Contexts for approaching validation: artefacts of information technologies[*]

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Introduction

This paper aims to discuss some aspects of a long term teaching experiment, carried out at the 9th - 10th grades. The teaching experiment's goal was to introduce pupils to proof. The experiment is framed in a joint research program on Semiotic Mediation in the mathematics classroom (Bartolini Bussi & Mariotti, in press) and we adopted the paradigm of research for innovation in the mathematics classroom (Arzarello & Bartolini Bussi, 1998).

According to the paradigm of research for innovation, where practice and theory nurture each other in a complex interlaced process, the implementation of the innovative didactic strategies is driven by a number of vague pedagogical assumptions. During the teaching experiment we tried to formulate and cyclically refine and clarify our theoretical hypotheses. Thus, over a long period of time, we developed a theoretical framework, which clarifies and frames our first vague intuition (Mariotti, 1996), placing it within a Vygotskian approach based on the key notion of semiotic mediation. In this paper I will present two examples of contexts for approaching proof, both of them centred on using a computer-based environment. The notion of semiotic mediation and its derived didactic model (Bartolini Bussi & Mariotti, in press) will be used to frame the discussion

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Artefacts and signs in a vygotskian perspective

Semiotic Mediation: tools of semiotic mediation

The role of tools and of their functioning as a source of knowledge was explicitly addressed by Vygotskij in the broader perspective that sees the evolution of human cognition as an effect of social and cultural interaction. Elaborating on the Vygotskian seminal idea of semiotic mediation a pedagogical model was set up with the aim of describing and explaining the functioning of artefacts' use in the teaching/learning process. The following short description of the model is strictly finalized to clarifying the discussion on the examples that constitute the core of this contribution (for a full discussion and more references see Bartolini & Mariotti, in press).

The semiotic lens was used by Vygotskij to interpret individual knowledge construction, in Vygotskian terms *internalization*, as a social endeavour. His basic assumption is that *the internalization process* is essentially *social* as well as directed by *semiotic processes related to the* communication dimension involving production and interpretation of signs, and living in what can be called the *interpersonal space* (Cummins, 1996).

According to a fundamental Vygotskian hypothesis within the social use of artefacts in the accomplishment of a task (that involves both a mediator and a mediatees) shared meanings are generated. On the one hand these meanings are related to the accomplishment of the task, in particular related to the artefact used, and on the other hand they may be related to particular mathematical content. In other terms, a semiotic potential resides in any artefact, and it consists in the double semiotic link that such an artefact has with both the personal meanings that are expected to emerge from its use, and the mathematical knowledge evoked by that use, as it can be recognized by a mathematics expert. That semiotic relationships hinged in the artefact may become the object of an a priori analysis, involving in parallel a cognitive and an epistemological perspective. This leads to the identification of what was called *the semiotic potential of an artefact with respect to particular educational goals* (Bartolini Bussi & Mariotti, in press). In this respect any artefact, either belonging to the set of new technologies or to the set of ancient technologies, may offer a valuable semiotic potential, although its identification might require different approaches (Bartolini Bussi, in this same issue).

Exploiting the semiotic potential of the artefact means for the expert (for instance, the teacher) to be aware of its potentialities both in terms of the emergent mathematical meanings and in terms of the emergent personal meanings. On the one hand, it means to orchestrate didactic situations where students face designed tasks that are expected to mobilize specific schemes of utilization and, consequently, situations in which students are expected to generate personal meanings. On the other hand it means to orchestrate social interactions with the aim of making personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives.

> Thus any artefact will be referred to as *tool of semiotic mediation* as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. […]
>
> (Bartolini & Mariotti, in press, p. 758)

According to the semiotic mediation theory, the complex semiotic processes of creation and evolution of personal meanings towards mathematical meanings can be developed through the design and implementation of the so called Didactical Cycle (op. cit., p. 758ff). This is an iterative cycle of the following kinds of activities: Activities with the artefacts, Individual production of signs, Collective production of signs.

