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PROOF AND PROVING IN MATHEMATICS EDUCATION

INTRODUCTION

Nowadays, differently to ten years ago, there seems to be a general consensus on the fact that the development of a sense of proof constitutes an important objective of mathematical education, so that there seems to be a general trend towards including the theme of proof in the curriculum. Take, for instance, the following quotation.

> Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied.
>
> (NCTM, 2000, p. 342)

I wonder whether these words would have been possible only a few years ago, and still now the idea of "proof for all" claimed in this quotation is not a view that most teachers hold, even in countries where there is a longstanding tradition of including proof in the curriculum. I'm thinking of my country, Italy, but also, as far as I know, France or Japan. In fact, the main difficulties encountered by most students have lead many teachers to abandon this practice and prompted passionate debate amongst math educators, which has produced a great number of studies. Proof has also been a constant theme of discussion in the PME community and at PME conferences, which has given rise to a large number of Research Reports, although not at the same rate every year (the reader can find a useful, although not yet complete, collection of references at the site: http://www.lettredelapreuve.it/).
The debate is far from being closed and has generated a number of research questions which have evolved in the decades, also in accordance with an evolution of the general trend of Mathematics education research. A quick overview of PME contributions on the theme of proof - probably not dissimilarly from what happened for other themes - shows a move away from early studies, focussed on students' (and more rarely teachers') conceptions of proof, and generally speaking on difficulties that pupils face in coping with proof and proving, towards more recent studies where researchers present and discuss opinions on whether and how is it possible to overcome such difficulties through appropriate teaching interventions. As a general trend, it is possible to observe a change in the methodology: reports on teaching experiments[1] have increased while reports discussing quantitative analysis based on questionnaires have decreased, though

[1] This term is used in a broader, but consistent, sense in respect to (Steffe & Thompson, 2000) to indicate research projects based on classroom practice.

questionnaires and quantitative analysis remain the main methods in large scale investigations, when nationwide investigations or cross cultural comparisons are carried out. We will return to this point in later.

An exemplar of earlier studies is the classic research report presented by Fischbein & Kedem (1982). Repeatedly quoted and subsequently re-discussed, this study focused on the crucial tension between the empirical and formal approach to proof.

Exemplars of recent studies centred on teaching experiments can be found among the number of reports concerning the use of Dynamic Geometry Environments: a specific section will be devoted to discussing these.

The variety and the complexity of the PME contributions, differently related to proof, required a drastic selection in order to give a reasonable account in the space of a chapter and within the limits of my capacity: I therefore apologize for the unavoidable incompleteness.

The discussion is organized according to three main streams of research, identified by three main categories of research questions, which I summarize as follows:

Proof at school. What is the status of proof at school? This quite general question is formulated differently in different studies, but the general characteristic aim consists in searching for a global view that captures widespread phenomena and possible correlations between them.

Students' Difficulties. The general issue concerns the study of students' difficulties, and it refers to two main questions, roughly corresponding to describing and to interpreting students' behaviours in proving tasks. What are the main difficulties that students face in relation to proof? Which might be the origin of such difficulties?

Teaching Interventions. Is it possible to overcome the difficulties that students meet in relation to proof? How can teaching interventions be designed? What general suggestions can be given?

Before starting the discussion I'd like to share with the reader some introductive reflections. As clearly pointed out by Balacheff (2002/04), the epistemological perspective taken by the researcher is not always made explicit, and this can be considered one of the main reasons for the failure of communication: instead of correctly fuelling the debate, contributions risk becoming blocked in the impasse of misunderstanding.

Epistemological issues are not often directly addressed in the Research Reports presented at PME Conferences. Nonetheless the centrality of these issues in the debate was clearly discussed by Gila Hanna at 20th PME Conference (Hanna, 1996) and two papers have explicitly dealt with them in the recent past (Godino & Recio, 1997) and (Reid, 2001). Both contributions focus, in different ways, on the differences in the meaning of the term proof as it appears in people's use of this word. The first paper takes a wide perspective, describing some of the meanings of the term proof in different contexts, such as Mathematics and mathematical foundations research, sciences, and Mathematics class. The second paper focuses on the domain of Mathematics education research, where different usages of the term proof are identified. While I refer to these papers, together with more recent contributions (Balacheff, 2002/04; Reid, 2005), for an explicit comparison between different perspectives, I will try to make my position explicit through a short introduction that might facilitate understanding of what I am going to present in the following sections.

Proof in the history of Mathematics culture

Taking a biologic metaphor, a historic and epistemological analysis throughout the centuries highlights different "mutations of proof", following the evolution and systematisation of Mathematics knowledge.
Although Mathematics cannot be reduced to theoretical systems, its theoretical nature certainly constitutes a fundamental component. As Hilbert and Cohn Vossen clearly point out in the introduction to their book "Geometry and the Imagination", Mathematics is characterised by a twofold nature: intuitive understanding on the one hand and a systematic order within logical relations on the other hand.

> In Mathematics … we find two tendencies present.
>
> On the one hand, the tendency towards abstraction seeks to crystallise the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner.
>
> On the other hand, the tendency towards intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the correct meaning of their relations.
>
> (Hilbert & Cohn –Vossen, 1932)

Actually a theoretical perspective of Mathematics has historic roots that lead us to the ancient book of Euclid's Elements and its particular way of presenting the 'corpus' of knowledge, which since then has become a paradigm of scientific discourse. Euclid's exposition was characterised by a particular modality, which appeared as a brilliant solution to a delicate problem, as Proclo expresses it:

> Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. (…) In all these ways Euclid 's system of elements will be found to be superior to the rest.
>
> (Heath, 1956, vol. I p. 115-116)

The crucial point seems to be a suitable order in which a corpus of knowledge may be expressed and communicated. The problem of the transmission of knowledge was solved by Euclid in a very peculiar way: rather than in terms of "revelation", the elements were transmitted according to "logic arguments", that is through what has been called a "proof". As already mentioned, since then the need for particular arguments in communicating mathematical knowledge became a shared practice and a relevant part of the Mathematics culture.
This basic reference to Euclid is not sufficient and it can be even misleading when used to outline the complex nature of what is and has been called proof in past centuries. Nevertheless, it is interesting to remember that, although crucial contributions were possible, mostly neglecting requirements of rigour, the need for validation within the frame of an acceptable script of arguments appeared recurrently. An interesting example is offered by the case of Thabit ibn Qurra (826-900) who, in his book entitled "*Correction of problems of algebra by geometric*

proofs" aims to provide acceptable, i.e. geometrical, justifications of the algorithms previously given by qu'al-Kawarizmi (Abdeljaouad, 2002).

In the history of Mathematics it is possible to observe a regularity in the successive phases of development in a given area: after an initial flourishing of ideas and methods a second phase occurs (sometimes separated from the first by a long gap) in which systematisation and reorganization is accomplished, aimed at making the corpus of knowledge acceptable and usable within the community of mathematicians. This social dimension, after the individual creative part of Mathematics development, has a crucial role, which has often been stressed and cannot be forgotten. The standards of acceptability are changeable and subject to different constraints which vary according to different variables, the first being the different areas of the discipline.

Focussing on Geometry, one of the salient features of the structure of Euclid's Elements is certainly its logical structure - the hypothetical / deductive structure - although this definitely differs from the "deductive structure of logic derivation within a formal theory" (Rav, 1999, p. 29). The main difference lies in the fact that, contrary to what happens within a formal theory, in the practice of Mathematics deduction depends on understanding and on prior assimilation of the meaning of the concepts from which certain properties are to follow logically. This is the sense in which Euclid's work must be interpreted: deductive arguments are to be considered as means of fostering understanding of the whole Geometry and relating new properties to indubitable facts.

