Mathematics teachers’ knowledge and beliefs

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1. Theory

1.1 Beliefs

There is no agreed definition of beliefs in mathematics education. Phillip (2007) is one of several authors who have provided an overview of the definitions related to beliefs and affect. He defines beliefs as ‘psychologically held understandings, premises, or propositions about the world that are thought to be true’ (p. 259). Beliefs are a part of affect together with emotions and attitudes, and beliefs are the most cognitive aspect out of these. Beliefs are also closely related to knowledge, as knowledge is defined as ‘beliefs held with certainty or justified true belief’ (p. 259). Wilson and Cooney (2002) hold the view that believing is a weaker condition than knowing. In the words of Leatham (2006):

Of all things we believe, there are some things we ‘just believe’ and other things we ‘more than believe – we know’. Those things we ‘more than believe’ we refer to as knowledge and those things we ‘just believe’ we refer to as beliefs (p. 92).

Furinghetti and Pekhonen (2002) explored the multitude of definitions when they asked eighteen mathematics educators to state their agreement or disagreement with nine different definitions of beliefs, and also to give their own characterization of beliefs. Both the nine different definitions and the answers from the mathematics educators illustrate the differences in definitions, and the authors state that it is unlikely that a complete agreement will be reached. As a result of this investigation they recommend considering two types of knowledge: objective and subjective. Objective knowledge is the knowledge that is accepted by the mathematical community, and individuals have access to this knowledge and ‘construct their own conceptions of mathematical concepts and procedures, i.e. they construct some pieces of their subjective knowledge’ (p. 53). Beliefs are connected to subjective knowledge. Their way of defining beliefs as connected to subjective knowledge clarify both the close relationship and the difference between the two, without using judgments of truth.

Beliefs are assumed to act as filters through which one sees the world (Pajares, 1992). A result is that teachers’ beliefs are thought to have a strong impact on their practice. Also, beliefs can present barriers and serve as affordances (Goldin, Röskén, & Törner, 2009). Beliefs can for example act as barriers against influence from external factors, such as curriculum changes or education. Thus, beliefs can preserve the teaching even if the curriculum and the use of mathematics in society change. Beliefs can also serve as affordances; for example, the belief that reasoning and argumentation are the most important aspects of mathematical knowledge may lead the teacher more often to situations where he learns about students’ mathematical thinking. The knowledge of how beliefs can serve as affordances is very useful in the development of teachers’ productive disposition (defined by Kilpatrick, Swafford and Findell (2001) as one of five components of ‘Proficient teaching of mathematics’).
Beliefs influence the decisions that individuals make and also serve as the best indicators of their decisions (Goldin, et al., 2009). As a result of such a view, a lot of research has been focused on the connection between beliefs and teaching practice. However, inconsistencies are often documented between the teachers’ practice and their beliefs about mathematics teaching. Inconsistent beliefs can be held simultaneously without becoming evident because they are connected to different contexts, certainty and consciousness (Törner, 2002). This raises the question of how important beliefs are for the teachers’ practice. Another perspective is to look at teachers’ beliefs as sensible systems, where apparent inconsistencies are instead thought to be caused by other beliefs ranking higher in certain situations (K. Leatham, 2006; Skott, 2009). These beliefs might be non-mathematical, for example related to a teacher’s wish to help the students succeed and feel well. Sometimes a teacher’s behavior looks inconsistent with the teacher’s beliefs because the researcher is not aware of all the beliefs at play during the observed practice. When one looks at teachers’ beliefs as sensible systems, such inconsistencies are investigated in order to understand which unknown beliefs decide this behavior.

1.2 Teachers knowledge

1.2.1 An overview
Several researchers have defined what a teacher needs to know to teach mathematics. There are for example several theoretical models describing what mathematical content knowledge is (for example Brekke, 1995; Kilpatrick, et al., 2001; NCTM, 1989, 2000; Niss & Højgaard Jensen, 2002). But even if there is an agreement that mathematical content knowledge is a prerequisite to be able to teach mathematics, there is also a growing understanding that this is not enough (Ball, 2002; Ball & Bass, 2003; Ball, Hill, & Bass, 2005; Niss, 2007). As an answer to this there are developed several models of teachers’ knowledge needed to teach mathematics, like the ‘content knowledge for teaching’ (Ball, Thames, & Phelps, 2007, 2008), ‘mathematical teacher competency’ by Niss and Højgaard Jensen (2002), ‘proficient teaching of mathematics’ by Kilpatrick, Swafford and Findell (2001) and ‘knowledge of teaching mathematics’ by The Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto, et al., 2008). The knowledge quartet (KQ) (Rowland & Turner, 2009) adds another perspective with the main focus on the knowledge that is observable in practice, and less focus on foundation that may not be useful to understand knowledge in practice.