The analysis of the semiotic potential of an artefact encompasses the analysis of both the personal and mathematical meanings related to such artefact, as well as the possible tasks which can be accomplished with it. What makes the analysis of the semiotic potential significant form a didactic point of view is its consistency with specific educational goals. The discussion of the following examples concern the common educational goal consisting in introducing students to a theoretical perspective and the common feature of exploiting the semiotic potential of a computer based environment.

First example: the semiotic potential of a DGE

My first example concerns a particular Dynamic Geometry Environment, Cabri-géomètre (Laborde and Bellemain, 1998) and the discussion of its didactic potentialities through the lens of a semiotic approach. My aim is to highlight the *semiotic potential* ***of this DGE*** with respect to the educational goal of introducing students to the idea of mathematical proof.

In a DGE figures can be constructed through tools in the menus, and the robustness of such figures can be tested through what researchers (Arzarello et al. 2002; Olivero, 2002) have defined the "dragging test". These figures can then be interpreted as constructions with a geometrical nature in the field of Classical Euclidean Geometry. The starting point of our analysis lies exactly in this evident/immediate relationship between Cabri figures and their corresponding geometrical constructions. Such relationship can be elaborated from the point of view of semiotic mediation through both an epistemological and a cognitive analysis leading to the definition of the semiotic potential of the artefact Cabri.

Euclidean Geometry, traditionally referred to as 'ruler and compass geometry', gives a central role to construction problems whose theoretical nature is clearly stated, in spite of their apparent practical objective, i.e. the drawing which can be produced on a sheet of paper following the solution procedure. As Vinner clearly point out in his Reviews of the Martin's book on Geometric Construction (Martin, 1998):

> The ancient Greek undertook a challenge which in a way represents some of the most typical features of pure mathematics as an abstract discipline. It is not related to any practical need.
>
> (Vinner, 1999, p.77)

In fact, as Euclid's Elements show, the use of ruler and compass generates a set of axioms defining a theoretical system, within which the correctness of the construction is *validated* by a Theorem.

Since their appearance DGEs triggered a new revival of the field of geometrical constructions, providing virtual tools that simulate the drawing tools of classic Geometry: lines and circles can be drawn intersect each other, nicely reproducing on the screen what for centuries was drawn on various supports sand, papyrus, paper. However, compared to the classic world of paper and pencil figures, the novelty of a dynamic environment consists in the possibility of direct manipulation of its drawn figures that can be achieved through the us the mouse. As a consequence, the stability (robustness, as some authors call it, for instance: Jones, 2000, p. 58) of the drawn figure in respect to dragging constitute the natural test of correctness for any construction task in the Cabri environment

> Dragging points of their constructions disqualifies purely visual strategies, by illustrating how constructions can be "messed-up" […].
>
> (Hölzl, Healy, Hoyles &Noss, 1994, p.)

In fact, the core of the dynamics of a DGE figure, as it is realized by the dragging function, consists in preserving its intrinsic geometric relationships. The elements of any figure in a DGE are related according the hierarchy of properties determined by its construction procedure. Such hierarchy of properties corresponds to a relationship of logical conditionality. The set of tools in a DGE is put in relationship with its correspondent set of constructing tools in Euclidean Geometry (Laborde & Laborde, 1995). This correspondence allows the control "by dragging" to be put in relationship with “proof and definition” within the system of Euclidean Geometry (Mariotti, 2000; Jones, 2000; Stylianides & Stylianides, 2005)[1].

In summary, as far as the Cabri tools are concerned, a double relationship is recognizable: on the one hand Cabri tools are related to the construction task that can be realized through them and it results in the existence of a Cabri figure on the screen, and on the other hand Cabri tools are related to the geometrical axioms and theorems that can be used to validate the corresponding construction problem within the Geometry theory.