The style of rationality introduced by Euclid has become a prototype for all kinds of science, and the power of this method may be related to its treatment of truth.

A twofold criterion of truth characterises the structure of Euclid's Elements: evidence, on which principles are assumed, and consistency, on which the truth of derived knowledge is based. In this framework a deep unity relates organisation of knowledge to understanding: organisation becomes functional to understanding, which consequently becomes strictly tied to the constraints of acceptability and validation shared within a given community that will become a scientific community.

A crucial point that I like to stress is that Euclid's *Elements* accomplish a twofold aim: on the one hand the need for understanding and on the other hand the need for validity, i.e. to be accepted by a community. These two main aspects seem to be recognised as being characteristic of a theoretical corpus and can be found in most of the discussions about the nature and the function of proof. Let us take for instance the following quotation from a classic paper by Hanna.

> Mathematicians accept a new theorem only when some combination of the following holds:
> They understand the theorem (that is, the concepts embodied in it, its logical antecedents, and its implications) and there is nothing to suggest it is not true;
> The theorem is significant enough to have implication in one or more branches of mathematics, and thus to warrant detailed study and analysis;
> The theorem is consistent with the body of accepted results;
> The author has an unimpeachable reputation as an expert in the subject of the theorem.

> There is a convincing mathematical argument for it, rigorous or otherwise, of a type they have encountered before.
>
> (Hanna, 1989, p. 21-22)

Despite the crucial change of perspective that led mathematicians at the beginning of the last century to a radical revision of the idea of truth, the relationship between understanding and acceptability of mathematical statements has not dramatically changed in the centuries and still constitutes a characterising element of this discipline.

The slow elaboration of the idea of rigour, which had its climax at the end of the nineteenth century, has a counterpart in the development of the ever more complex relationship between two fundamental moments of the production of mathematical knowledge: the formulation of a conjecture, as the core of the production of knowledge, and the systematisation of such knowledge within a theoretical corpus (Mariotti, 2001b). This leads one to recognise a deep continuity between the development of knowledge and its systematisation within a theoretical "corpus"; between aspects relevant in the communication process, such as the need for understanding, and aspects related to the fact that knowledge is a shared cultural product, such as the need for acceptability.

Certainly the issue of understanding, and in particular the issue of the relationship between proof as a hypothetical-deductive argument and its explaining function, is crucial. Different opinions are possible, according to the relevance given to the distance between the semantic level, where the truth (the epistemic value) of a statement is fundamental, and the formal / logical level, where only the logical validity of an argument is concerned.

Although the logical dependence of a statement in respect to axioms and theorems of the theory is considered, the issue of understanding arises inasmuch as it refers to the links between the meanings involved in both the statement and the arguments. On the one hand, these links may not necessarily be expressed through the structure of logic consequence; on the other hand, when required, it is impossible to formulate and prove the logical link between two statements without any reference to meanings.

In other terms, in spite of the fact that the epistemic value has no relevance at the theoretical level, it is impossible to conceive a practice of mathematical proof without any reference to the semantic level. From both an epistemological and a cognitive point of view, it seems impossible to make a clear separation between the semantic and the theoretical level, as required by a purely formal perspective.

> To expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge.
>
> (Lolli, 1999)

As a consequence, from the cognitive point of view, in spite of the fact of its theoretical autonomy, proof is strictly related to semantics. The explanation function of proof is fundamental, because it provides the support needed for understanding, but this function depends on the semantics of the statements and the truth value attributed to them.

The two basic elements of proof, arising from this short discussion, the explanation function and the need for acceptability, are expressed very effectively by Maturana, as quoted by Reid (2005):

> […] the observer accepts or rejects a reformulation […] as an explanation according to whether or not it satisfies an implicit or explicit criterion of acceptability […]. If the criterion of acceptability applies, the reformulation […] is accepted and becomes an explanation, the emotion or mood of the observer shifts from doubt to contentment, and he or she stops asking over and over again the same question. As a result, each […] criterion for accepting explanatory reformulations […] defines a domain of explanations, […].
>
> (Maturana, 1988, p. 28)

The discussion so far aimed to share with the reader the deep sense of complexity and, to some extent, the fleetingness of the idea of proof, when thought of as a living phenomenon. Such complexity, together with its centrality in mathematics practice, has fuelled a passionate debate, but has also generated a rich and varied collection of research studies. As anticipated in the introduction, the following discussion is divided into three directions, which I'd like to propose as different paths by which to enter the domain of research on proof and proving.

PROOF IN THE CURRICULUM

The first direction concerns contributions that aim to provide a description of the status of proof at school and in relation to the curriculum. These contributions are integrated and sometimes intermingled with those concerning students' conceptions of proof, but they seem to be born of the urgency to capture and describe the relevance of particular phenomena in terms of frequency or diffusion across different contexts.

Although there are few studies concerning teachers and these are very often limited to prospective teachers, currently most of the studies in this area report on students' responses to proof tasks. Some of them are more oriented towards giving a general picture of students' views on proof, through large-scale nationwide investigations (see for instance the seminal work of Hoyles (1997) and (Healy & Hoyles, 1998), followed by a number of successive research projects (Kükemann & Hoyles, 2001, 2002; Hoyles et al, 1995/2003). The main results of these studies highlight the recurrent difficulty that students face with proof: the difficulty emerges of controlling the complex relationship between mathematical validation, rooted in the frame of a theoretical system, and common sense validation, rooted in empirical verification (facts versus logical implications). Not only young pupils, but also high school and university students do not seem able to give mathematically adequate answers. In describing and classifying the answers, the effect of schooling has been investigated, and interesting aspects concerning the influence of the curriculum emerged (Hoyles, 1997).

The case of the research projects carried out in the UK represents a paradigm, but also an exception; only a few examples exist of large-scale nationwide studies of such a kind. Investigations in this direction are highly demanding due to the huge amount of data to be processed, but also due to the intrinsic complexity of their design, which needs to express both the specificity of a given country and the generality of comparable questions.

A recent effort in this direction was presented at 29^{th} PME Conference (Lin, 2005). Generally speaking, studies aimed at accomplishing cross-cultural comparison in my opinion constitute a promising direction of investigation. In this respect, other interesting approaches have been proposed (Knipping, 2001) that are complementary to large scale investigations and focalised on comparing specific teaching cultures regarding proof, for further discussion see (Mariotti, to appear).

Aimed at sketching a global picture, large-scale studies have been profoundly inspired by previous studies on students' conceptions of proof and at the same time provide new and rich data on this issue. In this sense, these studies can be considered strictly related to more analytic investigations aimed at obtaining a better insight into cognitive processes related to "proof" and "proving". The following section is devoted to this theme.

PROOF IS DIFFICULT: DIFFERENT APPROACHES TO STUDENTS' CONCEPTION OF PROVING

As it is not possible to give a comprehensive account of the different contributions, what I will try to do is discuss two main perspectives that generate different directions of research. In so doing it is my intention to offer a possible framework for a virtual debate with the reader.

- On the one hand, the researcher takes a broad perspective, according to which different ways of thinking are described and classified as "proof". The main source is the analysis of students' production in solving problems where recurrent behaviours can be identified.
- On the other hand, starting from an explicitly epistemological perspective which recognizes a specific status of a *mathematical formal proof*, the researcher describes and explains difficulties and obstacles encountered by students in relation to this specific idea of proof,

Analysis and classification of students' reasoning: the proof schemes

A paradigmatic example of the first type of analysis is provided by the research studies carried out by Harel & Sowder and based on the notion of a "proof scheme", first presented in a research report at the 20^{th} PME (Harel & Sowder, 1996).

