in this internalisation process is language in social interactions, which allows the required transformations. Moreover such transformations ‘curtail’ the linguistic register of the speech into a new register of what Vygotsky calls *inner speech*, which has a completely different structure. This has been analysed by Vygotsky in the last (7th) chapter of *Thought and Language* (Vygotsky, 1986), whose title is *Thought and Word*. Vygotsky distinguishes two types of properties that feature the inner from the outer language: he calls them *structural* and *semantic properties*. The structural properties of the inner language are its *syntactic reduction* and its *phasic reduction*: the former consists in the fact that inner language reduces to pure juxtaposition of predicates minimising its syntactic articulation; the latter consists in minimising its phonetic aspects (⁶), namely curtailing the same words.

The semantic properties of the inner language are based on the distinction made by the French psychologist Frederic Pauhlan(⁷) between the *sense* and the *meaning* of a word and by “the preponderance of the sense [smysl] of a word over its meaning [znachenie]” (Vygotsky, 1986, p. 244):

“the sense is...the sum of all the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole, which has several zones of unequal stability. Meaning is only one of the zones of sense, the most stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense.” (ibid., p. 244-245).

In inner language the sense is always overwhelming the meaning. This prevailing aspect of the sense has two structural effects on inner language: the *agglutination* and the *influence*. The former consists in gluing different meanings (concepts) into one expression(⁸); the latter happens when the different senses flow together⁶ into one unity. To explain the properties of inner speech, Vygotsky uses analogies that refer to the outer speech and these give only some idea of what he means: in fact he uses a semiotic system (written or spoken language) to describe something, which is not a semiotic system, but perhaps only a semiotic set (the inner language is not so structured as the external language). In fact, the grounding metaphors through which

Vygotsky describes inner speech show its similarity with semiotic sets: properties like *agglutination* and *influence* make inner speech akin to some semiotic sets, like drawings, gestures and so on. Also the syntactic phenomena of syntactic and phasic reduction (see above) mean that the so called linear and compositional properties of semiotic systems are violated. At the state of the research, it is still an open problem whether inner speech really *is* a semiotic set. E.g. it is not so clear in which sense the results of introspection, through which we become aware of our inner language, can be accepted as true signs, which presuppose intentionality.

D. Peirce's semiotics, the relevance of diagrammatic reasoning, and abduction

C.S. Peirce points out a “paradoxical feature” of mathematics, which distinguishes it from the other scientific disciplines:

It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (Peirce, C.P., 3.363: quoted in Dörfler, 2005, p. 57)

Mathematics happens to live in a tense dynamics between its *deductive nature* and those elements of *observation* that lead to discoveries and development. The linking ring, suggests Peirce, is constituted by *signs*⁶ and by what he calls *diagrammatic reasoning*.

Peirce defines diagrammatic reasoning as a three-step process:

- (a) constructing a representation;
- (b) experimenting with it;
- (c) observing the results.

Through the diagrammatic reasoning he could overcome the epistemological paradox illustrated in the quotation above, underlying the relevance of the perceptual components in mathematical activities. As Radford points out:

Diagrammatic thinking is a central piece in Peirce's endeavour to rescue the epistemological import of perception. It is strongly linked to a heuristic process that exhibits, via intuition (i.e., in a sensual manner), some aspects of the object under scrutiny, thereby making these aspects available for observation, in order to help us discover new conceptual relations. The epistemological potential of diagrammatic thinking rests then in making apparent some relations that have thus far remained hidden from perception or beyond the realm of our attention. (Radford, 2008)

⁶ According to Peirce (1931-1958), a *sign* is a triad composed by the sign or *representamen* (that which represents), the *object* (that which is represented), and the *interpretant*: “It [The sign] addresses somebody, that is, creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object, not in all respects, but in reference to a sort of idea.” (C.P. 2.228)

In the examples we will discuss how diagrammatic reasoning allows a subject to develop what we call *semiotic control* in a mathematical argumentation, and how it is intertwined with what we call *theoretic control*. Schoenfeld (1985) defines control in problem solving activities as “Global decisions regarding the selection and implementation of resources and strategies”. It entails actions such as: planning, monitoring, assessment, decision-making, and conscious metacognitive acts. We speak of *semiotic control* when the decisions concern mainly the selection and implementation of semiotic resources, namely when the decisions concern activities featured by the treatment of signs, according to a wide Peircean interpretation of what a sign is (from the symbols and graphs of mathematics to the drawings sketched by the subjects, to their gestures, etc.). On the other hand, the control is *theoretic* when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it, and is accomplished through a more or less organised cluster of properties, algorithms and possibly conceptions that subjects activate to elaborate an argument or a proof. For example, a semiotic control is necessary to choose a suitable semiotic representation for solving a task (e.g. an algebraic formula Vs a Cartesian graph), while a theoretic control intervenes when a subject decides to use a theorem of Calculus or of Euclidean Geometry for supporting an argument. Of course the two modalities are often intertwined in the concrete actions of subjects who solve a task and are here distinguished for the sake of analysis.