1.2.2 Content knowledge for teaching
Not surprisingly, research has confirmed that teachers need to know the mathematics they are teaching both deeply and thoroughly. More surprisingly, research has also shown that there is no clear relationship between the teachers’ formal mathematical education and their students’ learning of mathematics (Askew, 2008; Ball, Lubienski, & Mewborn, 2001; Perrin-Glorian, Deblois, & Robert, 2008). The reason that such a relationship has not been established may be that measuring teachers’ mathematical knowledge just in terms of their level of formal education (number of courses or ECTS) is not precise enough because there are probably aspects of such knowledge that are more important than others. One suggestion in the search for aspects of teachers’ mathematical knowledge that matter for students’ learning is to use Shulman’s (1986) distinction between subject matter knowledge and pedagogical content knowledge. One recent development resulting from this suggestion is the framework ‘content knowledge for teaching’ (Ball, et al., 2008). This framework divides both subject matter knowledge and pedagogical content knowledge into three parts, as illustrated by figure 1 (Ball, et al., 2007, p. 42):
Common content knowledge are defined as ‘the mathematical knowledge and skill used in settings other than teaching’ (Ball, et al., 2007, p. 32). It is questions that commonly are answerable by people who know mathematics, and therefore not knowledge unique to the teachers. Specialized content knowledge is defined as ‘the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work ... and therefore not commonly needed for purposes other than teaching’ (Ball, et al., 2007, p. 34). Also this specialized content knowledge is purely mathematical and does not require knowledge of students or teaching. But it is not the mathematics learned at the university mathematics courses, it is ‘mathematics in its decompressed or unpacked form’ (Ball, et al., 2007, p. 36). The knowledge of content and students is defined as ‘knowledge that combines knowing about students and knowing about mathematics. Teachers need to anticipate what students are likely to think and what they will find confusing’ (Ball, et al., 2007, p. 36). Knowledge of common student conceptions and misconceptions are a vital part of this domain. Knowledge of content and teaching is defined as ‘knowledge that combines knowing about teaching and knowing about mathematics’ (Ball, et al., 2007, p. 38). Included in this domain are for example the selection of examples, the sequencing of the content, the advantages and disadvantages of different representations, which student contributions could be used and how, when the whole class needs clarification, when to take it with single students or small groups, and the decision of what method or algorithm to introduce. Schulman’s curricular knowledge is placed within pedagogical content knowledge and re-named to curriculum content knowledge. Horizon knowledge is ‘an awareness of how mathematical topics are related over the span of mathematics included in the curriculum’ (Ball, et al., 2007, p. 42).

2. Research questions
My research is focused on mathematics teachers’ knowledge and beliefs. As I write articles I have one research question for each article. The constructs I use (SCK, CCK, rules, reasoning) are explained in part 3. The main research question for the whole project is: How can the teachers’ knowledge and beliefs explain their practice?

Article 1 (published): Are there any evidence for or against the existence of specialized content knowledge (SCK) and common content knowledge (CCK) as two separate constructs among Norwegian primary and lower-secondary teachers of mathematics?
Article 2 (submitted for review): The main focus is on exploring the interplay between their mathematical knowledge and both their mathematical teaching experience and their mathematical education. Are these connections dependent on different types of beliefs? If so, what can the differences explain?

Article 3 (not finished): The knowledge constructs (SCK and CCK) and the beliefs constructs (rules and reasoning) are only important if they have any impact on the teachers practice. To find out this the following research question comes naturally: How can the knowledge and beliefs constructs (SCK, CCK, rules, reasoning) explain the teachers’ practice?

3. Research design and methods

3.1 Article 1 – Exploring mathematical knowledge for teaching

The first article reports from the results of a multiple choice test and a questionnaire answered by 356 teachers from primary and lower secondary school. The test consisted of 20 tasks with a total of 46 questions originally developed by the Learning Mathematics for Teaching project (LMT, 2009). The test results verified the existence of the constructs of specialized content knowledge (SCK) and common content knowledge (CCK) (Drageset, 2009). SCK has a reliability of 0,77 with 27 items (questions), and CCK has a reliability of 0,72 with 12 items. The observed correlation between the SCK and the CCK construct is 0.58 and the latent correlation is 0.80. This confirms that there are two different constructs measured since the constructs turn out to be sufficiently different empirically.

3.2 Article 2 – The interplay between the knowledge and the beliefs of mathematics teachers

The second article studies the connections between two knowledge constructs (SCK and CCK) and two beliefs constructs (rules and reasoning), based on the same data as the first article.