The individual's perspective is taken and the researchers finally outline a comprehensive map summarizing the full description of the taxonomy (see Harel & Sowder (1998), and Harel (1999, 2001), for further explanations). In the last elaborations of the model the categories emerging from the interpretation of empirical data have been supported by historical and epistemological analysis [2].

The interesting characteristic of this kind of investigation is the combination of a detailed analysis of students' production with the need to maintain the unity of the

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[2] It is important to remind that, starting from this model and according to general pedagogical assumptions, a complex teaching project was set up, aiming to make students' proof schemes evolve (PUPA- Proof Understanding, Production and Appreciation).

issue in focus. The successive elaborations of this model describe students' ways of reasoning, mainly in the solution of given problems, thus providing a highly refined classification of what are called "proof schemes". The authors formulate an explicit definition of the process of proving (of which a proof can be considered a product).

> *Proving is the process employed by an individual [...] to remove or create doubts about the truth of an assertion.*
>
> The process of proving includes two sub-processes: *ascertaining* and *persuading*.
>
> (Harel & Sowder, 1998, p. 241)

As the authors explain, ascertaining is the process an individual employs to remove her or his own doubts, whilst persuading is the process employed to remove others' doubts.

According to this definition, different types of reasoning are described. A proof might analyse a single case, establish the formal truth of a theorem, explain a statement, test a problematic hypothesis or be purely a ritual procedure devoid of any sense except maybe its status of expected answer.

Different observable phenomena are related to each other, so that different students' answers are interpreted as sharing common features, leading the authors to define them as different ways of thinking, each of which is referred to a proving process. As pointed out by Harel, the term *proof* is largely referred to any kind of argument, and "need not connote mathematical proof" (Harel & Sowder, 1996, p. 60). Thus, different kinds of arguments are all recognized as strictly related so as to be classified as products of particular *proof schemes*.

Classic epistemological perspective

In contrast to what I have just presented, other studies originate in the epistemological distinction between mathematical proof and other forms of proving processes. As already mentioned, one of the classic contributions on this theme was by Fischbein & Kedem (1982). In the same vein as other studies investigating the tension between intuitive and mathematical thinking (Fischbein, 1987), the issue of proof was addressed in the general perspective, concerning the potential conflict between an **empirical and a formal approach.**

Besides direct acquisition of information that is mostly related to factual evidence and attained through experience, human culture has developed a complex way of obtaining information and knowledge, which is not direct, but is rather mediated by means such as language, logic and reasoning. As a consequence of this mediation, the structural unity between cognition and adaptive reactions has been broken:

> Knowledge through reasoning becomes a relatively autonomous kind of activity, not directly subordinated to the adaptive constraints of the behaviour of human beings.
>
> (Fischbein, 1987, p.15)

In particular, as claimed by the authors (Fischbein & Kedem, 1982; Fischbein, 1982), crucial differentiation occurs between empirical verification and logical deduction, so that their relationship becomes very problematic.

The comparison between truth evaluation in terms of factual verification and logical validity in terms of deductive inferences leads one to consider the effect of a factual confirmation of the validity of a statement; of course different attitudes can be described according to an empirical approach and to the theoretical approach: despite the fact that a formal proof confers a general validity to a mathematical statement, further checks seem to be desirable, in order to confirm that validity (Fischbein, 1982). The reasonable conclusion is that the discrepancy between empirical verification (typical of common behaviour) and deductive reasoning (typical of theoretical behaviour) is recognized as a source of difficulties. These findings have been confirmed by other studies (for instance, Vinner, 1983) and the tension between empirical and theoretical ways of supporting or rejecting the truth of a statement have been investigated further. The issue of generality clearly emerged from these studies in its twofold complexity: not only do students seem not to realize that empirical verifications (measures, for instance) cannot be directly generalized, but conversely they may not grasp the generality of a deductive argument, strictly related to the use of a single drawing. A first discussion on the potentialities of a Dynamic Geometry Environment in respect to these issues can be found in (Chazan, 1988); as we will see in a later section, this theme has been further elaborated and discussed in the subsequent years.

Argumentation versus proof

The possible discrepancy, described above, between the empirical and the theoretical point of view has been developed and radicalised by Duval (Duval, 1989, 1992/93), who stresses the distinction between different approaches to proof, outlining an opposition between *argumentation* and *proof*.

In one of his classic papers (Duval, 1992/93), the author carries out an accurate analysis of the nature and the status of the discourse that people use to support the truth or the falsehood of a particular statement; the comparison between discourses in different domains, and in particular in mathematics practice. The author claims a neat distinction between what is commonly referred to as *argumentation* and what is referred to as *mathematical proof*. Argumentation may be regarded as a process in which the discourse is developed with the specific aim of making an interlocutor change the epistemic value given to a particular statement; in short, argumentation consists in whatever rhetoric means are employed in order to convince somebody of the truth or the falsehood of a particular statement. On the contrary, proof consists in a logical sequence of implications that derive the theoretical validity of a statement.

> Une argumentation ne fonctionne pas d'abord sur le statut des propositions mais sur leur contenu. [An argumentation does not firstly function on the status of the propositions, rather on their content].
>
> (Duval, 1989, p. I -234)

Duval certainly holds a very radical position, nonetheless he focuses on a crucial point: the difference between the semantic level, where the epistemic value of a statement is fundamental, and the theoretical level where, in principle, only the validity of a statement is concerned. The assumption that at the theoretical level the logical dependence of a statement in respect to axioms and theorems of the theory is independent from the epistemic value that one attributes to the propositions in play leads Duval to recognize a cognitive rupture between *argumentation* and *proof*. Coherently, Duval stresses the relevance of this issue from an educational point of view, both in explaining difficulties met by students and inspiring the organization of a coherent educational activity (Duval, 1991).

According to Duval, the rupture between the two levels (the semantic and the theoretical) may be irretrievable, so that the conception of proof as a process that aims to convince the interlocutor (as the author says, "to affect the epistemic value of a statement") may conflict with the requirements of a mathematical proof.

As clearly discussed by Balacheff (1999), such a conflict may become an epistemological obstacle (Brousseau, 1997) that students have to overcome in order to grasp the very idea of proof in mathematics. In fact, the learner has to make sense of the difference between argumentation and proof, without rejecting one for the other. Argumentation as experienced in everyday practice has to be consciously brought back into the mathematical classroom; but achieving a theoretical perspective means becoming aware of the particular nature of mathematics validation, so that particular argumentative competencies that naturally emerge in social interaction might appear inadequate and, for this reason, are likely to be overcome.

Overcoming the dichotomy: the notion of Cognitive Unity

The main point that emerges from the previous discussion seems to be how to manage the possible rupture and the consequent possible conflict between argumentation and proof.

Of course a preliminary question arises: is it possible to overcome this rupture?

A critical reflection is needed to at least understand, if not conciliate, different epistemologies.

In this perspective, interesting studies have been carried out with the aim of clarifying the relationship between mathematical proofs and the process of producing arguments. These studies do not deny the distinction between argumentation and mathematical proof, but are centred on the idea of a possible continuity rather than a rupture between them. This idea has subsequently been elaborated into the notion of Cognitive Unity, which seems to present great potentialities and which I'd like to discuss below.