A further tool from Peirce is abduction, specifically the so-called *sylogistic abduction* (C.P. 2.623), according to which a *Case* is drawn from a *Rule* and a *Result*. It is well known his example about beans:

<i>Rule:</i>	All the beans from this bag are white
<i>Result:</i>	These beans are white
<i>Case:</i>	These beans are from this bag

As such an abduction is different from a deduction that would have the form: the Result is drawn from the Rule and the Case, and it is obviously different from an induction, which has the form: from a Case and many Results a Rule is drawn. Of course the conclusion of an abduction holds only with a certain probability (in fact Polya, 1954 called this abductive argument an ‘*heuristic syllogism*’). The conclusion is a plausible hypothesis, and in fact Peirce (1931-58) called ‘*hypothesis*’ the abduction. Moreover, as pointed out by Peirce (C.P. 5.14-212), three aspects determine whether a hypothesis (abduction) is promising: it must be *explanatory*, *testable*, and *economic*. A hypothesis is an *explanation* if it accounts for the facts; its status is that of a suggestion until it is verified, which explains the need for the *testability* criterion. The motivation for the *economic* criterion is twofold: it is a response to the practical problem of having innumerable explanatory hypotheses to test, and it satisfies the need for a criterion to select the best explanation amongst the testable ones. Peirce points out that abductive reasoning is essential for every human inquiry. It is intertwined both with perception and with the general process of invention: "It [abduction] is the only logical operation which introduces any new ideas" (C.P. 5.171). In short, abduction becomes part of a *process of inquiry* in which abduction, induction, and deduction play particular roles. Eco (1983) describes abduction as the search for a general rule from which a specific case would follow. If there are multiple general rules to be selected from, Eco calls the abduction ‘*undercoded abduction*’ (p. 206). It is different from the ‘*overcoded abduction*’ (only one rule) and from the ‘*creative abduction*’ (no rule to select).

E. Toulmin’s model for argumentation

Toulmin (1958/2003) pointed out a model to analyse the *structure* of argumentations, independently of their fields. According to the Toulmin’s model, every argumentation is composed by a statement

or *claim*, some *data* justifying the claim, and some inference rule (called *warrant*) allowing to link the data to the claim. This basic structure may be enriched by auxiliary elements: a *qualifier*, indicating the strength provided by the warrant; a *rebuttal*, which indicates the exceptions to the rule of the warrant; the *backing*, i.e. the grounding of the warrant. Whereas the warrant is usually mentioned in explicit way, the backing remains often implicit. To illustrate the Toulmin's model, let us consider an example provided by Toulmin himself (Toulmin, 1958/2009, p. 92 ff.). '*Harry is a British subject since he was born in Bermuda*'. 'Harry is a British subject' is a claim that is supported by the data 'Harry was born in Bermuda', according to the warrant 'a man born in Bermuda is a British subject'. This warrant lies on a certain backing, which refers to the British laws. A possible rebuttal could be, for instance, the fact that Harry has become a naturalized American.

Toulmin's analysis is aimed to "characterize what may be called 'the rational process', the procedures and categories by using which claims-in-general can be argued for and settled" (*ibid.*, p. 7). He is interested in studying the structure of arguments as a product rather than the processes through which they are generated. Following other researchers (see for instance Pedemonte, 2007; Inglis, Mejia-Ramos & Simpson, 2007; Jahnke, 2008), we will adopt the Toulmin's (complete) model to study argumentation and proof not only as products but also as processes. An innovative contribution of our work will be complementing the structural analysis that makes use of such a model with a semiotic analysis that takes into account the specificity of the content of the argumentation. This operation is not stranger to Toulmin's ideas, who rejected models based on classical logics to describe argumentative processes for their claim to be universal. In fact, he argued for the strong dependence of argumentations on their specific 'field': "the standards for judging the soundness, validity, cogency or strength of arguments are in practice field-dependent" (Toulmin, 1958/2009, p. 137). As mentioned above, we will complement the Toulmin's model for argumentation (focusing on the structure of argumentation) by considering the semiotic resources used by the subjects, and the theoretical ones (the content of argumentation). This point will be discussed in the next chapter.