Hence, the semiotic potential of the Cabri environment resides in its relationship between the meanings of the construction emerging from the use of its virtual tools - that simulate the drawing tools of classic Geometry - for solving construction problems controlled by the dragging test, and the theoretical meaning of a geometrical construction as it is defined within Eucliedean Geometry.

Exploiting this semiotic potential, centred about the artefact Cabri, became the key pedagogical assumption inspiring a long term teaching experiment. The pedagogical plan was to design (for details see Mariotti, 2000; 2001), following the structure of a Didactic Cycle, activities involving the use of the artefact and semiotic activities aimed at individual and social elaboration of sings (Bartolini Bussi & Mariotti, in press; Bartolini Bussi, this volume).

Activities in the Computer lab consisted primarily in a *construction task* in which students were asked:

1. to produce a Cabri figure corresponding to a Geometric figure;
2. to write the description of the procedure used to obtain the specific Cabri figure;
3. to produce a justification of the 'correctness' of such construction.

Thus the task was composed of two types of requests, the former corresponding to acting with the artefact, the latter to reporting on such actions through written text. Note that producing written text consists in both describing and commenting the procedure carried out. The request of *justifying* the solution made sense with respect to the Cabri environment corresponding to the need not only of validating one's own construction, but also of explaining and getting insight into the reason why the figure on the screen passes the dragging test.

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[1] Actually a DGS provides a larger set of tools, including for instance "measure of an angle", "rotation of an angle" and the like. That implies that the whole set of possible constructions do not coincide with that attainable only with ruler and compass, see (Stylianides & Stylianides, 2005) for a full discussion.

Once reported, the different solutions were compared within collective discussions, that became true Mathematics Discussions (Bartolini Bussi, 1998; Mariotti, 2001) where the main motive was focused on the evolution of the meaning of the term *construction*. At the very beginning 'construction' made sense only in the field of experience of Cabri, that is in relation to using particular Cabri tools and to passing the dragging test. Later the meaning slowly evolved, acquiring the theoretical meaning of Geometrical construction.

Such evolution could be accomplished exploiting the correspondence between Euclidean Axioms and specific Cabri tools and their modes. Starting from an empty menu, under the guidance of the teacher, the choice of the appropriate tools to start with was discussed as well the correspondence with the Construction axioms constituting the first core of the Geometry Theory. Then, as long as new constructions were produced, the corresponding new theorems were validated and added to the theory. Following a parallel process of evolution, pupils participated, on the one hand, in the development of a geometry system, and on the other hand to the enlargement a corresponding Cabri menu. In so doing students not only appropriate the new theorems but they also become aware of how the theory develops. Results of longstanding teaching experiments attested to the emergence of intermediate meanings, rooted in the semantic field of the artefact, and their evolution into mathematical meanings, consistent with the Euclidean Geometry (for details see Mariotti, 2001).

These teaching experiments were designed, implemented and repeated for several years. The analysis of the data led us to reflect upon the semiotic potential and, consequently, to refine the epistemological analysis related to what we had generically called a theoretical perspective. Thus we achieve a more articulated description of the semiotic relation linking the use of particular tools, available within Cabri, and the mathematical meanings related to them. In particular, the experience in the classroom highlighted the importance of rooting the sense of proof in the sense of theory. The constrained world of the DGE was effective in developing and interlacing these two meanings. In a DGE, the use of any single tool mediates the meaning of application of an axiom, while the set of available tools mediates the meaning of theory, its conventionality and its evolutionary nature. Moreover, by exploiting the possibility of personalizing the menu by selecting the tools to be used, it was possible to make the students experience establishing and developing Geometry Theory. The epistemological considerations rising from classroom observation, and in particular the importance of the focus on the development of a theory, led us to further elaborate our educational goal focussed on "introducing students to a theoretical perspective".

Theoretical and metatheoretical considerations

A didactic definition of Theorem

In the current literature in the field of mathematics education we are used to the issue of proof being considered in itself. This habit, although comprehensible with respect to the mathematics practice, may reveal its limits when one takes an educational stance. Generally speaking, it is impossible to grasp the sense of a *mathematical proof* without linking it to the two other elements: a *statement* and overall a *theory*, that is a proof is a proof when there is a statement to which it provides support but also when there is a theoretical frame within which this support makes sense.