In the context of a long term teaching experiment that exploited the semantically rich field of experience (Boero et al., 1995) of sun shadows, interesting results came to light concerning students' production of conjectures (Boero, 1994; Boero et al. 1995) and the argumentation accompanying it. The teaching experiments were part of a long-term experiment aiming to introduce pupils to Geometry and were based on the solution of an open-ended problem, requiring both a conjecture and its proof. Clear evidence was found of different kinds of argumentative processes appearing in the solution. Further investigations demonstrated that when the phase of producing a conjecture had shown a rich production of arguments that aimed to support or reject a specific statement, it was possible to recognize an essential continuity between these arguments and the final proof; such continuity was referred to as Cognitive Unity.

> During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices.

During the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain.

(Boero et al, 1996, p. 113).

Although the notion of Cognitive Unity emerged within the limits of a very particular teaching experiment and as part of a specific teaching project, it immediately showed its potential, suggesting new directions of investigation. If the first results supported a sort of "continuity" (as the authors called it), further investigation brought evidence of a possible gap between the arguments supporting the production of a conjecture and the proof validating it.

> We defined as *gap between the exploration of the statement and the proving process* the distance between the arguments for the plausibility of the conjecture, produced during the exploration of the statement, and the arguments, which can be exploited during the proving process.
>
> (Garuti et al., 1998, 347)

The term Cognitive Unity, firstly coined to express a hypothesis of continuity, was later redefined to express the possible congruence between the argumentation phase and the subsequent proof produced, clearly assuming that congruence may or may not occur.
The main strength of this construct is that of providing a way to escape the rigid dichotomy setting argumentation against proof: the possible distance between argumentation and proof is not denied but also not definitely assumed to be an obstacle; in this perspective, the essential irretrievable distinction between argumentation and proof is substituted by focussing on analogies, without forgetting the differences. This position opened the way to a new approach where the complex relationship between argumentation and proof is taken as an object of research and the notion of Cognitive Unity can be used as a means to structure the investigation.

Proof and theory

A further element aimed at clarifying the status of proof in relation to argumentation has been elaborated (Mariotti et al., 1997). Proof is traditionally considered in itself, as if it were possible to isolate a proof from the statement to which it provides support, and from the theoretical frame within which this support makes sense. When one speaks of proof, all these elements, although not always mentioned, are actually involved at the same time, and it is not possible to grasp the sense of a *mathematical proof* without linking it to the other two elements: a *statement* and overall a *theory*.
In their practice, mathematicians prove what they call "true" statements, but "truth" is always meant in relation to a specific theory. From a theoretical perspective, the truth of a valid statement is drawn from accepting both the hypothetical truth of the stated axioms and the fact that the stated rules of inference "transform truth into truth".

> A statement B can be a theorem only relatively to some theory; it is senseless to say that it is a theorem (or a truth) in itself: even a proposition like '2+2=4' is a theorem in a theory A (e.g.i.e. some fragment of Arithmetic)”.
>
> (Arzarello, in press)

Generally speaking, for students it does not seem spontaneous to reach such a theoretical perspective on truth, which on the contrary becomes automatic and unconscious for the expert. For a mathematician, the existence and the reliability of a theoretical framework within which the proof of a statement is situated is unquestionable and tacitly assumed, even when it is not made explicit. On the contrary, for novices, the idea of a truth as *theoretically situated* may be difficult to grasp; however, this way of thinking cannot be taken for granted and its complexity cannot be ignored. In particular, the confusion between an *absolute* and a *theoretically situated* truth, corresponding to the two main functions of proof - explication and validation - may have serious consequences.

> In my eyes it would even be an error of epistemological character to let students believe, by a sort of Jourdain effect, that they are capable of producing a mathematical proof when all they have done is argue.
>
> (Balacheff, 1999)

Thus, in order to remember the contribution given by each of the different components involved in producing a theorem, the following characterization of Mathematical Theorem was introduced, where a proof is conceived as part of a system of elements:

> The existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules are so intimately interrelated so that what characterises a Mathematical Theorem is the <u>system of statement, proof and theory.</u>
>
> (Mariotti et al. 1997, 1, p. 182)

Balacheff clearly indicated a possible articulation between the notion of Cognitive Unity and that of Mathematical Theorem, recently introduced, proposing an audacious parallel in the study of proof, which has become a plan of investigation:

> I will give a capsule description of the place I think possible for argumentation in mathematics, using the sense of the concept of Cognitive Unity of theorems coined by our Italian colleagues:
>
> Argumentation is to a conjecture what mathematical proof is to a theorem.
>
> (Balacheff, 1999)

Following this parallel, Pedemonte (2001, 2002) carried out a systematic investigation that aimed to study the relationship between the process of producing a conjecture and the related proof.

According to the first studies, Cognitive Unity was mainly discussed considering the "referential system"[3] of argumentation and proof. There is Cognitive Unity if it is possible to recognize elements which students used during the argumentation activity in the proof, i.e. in the chain of theorems and definitions used. It is usually possible to identify expressions or particular words that refer to the objects and the properties expressed by theorems and definitions.

Cognitive Unity and structural continuity

Taking into account the definition of Mathematical Theorem introduced above, it is possible to outline a correspondence between the reference system, i.e. the system of conceptions from which a conjecture emerges and within which it is formulated and supported by more or less explicit arguments, and the theory, i.e. available theorems and definitions within which the proof is produced. As a consequence, Cognitive Unity is recognizable when there is congruence between the system of conceptions on which the conjecture and its supporting arguments are constructed and the theory within which the proof is produced.

Nevertheless, although continuity in content is often recognizable, it sometimes happens that the construction of a deductive chain that correctly relates the theoretical elements involved may be difficult to achieve. In fact, differently from what happens in the case of content, continuity between the structure of the argumentation and that of the proof may be problematic, leading to errors or inconsistencies.

For example, abduction is a very common structure in argumentation, but in order to construct a proof, students must change the abductive structure into a deductive structure. In this case, continuity would lead the student to fail, while the construction of a correct mathematical proof has to overcome a structural distance.

For this reason, it becomes useful to introduce a distinction between ***referential*** and ***structural*** Cognitive Unity (Pedemonte, 2002).

Structural cognitive unity is the cognitive unity derived from a structural continuity, i.e. the continuity between the structure of the argumentation and the structure of the proof.

The examples discussed by Pedemonte (2002) are very interesting, as is the theoretical tool used to accomplish this analysis. The ternary model introduced by Toulmin (1958) allows the author to interpret the argumentation supporting a conjecture and to outline its whole structure.

On the base of such a structural analysis, the author can compare an argumentation, supporting the statement of a conjecture, and its final proof; as far as argumentation is concerned, different structures can be recognized, such as deduction, abduction, induction. Not all of them are consistent with a mathematically acceptable logical structure, thus the transposition of a given argumentation into a mathematical proof cannot always be accomplished straightforwardly. As previously mentioned, transforming an abductive argument

[3] The use of the term "referential system" may require some clarification. It generally refers to the knowledge system available to the subjects and used during the construction of a conjecture and the consequent proof (Pedemonte, 2002). Sometimes, mainly when the representational system (such as language, drawings, etc) is in focus, one can refer to the notion of *cadres* (Douady, 1986), and other times to the notion of *in act concept* (Vergnaud, 1990).

into a deductive proof requires its structure to be reversed, so that "the transition from argumentation to proof may demand relevant (and sometimes difficult to perform) changes concerning structures" (ibid., p. 35).