Notes to Appendix 1

(5) *Semiotic Activity* is classically defined as any communicative activity utilizing signs. This involves both sign 'reception' and comprehension via listening and reading, and sign 'production' via speaking, writing or sketching.

(6) To make an analogy with the outer language, Vygotsky recalls an example, taken from Le Maitre (1905), p. 41: a child thought to the French sentence "Les montagnes de la Suisse sont belles" as "L m d l S s b" considering only the initial letters of the sentence. Curtailing is a typical feature of inner language.

(7) It does not seem that Vygotsky was aware of the work of G. Frege between Sinn and Bedeutung (Frege, 1892).

(8) Vygotsky makes the analogy with the outer language recalling so called agglutinating languages, which put together many different words to constitute a unique word.

APPENDIX 2

The notion of semiotic mediation

The notion of *semiotic mediation*⁷ comes from Vygotsky. He pointed out a functional analogy between tools⁸, which can support the human labour, and signs, which can uphold the psychological activities of subjects:

⁷ It is described in Vygotsky, 1978, especially p. 40 and ff.

“... the invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose an so on) is analogous to the invention of tools in one psychological respect. The signs act as instrument of psychological activity in a manner analogous to the role of a tool in labour.” (Vygotsky, 1978, p. 52).

The notion of semiotic mediation gives a unitary approach to signs and artefacts. In fact, Vygotsky introduced both a functional analogy and a psychological difference between signs and artefacts. The analogy is illustrated by the following quotation, which stresses their semiotic functions:

“... the basic analogy between signs and tools rests on the mediating function that characterizes each of them” (ibid., p. 54).

The difference between signs and tools is so described:

“the tool’s function is to serve as the conductor of human influence on the object of activity; it is externally oriented...The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself: the sign is internally oriented.” (ibid., p. 55)

This distinction is central in the Vygotskian approach, which points out the transformation from externally oriented tools to internally oriented tools (often called *psychological tools*), through the process of *internalisation*. According to Vygotsky, in the process of internalisation interpersonal processes are transformed into intrapersonal ones. The process of internalisation (through which the ‘*plane of consciousness*’ is formed, see Wertsch & Addison Stone, 1985, p.162) mainly occurs through semiotic processes:

“... the Vygotskian formulation involves two unique premises... First, for Vygotsky, internalisation is primarily concerned with social processes. Second Vygotsky’s account is based largely on the analysis of the semiotic mechanisms, especially language, that mediate social and individual functioning ... Vygotsky’s account of semiotic mechanisms provide the bridge that connects the external with the internal and the social with the individual... Vygotsky’s semiotic mechanisms served to bind his ideas concerning genetic analysis and the social origins of behaviour into an integrated approach... it is by mastering semiotic mediated processes and categories in social interaction that human consciousness is formed in the individual” (Wertsch & Addison Stone, 1985, pp.163-166)

As Bartolini Bussi & Mariotti (2008) point out, Vygotsky stresses the role and the dynamics of semiotic mediation: first, externally oriented, a sign or a tool is used in action to accomplish a specific task; then the actions with the sign or the tool (semiotic activity, possibly under the guidance of an expert), generate new signs (words included), which foster the internalisation process and produce a new psychological tool, internally oriented, completely transformed but still maintaining some aspects of its origin. Vygotsky describes such dynamics without any reference to mathematics; hence his observations are general; many recent studies have adapted his frame to the specificity of Mathematics (e.g. see Radford, 2003; Bartolini & Mariotti, 2008).

In particular Bartolini Bussi and Mariotti define as semiotic mediation the action of the teacher when her/his goal is to coach the didactic situation so that the personal senses that the students give to a sign can evolve towards the scientifically shared sense. Our approach is within this last perspective.

APPENDIX 3

(Information on the videoclip of question 2 below)

An example to discuss

The activity concerns students attending the third year of secondary school (11th grade; 16-17 years old). They attend a scientific course with 5 classes of mathematics per week, including the use of computers with mathematical software.

⁸ In the Cambridge Dictionary a tool is defined as “*something that helps you to do a particular activity*”, an instrument is “*a tool that is used for doing something*”, while an artefact is an “*object*”. In this paper, we follow this definition and consider the instrument as a specific tool.