The first beliefs construct is based on a view that instrumental understanding (as defined by Skemp, 1976) of mathematics is the most important aspect of mathematical knowledge. The construct consists of ten statements that were answered along a four-point Likert scale (reliability of 0.71, measured by Cronbach’s alpha, see Crocker and Algina (1986)). Those agreeing with these statements emphasize formal mathematics and the learning of rules as most important, without focusing on explanations or connections. Their focus is on solving the tasks, and they do not emphasize connections or explanations. This beliefs construct is called ‘rules’ (see attachment 3), and has clear similarities with a construct that Nisbet and Warren (2000) call ‘a static view of mathematics’. The teachers that emphasize rules consider rules and the correct answer to be the most important aspects of mathematics, and feel that mathematics is best learned by rote and by trying to imitate examples. The rationale behind this belief can be that the teacher wants to reduce the complexity and ambiguity of mathematics. Another reason can be that the teacher avoids taking risks because of uncertainty in his own mathematical knowledge.

The second beliefs construct consists of eleven statements along a four-point Likert scale (reliability of 0.81). This construct is based on a view that reasoning competency (and to some extent problem solving competency) (as defined by Niss & Højgaard Jensen, 2002) are important aspects of mathematical knowledge. The construct represents a belief that
reasoning, argumentation and justification are more important than the answer. This beliefs construct is called ‘reasoning’ (see attachment 4), and is connected to a dynamic view of mathematics. A focus on reasoning, argumentation and justification in practice would involve a risk of not being able to follow the students’ thinking.

The most important findings are that different emphases on beliefs are connected to different aspects of mathematical knowledge. An emphasis on reasoning seems to be an affordance for the learning of SCK, and the learning of SCK seems to be an affordance for an emphasis on reasoning (or a barrier against not emphasizing reasoning). Both a lack of emphasis on reasoning and an emphasis on rules seems to act as an affordance for the learning of CCK and a barrier against the learning of SCK from education, while learning more CCK and less SCK (than the whole group) from education seems to act as an affordance for not emphasizing reasoning and for an emphasis on rules (or as a barrier against an emphasis on reasoning or against not emphasizing rules).

A consequence of these findings is that there is a need to consider beliefs and knowledge not only as connected, but as elements that strengthen each other.

3.3 Article 3 – Observation of practice

To be able to answer the research question I have videotaped the practice of teaching. I have picked five teachers, and videotaped all their mathematics teaching for one week. This means 3-5 lessons (between 130 and 220 minutes) for each teacher. They are all teaching in upper primary school (age 10-13), and the topic is fractions. The teachers are selected based on their profile from the four constructs of SCK, CCK, rules and reasoning. I have chosen varied profiles, but of ethical reasons not the from the 20% lowest results on SCK and CCK.

My coding manual is divided in three parts. The first part use elements of an existing coding manual called ‘Mathematical quality of instruction’ (see for example Hill, et al., 2008). This coding manual is developed of some of the same people that developed the tasks I used in my test (Hill, Schilling, & Ball, 2004), and the theory of ‘Content knowledge for teaching’ (Ball, et al., 2008). This coding manual is based on coding, not transcription. They split the video into parts of five minutes, and code each part separately (but not isolated). The result of the analysis is used quantitatively. As I will not use it quantitatively, I consider splitting the video into more natural segments. Then some segments can be two minutes, other ten minutes.

The second part is based on ‘the knowledge quartet’ (Rowland & Turner, 2009), which is a theory that focus on knowledge that is observable in practice. In the knowledge quartet there are four categories of knowledge. The first one (foundations) are not possible to observe directly. But the three others (transformation, connection and contingency) are collections of under-categories that the researchers observed in practice. There are no coding manual for this theory, so I have to develop one myself. I will then use the same segments as I do in the first part.

The third part is some categories I have developed myself to capture some more specific parts of the beliefs constructs. One code is on the standards for truth (authority, single arguments, reasoning) based on ideas from Brousseau (1997) and Carpenter (2003), the second code is on the acceptance of answer (if the teacher accepts answers without asking, or tries to understand the students thinking), and the third is on the teachers view on mathematics based on findings from Cross (2009). Also here I plan to use the segments from the first part when I code.
To explain how I plan to use these codes I will provide an example. One code is about the use of manipulatives (like fraction bars). In the first part of my analysis I will code each segment. Then I know in which segments the use of manipulatives is present or not. One teacher might use manipulatives in 15 of 30 segments, while another might use manipulatives in 5 of 40 segments. This first part will only be used to identify which codes I will study closer. In the second part of my analysis I will characterise how the teachers work within one or several codes. One example from the use manipulatives is a teacher that gives the students the task to compare 1/3 and 2/5. To help them, the teacher gives them fraction bars. Then the students take the bar divided in three and the bar divided in 5 and place them besides each other. The students will then directly see that 2/5 is bigger than 1/3. This use of manipulatives can be compared with the use of calculators, where the students just write the numbers and the calculator solves the task. With this ‘calculator’ use of manipulatives the students solve the task, but learn nothing about converting fractions and common denominators. Another teacher consequently uses manipulatives to start discussions about the fraction concept, and never as a ‘calculator’. When studying all the segments that have some use of manipulatives I will describe such patterns and try to make categories. These categories will then be used to characterise the practice of each teacher and to compare them.