The interest in the case of abduction lies in the fact that, according to experimental evidence, exploration supporting a conjecture is very often accompanied by arguments showing this structure, so that passing from conjecturing to proving would require transformation from an abductive into a deductive structure. This passage presents difficulties that seem to require specific didactical treatment to overcome (Mariotti & Maracci, 1999).

The inductive type of argumentation is also quite common. There are different kinds of arguments that can be generally classified as 'inductive' but that actually present essential differences. Mathematicians have come to an agreement about the acceptability of a particular type of inductive argument, commonly called *Mathematical Induction*. The development of this type of proof has been difficult and not free from debate[4], however, nowadays the Principle of Mathematical Induction is commonly used and taught in advanced mathematics courses.

This type of proof is recognized as difficult; let us see for instance what can be read on the web, on one of the many sites offering explanations on Mathematical Induction:

> First off, don't get mad at your instructor. Induction makes perfect sense to him, and he honestly thinks he's explained it clearly. When I took this stuff, I know I had a good professor, so I know I got a "good" lesson. But I still didn't trust induction. So I'll try to pick up where your instructor left off.
> (http://www.purplemath.com/modules/inductn.htm- 27.03.05)

I'd like to present a brief discussion on this type of proof because it offers a good opportunity to coordinate different research studies, one shedding light on the other.

The case of Mathematical Induction

According to Harel & Sowder's classification of proof schemes, two different types of arguments can be recognized as referring to 'induction' and can be explained as follows:

- Generalization is achieved by recognizing a general pattern in the result itself (*result pattern generalisation*). For instance, after a series of calculations, the subject observes the regularity of the result of a calculation.
- Generalization is derived from the process *(process pattern generalisation)* that leads to the results. For instance, after a series of calculations, the subject observes the regularity of the process used to get the result, i.e. one may observe a particular chain of steps interrelating the results.

From the point of view of Harel & Sowder's classification, the former argument is a case of an *empirical* proof scheme, whilst the latter is a *transformational* proof scheme. The analogy between the arguments consists in the fact that, in both cases, the support of the general statement is obtained from an *inductive* process, i.e. from the verification of a limited number of particular cases. What makes the arguments

[4] Though the utilization of this kind of reasoning may be dated earlier, it was Dedekind (1870) who explicitly formalized it, introducing the Induction Axiom.

different is the fact that the elaboration of the particular cases is different. In the first case, the examples function as generic elements on which the proving arguments can be applied; in the second case, the passage from one case to the following is focused on and generalization applies to this passage.

In spite of the apparent analogy, the two types of inductive argumentations have quite different relationships with the type of proof by Mathematical Induction.

Using the principle of Mathematical Induction requires a shift of attention from proving the general statement P(n) to proving the inference [P(n-1)➔P(n)], conceived as a generic step in a chain of inferences. This difference is certainly crucial, and leads to difficulties that cannot be underestimated.

Such difficulties become even more evident if described in terms of Cognitive Unity. In fact, when a conjecture is generated by *result pattern generalization*, producing a proof by Mathematical Induction requires shifting from one type of argument to the other, meaning that the student has to control the gap between the two types of arguments, but this is not all. If a student has never experienced the production of arguments of a *process pattern generalisation* type, it might be difficult for him/her to grasp the sense of a proof by Mathematical Induction. Thus, looking at it from a pedagogical perspective, one can assume that promoting process pattern generalization could facilitate students' introduction to Mathematical Induction and consequently design a didactic intervention.

The brief discussion on the case of Mathematical Induction can be enlarged and adapted to other cases (interesting examples in the case of abduction are given by Arzarello et al. 1998), where it is possible to identify a gap between arguments produced by the students – in particular, supporting their production of a conjecture - and arguments used by and acceptable to experts, in mathematics practice.

In this same vein, the possible discrepancy between the individual/private and the social/public perspectives is highlighted by the results presented in the doctoral dissertation by Raman (2002). The examples discussed in the dissertation are interesting and deeper insight could come from a more detailed analysis; the use of the Toulmin's model, for instance, could reveal the potential continuity between the private and public aspects of proof, as well the potential gap between them.

Social versus individual perspective

The distance between different types of arguments exists and is certainly stressed by the fact that, in the case of mathematical proof, the standards of acceptability for an argument limit them to a well defined and clearly stated set of paradigms: some of them are commonly used, some of them are not very frequent, and some of them are definitely extraneous to the common practice of argumentation. A detailed analysis of the types of arguments ("proof schemes" in the terminology of Harel & Sowder) and a critical comparison with mathematical proofs brings a crucial aspect to light: consciousness required to manage the distance between argumentation and proof belongs to a meta-level where arguments in themselves become an issue to inspect.

In fact, in order to decide about the acceptability of a proof rather than its content – for instance whether to accept an argument by Mathematical Induction or reject an argument by abduction - the proof method itself becomes the subject of discussion, in order to decide on its acceptability.

Generally speaking, this corresponds to taking the fundamental characteristic of proof into account, namely its social dimension, meaning that proof makes sense in respect to a community that shares (more or less implicitly) the criteria of acceptability of the arguments in play.

At school, the social dimension related to the community of mathematicians must be coordinated with the social dimension related to the classroom community: the crucial role of the teacher comes to the forefront, representing contemporaneously the guarantor of the mathematics community and the guarantor of the classroom community. In short, the teacher has to become a cultural mediator and introduce students to the standards of mathematical validation.

It is important to remark the fact that the tasks proposed in the experimental investigation are often of the "proof that" type, meaning that students are asked to prove the validity of a given statement. This kind of task does not seem to be as effective in triggering the production of arguments as the task requiring the production of a conjecture. In the latter case it is plausible to expect that arguments arise to fuel the reasoning and this type of situation was often suggested as useful to approach the theme of proof at school for this reason (Balacheff, 1987; Boero et al., 1996).

This remark leads us to the third direction of research, concerning didactic proposals for introducing pupils to proof.

PROPOSALS FOR INTRODUCING PUPILS TO PROOF

Different research projects at different age levels in different countries have designed and implemented possible approaches to proof. Naturally, this has been done assuming different epistemological perspectives and different cultural contexts in respect to proof.

Results of survey studies and research work focused on students' conceptions of proof often motivated and, at the same time, inspired the development of innovation projects aimed more or less directly at introducing pupils to proof. A clear indication from the research studies focused on the need for an early start in proving practice and, consistently with this indication, seminal work can be found at the primary school level, dating back to a long time ago. A common feature that characterizes most of these innovation projects is immediately recognizable: attempts to foster the development of mathematical meaning are widely based on thoughtful mathematical activities investigated by children. In this case one of the basic aims is very often that of establishing a "mathematical community in the classroom" (Bartolini Bussi, 1991, 1998; Arsac, 1992; Maher & Martino, 1996; Yackel & Cobb, 1996).

Arguments and proof in the construction of knowledge

According to a widely shared point of view in mathematics education research, different approaches express the need to coordinate psychological and sociological perspectives, coherently with the interpretation of education as a way to encourage students participate in a culture rather than a way to transmit a piece of knowledge. Although not always made explicit, there seems to be a widely shared assumption (Boero et al, 1995) that reasoning and argumentation contribute to knowledge construction.

In the theory of didactical situations (Brousseau, 1997) this assumption is recognizable in the coordination between the three main types of situations, related respectively to action, formulation and validation. From this perspective, validation actually represents the key issue of learning, whose goal is envisaged as making the subject place action (eventually involving manipulations of representation systems) under the control of his/her mathematical knowledge. The functioning of the didactical system is accurately described and the metaphor of the *game* used by the author (Brousseau, 1997) clearly expresses the functioning of validation in relation to a system of rules.