Methodology. These students are (early) introduced to the fundamental concepts of *Calculus* since the beginning of high school (9th grade), because their teacher is a teacher-researcher of our team. They become gradually acquainted with the concepts of functions and of their variations (first in a qualitative and then in a quantitative way) through the modelling of motion phenomena using a CBR (Computer Based Ranger, Texas Instruments[®]), which collects data of moving persons and objects in real time and gives representations of them (graphs, tables) on the screen of a calculator (TI-92, Texas Instruments[®]). Moreover they have the habit of using different types of software (Excel, Derive, Cabri_Géomètre, TI-Interactive, Graphic Calculus) to represent functions, both using their Cartesian graphs and their algebraic representations. The approach to the derivative concepts is pursued introducing finite differences and using Graphic Calculus. This software, given the equation $y = f(x)$ of a function, produces: its graph; a moving point $P(x, f(x))$ that traces it; the moving secant between P and $P'(x+h, f(x+h))$, where h is a parameter chosen as small as one likes; and the graph of the slope of the secant (the graph is drawn while the secant PP' is moving). Making h 'almost 0' this secant becomes 'almost' the tangent in P : hence the graph of its slope corresponds 'almost' to the graph of the derivative of f : see Fig. 1.

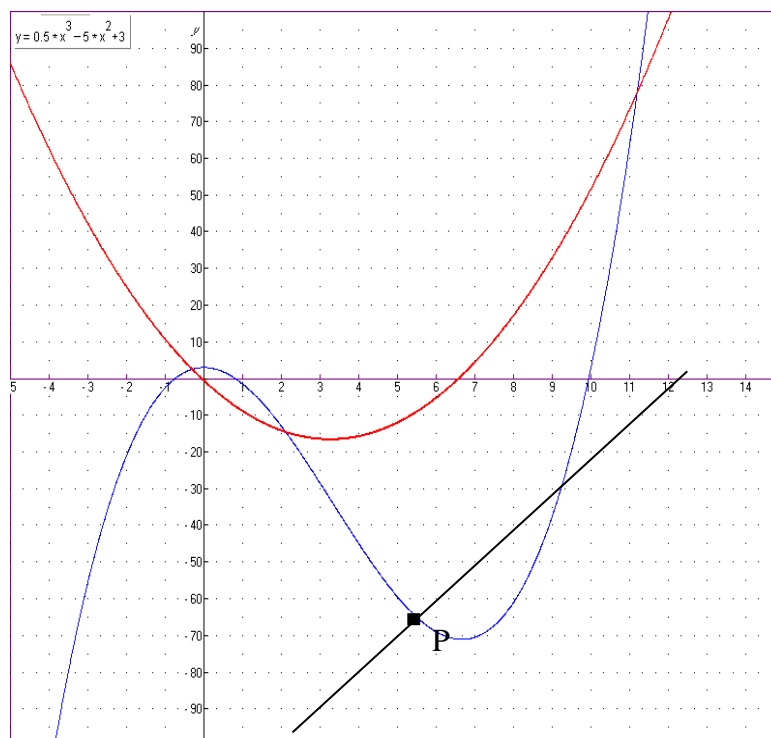


Figure 1

The blue graph is that of the function $f : x \rightarrow 0.5x^3 - 5x^2 + 3$; the red one is the graph of the slope of the 'quasi' tangent to the function. Graphic Calculus allows to see the genesis of the two graphs through a point P , which moves in time and traces the graph of the function f . In the same time one can see the corresponding moving 'quasi' tangent and the graph of its slope, which is drawn in real time while P is moving.

Students are familiar with problem solving activities, as well as with interactions in groups. They work in groups in accordance with the didactical contract that foresaw such a kind of learning. The methodology of mathematical discussion is aimed at favouring the social interaction and the construction of shared knowledge. As part of the didactical contract, each group has been also asked to write a description of the process followed to reach the solution, including doubts, discoveries, heuristics, etc. Students' works and discussions have been videotaped and their written notes collected.

The video concerns the activity of a group of three students, who discuss among them and with the teacher the graphs of Fig.1. The students are answering the following specific question (asked by the teacher while discussing with them): *Imagine that you have not the red curve, but you see the tangent while moving. Can you have information on the concavity of the slope function?*

The students have made experiences with the function concept and know the concepts of increasing/decreasing functions (having solved problems using first and second finite differences of functions), but they do not yet know the notion of derivatives. Moreover they are able in using Graphic Calculus and know that the 'quasi' tangent is not the real tangent, because of discrete approximations.