The real challenge is to connect this analysis to the constructs from the test and questionnaire. I am aware that it is impossible to prove any connection. At the moment my plan is to use the characterisation of each teacher and see if their knowledge and beliefs can give any explanation of the practice. If the teacher that uses manipulatives to start discussions about the concept also emphasise reasoning this connection is probably not accidental. Then this can be an example of the practice of a teacher that emphasise reasoning.
Attachment 1 – Example of an SCK task
(Learning Mathematics for Teaching, 2006)
Imagine that you are working with your class on multiplying large numbers. Among your students’ papers, you notice that some have displayed their work in the following ways:

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
<th>Student C</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>35</td>
<td>35</td>
</tr>
<tr>
<td>x 25</td>
<td>x 25</td>
<td>x 25</td>
</tr>
<tr>
<td>125</td>
<td>175</td>
<td>25</td>
</tr>
<tr>
<td>+7 5</td>
<td>+7 00</td>
<td>1 50</td>
</tr>
<tr>
<td>8 75</td>
<td>8 75</td>
<td>8 75</td>
</tr>
</tbody>
</table>

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

<table>
<thead>
<tr>
<th>Method would work for all whole numbers</th>
<th>Method would NOT work for all whole numbers</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Method A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>b) Method B</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>c) Method C</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Ms. Whitley was surprised when her students wrote many different expressions to represent the area of the figure below. She wanted to make sure that she did not mark as incorrect any that were actually right. For each of the following expressions, decide whether the expression correctly represents or does not correctly represent the area of the figure. (Mark REPRESENTS, DOES NOT REPRESENT, or I’M NOT SURE for each.)

![Diagram of a figure with dimensions a and a + 5.]

<table>
<thead>
<tr>
<th></th>
<th>Correctly represents</th>
<th>Does not correctly represent</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$a^2 + 5$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>b)</td>
<td>$(a + 5)^2$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>c)</td>
<td>$a^2 + 5a$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>d)</td>
<td>$(a + 5)a$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>e)</td>
<td>$2a + 5$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>f)</td>
<td>$4a + 10$</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Attachment 3 – The rules construct

Please indicate to which extent you agree with the statements below
(disagree entirely – disagree somewhat – agree somewhat – agree entirely)

A1 The most important aspect of mathematics is to know the rules and to be able to follow them
A2 Mathematics means finding the correct answer to a problem
A4 The best way to learn mathematics is to see an example of the correct method for solution, either on the blackboard or in the textbook, and then to try to do the same yourself
A5 If you cram and practice enough, you will get good at mathematics
A6 Those who get the right answer have understood
A8 Mathematics should be learned as a set of algorithms and rules that cover all possibilities
A10 What you are able to do you also understand
AD3 In mathematics, it is more important to understand why a method works than to learn rules by heart [opposite]

Please indicate how important you think each element below is for the pupils
(not very important – somewhat important – important – very important)

A11 Learning rules and methods by heart
A12 Learning formal aspects of mathematics (e.g. the correct way to write out calculations) as early as possible

Attachment 4 – The reasoning construct

Please indicate to which extent you agree with the statements below
(disagree entirely – disagree somewhat – agree somewhat – agree entirely)

C1 The pupils learn more mathematics from problems that do not have a given procedure for solution, where instead they have to try out solutions and evaluate answers and procedures as they go
C2 It is important to be able to argue for why the answer is correct
C6 Solving mathematical problems often entails the use of hypotheses, approaches, tests, and re-evaluations
D1 The pupils learn from seeing different ways to solve a problem, either by pupils presenting their solutions or by the teacher presenting alternative solutions

Please indicate how important you think each element below is for the pupils
(not very important – somewhat important – important – very important)

C12 The ability to explain their answers
C13 The ability to argue for their procedures and answers
C14 Being able to explain their reasoning
C15 Being able to evaluate other procedures than their own
C16 Being able to follow the reasoning of another pupil
D11 The ability to solve complex problems where the pupils have to use several aspects of mathematics
D12 Teaching must focus on understanding as much as possible so that the pupils can explain methods and connections


Törner, G. (2002). Mathematical beliefs - A search for common ground: Some theoretical considerations on structuring beliefs, some research questions, and some phenomenological observations. In G. C. Leder, E. Pehkonen & G. Törner (Eds.),