The metaphor of game

When the pupil is playing, s/he develops strategies; this means that actions are selected from a set of potentialities according to intuitive or rational reasons; the feedback produced by the environment allows the subject to check the effectiveness of her choice and may consequently lead her to accept or reject it. The sequence of interactions between the student and the environment (*milieu*) constitutes what is called the 'dialectic of action'. Continuing in the game and the student passes through what is called the 'dialectic of formulation' that consists in "progressively establishing a shared language", making "possible the explanation of actions and modes of action". During this phase, according to Brousseau's model,

> […] it can happen that one student's propositions are discussed by another student, not from the point of view of the language […] but from the point of view of the content (that is to say, its truth or its efficacy). […] these spontaneous discussions about the validity of strategies are usually referred to as "validation phases".
>
> (Brousseau, 1997, p. 12).

The means of convincing the interlocutor may vary widely and they may remain beyond the student's control.

It is only by entering what the author calls the *dialectic of validation* that the student is motivated to discuss a situation and encouraged to express his/her reasons, which might previously have remained implicit. Although the reasoning may be insufficient or incorrect and adopt false theories (ibid., p. 17), the features of the milieu[5] should prevent "illegitimate" means from being used to obtain the agreement of the other, such as authority, force, seduction (ibid., p. 70) and favour students to use mathematical knowledge, and in any case to root it in rational thought. The dialectics of contradiction and the role counter-examples can be interpreted in this sense, as introduced and successively elaborated by Balacheff (1985, 1987, 1991).

Stating and developing socio-mathematical norms

Though related to the individual and to his/her personal relationship with the specific context, the effectiveness of the chosen strategy can be established depending on its intrinsic functioning: social interaction constitutes a basic factor that affects, motivates and fuels a dialectic of validation. Collaborative work,

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[5] In particular, if the characteristic of the milieu is *being a-didactical*, that is not directly controlled by a didactic goal, it becomes possible for the student to enter a correct dialectic of validation.

amongst peers or in small groups, seems to be a favourable social context in which to make cognitive conflicts arise as they are naturally brought to students' consciousness in confronting answers and arguments. Nonetheless, the emerging knowledge still needs to be related to the mathematical domain. In other words, in order to enter a mathematical perspective, students have to share a system of social norms (Yackel & Cobb, 1996) that state what is considered acceptable and, in particular, mathematically acceptable. The introduction of young children to the practice of "mathematical argumentation" has been the key objective of a number of research projects (Maher & Martino, 1996: Zack, 1997) designed to create classroom environments within which teachers can develop cultural norms, favouring the emergence of argumentation and proof making in children's discourse.

This issue becomes even more crucial when young children are involved. In fact, at the primary school level it seems difficult for a child to recognize something as a mathematical 'object' if nobody tells the child what "mathematical" might mean. As Cobb & Yackel (1996) clearly point out, it is necessary to establish what makes certain arguments less mathematical than others. Thus, beyond the social norms controlling what students are expected to do, it becomes crucial to establish **socio-mathematical** norms in the classroom: not only the practice of supporting one's own statements is introduced, but the criteria for acceptability of mathematical arguments are also negotiated in the classroom.

> […] the understanding that students are expected to <u>explain</u> their solutions is a <u>social norm</u>, whereas the understanding of what counts as an acceptable mathematical explanation is a <u>socio-mathematical norm.</u>
>
> (Yackel, 2001, p. 12)

> The school and classroom learning site is a community of practice which Richards (1991) has called inquiry math; it is one in which the children are expected to publicly express their thinking, and engage in mathematical practice characterized by conjecture, argument, and justification (Cobb, Wood & Yackel, 1993, p. 98).
>
> (Graves & Zack, 1996, p. 28)

The introduction of the mathematical perspective is clearly a responsibility of the teacher, as described in the following passage:

> […] the teacher initiated both the diagram and the verbal elaboration of Louis' solution. Her actions accomplished several goals. One was to call attention to the argumentative support for the conclusion. Second, she contributed to the class' understanding of what is taken as argumentative support.
>
> (Yackel, 2001, p. 16)

The experiences carried out and discussed by Bartolini Bussi (1996; Bartolini Bussi et al, 1999) are consistent with this perspective but more explicitly oriented not only towards introducing pupils to mathematical arguments, but also towards framing mathematical arguments within a theoretical system. In the frame of long term teaching experiments, young pupils are introduced to what the authors call "germ theories", meaning that, starting from a set of explicit assumptions,

arguments are produced to support specific statements. A basic characteristic is that assumptions[6] are collectively negotiated in the modelling process and triggered by a problem solving activity.
Analogously, the field of experience of shadows has functioned to foster a rich context in which the need for explanation leads to modelling and conceptualizing (Boero et al., 1995, Boero et al., 1996). As mentioned above, the cognitive analysis of students' arguments has highlighted interesting aspects concerning the process of producing conditional statements and complex arguments.
The relationship between empirical evidence and theoretical perspective has been a longstanding issue in the field of math education, which has mainly stressed the contraposition between them; however, a new approach to empirical evidence and its link with the mathematical validity emerged from these projects.
In the same vein the empirical, or rather the *pragmatic* perspective, as the authors call it, is widely discussed by Hanna & Jahnke (1993). The authors propose a distinction between the *pre-formation* and the *established* phase of a theory, and stress the need to let pupils have their own experience with the process of establishing a theoretical perspective. The need to state assumptions in relation to answering the question "why?", and the need for a reasonable empirical base for these assumptions is part of the "theoretical physicist" approach, that Hanna & Jahnke (2002) promote and hold to be indispensable in the development of a sense of proof.
The distance between the established and the pre-formation phase of a theory should not be ignored. In fact, the inability to realise that student and teacher are arguing from completely different points of view may be the origin of serious misunderstandings about the sense of proof, and the sense of mathematics in general.
In summary, different studies have supported the proposal that proof might be rooted in a 'culture of why questions' (Jahnke, 2005) and students might be introduced to a theoretical perspective that emerges as a way of describing and explaining experienced phenomena through a modelling process.
These kinds of modelling contexts reveal their power, in fact in that case the need to share assumptions springs out of the modelling process, as accomplished in classroom activities, while the specific status of such assumptions in respect to possible consequences requires a clear definition of their mutual relationships. In this sense, the coordination between constructing a model and reflecting on this construction offers the opportunity to make sense of a theoretical environment. In such a context, arguments are elaborated to explain what is observed, as well to foresee new properties, so that the rigour of a hypothetical – deductive system can be assured, without loosing the connection with the original meaning.
Taking this approach, the choice of an experiential context within which the modelling process can be realized may become crucial, and therefore research studies aimed at exploring the potential of particular contexts are needed in order to shed light on the general characteristics required.
Specific contexts related to the use of new technologies, among others, have been at the core of a number of studies; in particular, the development of Dynamic Geometry Environments has brought new life to Geometry classes and revitalized

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[6] Although they are not explicitly called Axioms, these assumptions have the theoretical status of Axioms.

classic mathematical activities that had nearly disappeared from the classroom. The following section will provide a brief discussion on some aspects concerning proof within a Dynamic Geometry Environment.