The excerpts in the video altogether last 95 seconds and are taken from two lectures, each of 50 minutes. There are the Teacher and 3 students: *Ciro* on the right, *Simone* in the middle and *Gabriele*, in the back. They are both clever students who participate to classroom activities with interest and active involvement. The teacher is not always with these students, but passes from one group to the other (the class has been divided into 8 small groups of 3-4 students each).

PART 1 (after 13')

1. *Ciro*: Practically one could say to explain that this straight line...
2. Teacher: Uh
3. *C*: That is it must join, ok, the X axis the interval...is [always?] the same
4. Teach.: The X interval is the same; delta-X [Δx] is fixed
5. *C*: Delta...eh, indeed, however there are some points where... to explain it one can say that this straight line must join two points on the Y axis, which are farther each other, hence it is steeper towards...
6. Teach.: Yes
7. *C*: Let us say towards this side. When, here, ...when ...however it must join two points, which are farther, hence there is less distance
8. Teach.: More distance?
9. *C*: Less far [he corrects what said in #7]
10. Teach.: Eh
11. *C*: On the Y axis I am saying
12. Teach.: Yes
13. *C*: It slants softly from this side; in fact here is the point, let us say...we may call it zero, since this is a real line like so.

PART 2 (after 43')

14. Teach.: Uh, uh, do you agree about this? If you wish to go at a mild speed with this one [he indicates the tangent in the Graphic Calculus window]..
15. *S*: Yes, yes
16. Teach.: Otherwise the other one is slower
17. *C*: That is if I understand, ...for this reason we say that seeing the tangent and its slope one can find...

PART 3 (after some minutes)

18. Teach.: Hence let us say, in this moment if I understood properly, with a fixed delta-X, which is a constant,...
19. *C*: Yes
20. *S*: Yes
21. Teach.: It... is joining some points with delta-Y, which are near
22. [overlapped with #21] *C*: In fact, while they [the points on the graph] are approaching each other

23. C: ...they [their ordinates] are less and less far. In fact, I do not know how to say it, ...the slope is going towards zero degrees.
24. Teach.: Uh, uh
25. C: Let us say so
26. Simone: Ok, Δ at a certain point here ΔY over ΔX reaches...
27. C: The points are less and less far
28. Teach.: Sure
29. S: ...a point, which is zero.

Note. Further information (in Italian) about the teaching project from which these excerpts are taken is in the website: <http://www.matematica.it/paola/Corso%20di%20matematica.htm>

Questions.

1. Consider the video-clip that you find at the following address in the web:

<http://www2.dm.unito.it/paginepersonali/arzarello/index.htm>, click on YESS Texts and Videoclips (the clip is in quick-movie format).

It contains three excerpts from a small group discussion. In Appendix 3 you find some information on the students and on the problem that they are solving. Analyse (at least one of) the three excerpts using your frame. After that you will see the analysis according to my frame that I'll sketch during the lecture, compare the two. Which elements are focussed by the one and not by the other?

2. Taking into account your answer to the previous question, comment the following quotation:

In innovative research paradigms, the fine grain analysis of short-term processes is coordinated with the analysis of long-term processes. How do researchers manage the many problems raised in research for innovation? They use a variety of methods that are not universal but functional to the particular research question.

For instance, when the focus is on the teacher's role and the aim is the modelling of the social processes in the mathematics classroom, Leont'ev's activity theory (1978) is one of the possible solutions, because, on the one hand, it comes from the vygotskian tradition where the cultural (and asymmetrical) role of the teacher is emphasised and, on the other hand, it offers a system of tools to relate the global level of activity developed over time to the individual operations (i. e. communicative strategies) realised by the teacher on the spot. Activity theory makes explicit (and functional) the distinction among different levels of phenomena (Activity - Action - Operation), which occur in different but interrelated periods of time.

...

Yet, when the focus is on the individual processes in problem solving, activity theory might not be the best solution.

...

The 'true life' observation of a long term teaching experiment is quite different from the observation of processes that take place in the psychology laboratory (or in the surrogate of a psychology laboratory given by a classroom when only short term processes are analysed without any reference to the global school activity), where the influence of the external events can be made as little as possible and where the different time variables can be separated.

...

Are we satisfied of the above juxtaposition of methods? Are other solutions possible?

One might say that it is untimely to look for a coordination of methods of analysis, because of the insufficient development of theoretical reflection.

...

Maybe the search for methodological purity has to be given up for some time, at least in innovative research paradigms. Yet we feel the pressure of the development of trends of research that overcome the distinction between theoretical and pragmatic relevance (Sierpinska 1993, Bishop 1998) producing results with a sound theoretical basis and with a deep impact on the practice of teaching in society.

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