PROVING IN A DYNAMIC GEOMETRY ENVIRONMENT

In spite of the doubts arising at the very beginning and concerning the new relationship between students, experience and mathematics, it has emerged from computational environments that a number of studies consider specific computer-integrated contexts as very promising[7]. Within the limits of this chapter, only the contribution of a very specific context has been considered - that of Dynamic Geometry Environments. In particular, it is a widely shared opinion that Dynamic Geometry Environments have opened new frontiers, linking informal argumentation with formal proof (Haddas & Hershcovitz, 1998, 1999; Hoyles & Healy, 1999; Olivero & Robutti, 2001).

> Our results show that the computer-integrated teaching experiments were largely successful in helping students widen their view of proof and in particular link informal argumentation to formal proof […].
>
> (Hoyles & Healy, 1999, p. 112)

The contribution of DGEs.

In the last years a number of different contributions on the theme of proof shared the choice of a DGE as context. The joint efforts of some members of the PME community resulted in a set of papers published in a special issue of the international journal "Educational Studies in Mathematics" (Jones, Gutiérrez & Mariotti, 2000).

While it was immediately clear that a Dynamic Geometry Environment could contribute in developing geometrical reasoning, particularly in supporting the solution of geometrical problems, the contribution that such a context could provide to fostering a culture of proof appeared to be more questionable.

Certainly, the availability of graphing capabilities "has given a new impetus to mathematical exploration, and has brought a welcome new interest in the teaching of Geometry", but also "Dynamic software has the potential to encourage both exploration and proof, because it makes it so easy to pose and test conjectures" (Hanna, 2000).

It seems clear that within a DGE, dragging provides the students with strong perceptual evidence that a certain property is true. Nevertheless, the contribution provided to finding a proof does not seem as clear (Mason, 1991), although the possible contribution to the main problem seems even more critical, i.e. that of introducing students to a theoretical perspective. In pragmatic terms, can DGE provide a context for a culture of why questions?

As often stressed, it may be natural and reasonable for the student to jump to the conclusion that exploration via dragging is sufficient to guarantee the truth of what can be observed (Mason, 1991), thus preventing the emergence of any "why

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[7] A chapter of this book is devoted to discussing the general issues concerning the impact of new technologies in mathematics education.

questions" and as a consequence, a possibility of access to a proof sense. Thus the critical point, concerning the relationship between empirical evidence and theoretical reasons returns, in the new context, in the case of a DGE.
Using the notion of milieu (Brousseau, 1997), Laborde clearly explains the didactic complexity of the organization of a learning context:

> […] a DGE itself without an adequately organized milieu would not prompt the need of proof. It is a common feature […] to have constructed a rich milieu with which the student is interacting during the solving process and the elaboration of a proof".
>
> (Laborde, 2000, p. 154)

In this respect, an interesting issue concerns the comparison between different possible contexts. For instance, the comparison between physical experience with concrete linkages and virtual experience with simulations in the Cabri environment was explicitly addressed in Vincent & McCrae (2001) and Vincent, Chick & McCrae (2002). Referring to the work of Bartolini Bussi (1993), the authors investigate the potentialities of virtual models of mechanical linkages as a context for the emergence of the need of proof. The realization of the linkages in a Cabri figure constitutes a first model of the concrete apparatus within a very particular representation context which, differently from a paper and pencil context, may offer a support orientated towards geometric reasoning. Cabri can offer accurate measurements, the possibility of tracing the trajectory of a point, and thus the possibility of obtaining a precise representation of the locus to be studied. In short, Cabri may support that modelling process which can prelude, as the author claims, the emergence of "why questions". As suggested by de Villiers (1991, 1998), by overcoming the verification function, that is the use of DGE for verifying the truth of a statement, a new specific function of a DGE emerges.
A large number of contributions have been presented at the successive PME Conferences, focussing on different aspects and contributing to clarifying the potentialities and limits of DGEs in relation to proof. It is not possible and may be not worthwhile to give a full account of the single contributions here; I would rather focus on a specific but in my view crucial point, namely the relationship between dragging tool and theoretical control and, more generally, the dialectics between action and proof in the very specific context of a DGE.

Dragging tool and logical control

DGEs contain within them the seeds for a Geometry of relations. In this sense there is an opposition between a DGE and the paper and pencil environment, where Geometry may emerge from the experience of the evidence of unrelated facts; in fact, entering a DGE offers the opportunity to experience the break between these two worlds and to experience this break at the level of actions (Laborde 2000). But I'd like to go further and remember the fact that actions are mediated by tools which, according to vygotskian perspectives, can become "semiotic tools" (Vygotsky, 1978), meaning that actions with tools may contribute to knowledge construction inasmuch as tools themselves may become semiotic mediators.
The use of computational tools – such as primitives, macro, or dragging - may be exploited by the teacher with the aim of developing students' personal meanings towards mathematical meanings, according to specific didactic objectives.

In particular, a semiotic analysis concerning the "dragging" tool highlights the link between this tool and the meaning of "theoretical control", thus opening an interesting perspective on the potentialities of this tool as an "instrument of semiotic mediation" (Mariotti, 2002). In fact, the test consisting in the dragging mode, by which a construction is validated, may be used as an instrument of semiotic mediation to introduce the meaning of theoretical control within a Geometry theory (Mariotti & Bartolini Bussi, 1998). In other terms, in the field of experience of Geometrical constructions in the Cabri environment, the coordinated use of the different tools offered by the microworld make them function as instruments of semiotic mediation to make the meaning of "Mathematical Theorem" evolve (Mariotti, 2000, 2001b, 2002).

In this case, personal meanings concern the idea of dependent movement as it emerges from pupils' own experience in the Cabri environment; general mathematical meanings concern the mathematical ideas of logical dependence between hypothesis and thesis, as expressed in a Mathematical Theorem (that is the system of mutual relationships among the three main components: a statement, its proof and the theory within which the proof makes sense).

It is interesting to remark that in a recent project, based on the notion of semiotic mediation, a similar approach has been used to promote a theoretical perspective in a completely different mathematical field, namely Algebra. Inspired by the case of the DGE Cabri, a microworld was designed and experimented with the aim of providing a suitable experiential context to mediate and develop the meaning of proving within an Algebra Theory (Mariotti & Cerulli, 2001).

Dragging and theorems

Let us focus on the case of a DGE. Evidence from different studies indicates that using DGE tools does foster students' access to the world of Geometry theory, for instance by giving sense to inclusive definitions and consequently to explanations based on the logical relationships between properties (see for instance Jones, 2000). In particular, the dialectics of conjecture and empirical findings may lead students to experience contradiction and uncertainty, opening the way to the need for explanations and overcoming the strength of empirical evidence (Hadas, Herschkowitz & Schwarz, 2000). Functional dependency related to the functioning of the DGE constitutes a key element: conditional motion is certainly one of the main features of DGEs. In fact, the sequential organization of tools used to produce any dynamic figure (Cabri figure) states a dependency between different elements of the figure: such a dependency may not be immediately grasped, but it becomes evident to a user as soon as the dragging tool is activated. In fact, two main kinds of motions are possible using the mouse: direct and indirect motion.

The direct motion of a basic element (for instance, a point) is related to the direct action on a particular point (or other Cabri object). It represents the variation of this element in the plane; this is the way of representing a generic element[8] in Cabri.

[8] Different movements are possible, realizing different kinds of variation. The tool "point on an object" represents the variation of a point within a specific geometrical domain, a line, a segment, a circle, and the like. This is a way to represent the relationship " … belongs to …", and consequently the general statement "For all points, P belonging to a line r" may find a counterpart in the construction of a point P on a generic line r.

The indirect motion of an element occurs when a **construction** has been accomplished; in this case, dragging the basic points from which the construction originates will determine the motion of the new elements obtained through the construction. According to the basic semiotic relation, constraints stated by the tools used in the construction correspond to geometrical properties relating to the geometrical objects involved, so that motion dependency corresponds to logical dependency between properties.

Actually, the use of dragging allows one to experience ***motion dependency*** that can be interpreted in terms of **logical dependency** within the DGE, but also interpreted in terms of logical dependency within the geometrical context (i.e. logical dependency between geometrical relationships within a Geometry Theory).

The difficulty of establishing a correct and effective interpretation of different kinds of movements is well documented (Hölzl et al. 1994; Jones, 2000; Talmon & Yerushalmy, 2004), nevertheless, such a 'correct' interpretation constitutes the basic element for effective use of the dragging tool for both conjecturing and proving, i.e. for producing conjectures and their mathematical proofs.

Consider the Cabri microworld[9]: any Cabri figure may be related to a Mathematical Theorem, as defined above. The correspondence is not unique, but can be identified as follows: a *statement* can be reconstructed considering the construction process, performed to obtain the Cabri figure; the relationships stated by the construction constitute the *hypothesis*, while the properties derived from them and appearing as invariants in the dragging mode constitute a possible *thesis*. Thus, when an open problem is given, constructing a figure and dragging points represents a way to make relationships between properties appear. In other words, conjectures may emerge from the coordination between properties used in the construction and properties highlighted as invariant by dragging mode. Such a coordination is easy to state but not easy to realize. Nevertheless, the observation of students using the mouse while solving open problems in the Cabri environment has shown the appearance of different dragging modalities, as described in (Arzarello et al., 1998; Olivero, 2003).

Beside the basic mode that the authors call **Wandering dragging** - moving the basic points on the screen randomly without a plan - other modes are described.

- **Bound dragging:** moving a semi-draggable[10] point (already linked to an object).
- **Guided dragging:** dragging the basic points of a figure in order to give it a particular shape.
- ***Dummy locus*[11] dragging**: moving a basic point so that the figure keeps a discovered property, meaning that you are following a hidden path (*Dummy locus*) even without being aware of this.
- **Line dragging:** drawing new points on the ones that keep the regularity of the figure.
- **Linked dragging:** linking a point to an object and moving it onto that object.
- **Dragging test:** moving draggable or semi-draggable points in order to see whether the figure keeps its initial properties. If so, then the figure passes the test;

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[9] In the following discussion, I will take the example of Cabri, although with slight differences the discussion can be transferred to other DGEs.

[10] A semi-draggable point is a point on an object that can be moved but only on the object it belongs to.

[11] In French: ***lieu muet***

if not, then the figure was not constructed according to the geometric properties you wanted it to have.

According to the experimental data reported by the authors, these dragging modalities can be successfully exploited during the exploration phase, leading to the formulation of a conjecture. Taking a perspective coherent with the notion of Cognitive Unity, it seems reasonable to hypothesize a link between the dragging modalities used by the solver in the exploration and the subsequent proof of the conjecture. The dragging modalities used by the subject may correspond to a sort of *instrumented arguments* supporting the conjecture produced, which can be compared with formal arguments that can be used in constituting a mathematical proof.

This hypothesis opens a new direction of investigation, raising a number of complex but fascinating research questions concerning the correspondence between motion dependency and logical dependency. In particular, the basic hypothesis can be further elaborated as follows:

Is it possible to foster cognitive continuity in students' performances, provided the teacher supplies students dragging modes as mediators?

CONCLUSIONS

As mentioned at the beginning, a general consensus has been achieved on the fact that the development of a sense of proof constitutes an important objective of mathematical education, and there seems to be a general trend towards including proof in the curriculum[12]: this objective is strictly linked to other objectives concerning the development of other mathematical competencies.

Besides the importance of proof and the need to include it in the mathematics curriculum, current research has shown the complexity of the idea of proof and the difficulties that teachers and students face when proof becomes part of classroom mathematical activities.

Proof clearly has the purpose of validation - confirming the truth of an assertion by checking the logical correctness of mathematical arguments - however, at the same time, proof has to contribute more widely to knowledge construction. If this is not the case, proof is likely to remain meaningless and purposeless in the eyes of students. Alternative approaches have been proposed for a long time and the crucial point that has emerged from different research contributions concerns the need for proof to be acceptable from a mathematical point of view but also to make sense for students. For instance, when Hanna spoke of explanatory proofs (1989a, p. 12), the main goal was that of achieving flexible thinking, moving from different functions of proof, and in particular from validating to explaining and vice versa.

Encouraging student engagement and ownership of the proving activity has to be integrated into explanatory proving in a social dimension, where students explain their arguments to a peer or to the whole class, including the teacher, also to convince themselves of their truth. It is in this vein that a number of teaching experiments and research projects have been taken up. Suggestions coming from these studies have highlighted key elements, mainly concerning how to choose and

[12] We like to think, with Hoyles (1997), that this trend emerged under the pressure of educational research.

organize meaningful contexts within which the different components of the sense of proof can be developed. Among others, Dynamic Geometry Environments have opened new perspectives, which appear very promising. Nevertheless, further investigation is needed concerning students' active production of proofs in order to design appropriate contexts; in particular, analysis of the cognitive processes involved in producing and proving conjectures seems to shed new light on students' difficulties as well on the possible source of these difficulties. The notion of cognitive unity, which addresses the link between spontaneous arguments and mathematically acceptable arguments, may provide a powerful tool of investigation and is open to further elaboration (a recent example in this direction can be found in Vincent et al., 2005).

The need to develop mathematical ideas in relation to arguments and provide effective argumentation that can become mathematical proof is strictly related to the potential congruence between conceptions and theorems: arguments produced to support one's own conjecture must be compared with arguments that are acceptable, i.e. that are already stated and shared in the mathematics community that the individual has to participate in, as Harel & Sowder say, arguments must be recognized as acceptable at the same time for both ascertaining and persuading.

Consistency between these two facets of the problem of proof may be considered a main educational objective. Nevertheless, there is an even more basic issue: to become aware of the existence of different points of view and the need to negotiate the relevance and the acceptability of the mathematical perspective in respect to other forms of argumentation requires a complex and delicate teaching intervention. The perspective of *cognitive unity* makes it clear why, in the relation between argumentation and mathematical proof, there must not be a mere rejection of the first in favour of the second. Meanwhile, as discussed above, the existence of an *epistemological obstacle* explains why the tension cannot disappear completely, and students can only be made aware of it.

All this corresponds to quite a demanding objective from the educational point of view, requiring the teacher to bring about development both at a cognitive and meta-cognitive level. In particular, students' attitude to proof is strictly and more generally related to their beliefs about mathematics. Thus, the horizon of the field of research has to be expanded to embrace the cognitive and the meta-cognitive perspectives; in particular, the teacher's role as a cultural mediator comes to the forefront. Further investigation is required in this direction, concerning both practice at school and the training of teachers. On the one hand more has to be known mainly about the potentialities of contexts to introduce students to the practice of proof. On the other hand, the delicate and complex role of cultural mediator to which the teacher is called requires careful preparation involving both the cognitive and the meta-cognitive level.

The evolution of a mathematical culture in the classroom is a long-term process, requiring specific strategies of intervention that begin very early and develop over a long period. In this respect, investigation cannot be detached from classroom reality and, generally speaking, from the school environment: classroom investigations are of great value, and, although they raise difficult methodological problems, they should be promoted both in the form of comparison between different cultural experiences and in the form of teaching experiments.

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