But what becomes automatic and unconscious for the expert does not seem spontaneous to reach for novices. The idea of *theoretical validation* may be difficult to grasp. Certainly this way of thinking cannot be taken for granted and its complexity cannot be ignored. In particular, the confusion between an *absolute* and a *theoretically situated* truth, corresponding to the two main functions of proof - explication and validation - may have serious consequences (for a full discussion Mariotti, 2006).

Thus, in order to express the contribution of each component involved in a theorem, the following characterization of Mathematical Theorem was introduced, where a proof is conceived as part of a system of elements:

> The existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules are so intimately interrelated so that what characterises a Mathematical Theorem is the system of statement, proof and theory.
>
> (Mariotti et al. 1997, 1, p. 182)[2]

In the traditional school practice, the last component of the Theorem, i.e. the Theory within which the proof make sense, is largely neglected and except for the case of Geometry, the theoretical context in which theorems are proved normally remains implicit. This is often the case, for instance, in Calculus courses and textbooks, where theorems are proved, but very rarely the axiomatic reference system is explicitly stated.

It is important to remark that what is shortly referred to as Theory, has a twofold component. On the one hand, Axioms and already proved Theorems constitute the means of supporting the single steps of

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[2] This definition has been widely used and further elaborated generating subsequent interpretation models for both conjecturing and proving (Mariotti, 2006; Pedemonte, 2002; Mariotti & Antonini, 2007, Mariotti & Antonini, in press).

a proof; on the other hand, meta-theoretical rules assure the reliability of the specific way to accomplish this support, in other terms, how Axioms and Theorems, belonging to a Theory, can be used to validate a new statement.

Actually, as clearly pointed out by Sierpinska, acting at a meta theoretical level constitutes the very essence of a theoretical perspective

> [T]heoretical thinking is not about techniques or procedure for well-defined actions, […] theoretical thinking is reflective in that it does not take such techniques for granted but considers them always open to questioning and change. […] Theoretical thinking asks not only, *Is this statement true?* but also *What is the validity of our methods of verifying that it is true?* Thus theoretical thinking always takes a distance towards its own results. […] theoretical thinking is thinking where thought and its object belong to distinct planes of action.
>
> (Sierpinska, 2005, pp. 121-23)

In the school context, the complexity of this meta theoretical level seems to be ignored[3]. It is commonly taken for granted that students' way of reasoning is spontaneusly adaptable to the sophisticated functioning of a theoretical system. Thus not much is said about it and in particular deduction rules and their functioning in the development of a Theory are rarely made explicit.

There are at least two aspects of acting at a meta level that need to be made explicit. One consists in the acceptability of some specific deductive means, the other in the fact that no other means, except those explicitly shared, is acceptable.

If these two aspects are left implicit, it may happen that students have no access to any control on their arguments. In this case the control remains completely in the hands of the teacher, resulting for students in a general feeling of confusion, incertitude and lack of understanding.

The following example illustrates the key elements of a research project focused on the design of a microworld, that could offer a semiotic potential consistent with our epistemological and didactic analysis. In other terms, we focused on the need of an environment where the use of specific tools could contribute to the evolution of the meaning of Theorem, as the unity of the three components: Statement, Proof and Theory).

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[3] An exception is that of mathematical induction, which is explicitly treated, and to which a specific training is devoted. But, very rarely, is mathematical induction presented in comparison to other modalities of proving, which are commonly considered natural and spontaneous ways of reasoning.

Second example: a microworld for Algebra Theory

Firstly, I will briefly explain in what sense I will speak of Algebra as a Theory and then I will illustrate how a microworld was designed in order to provide tools of semiotic mediation related to the idea of Theorem with respect to an Algebra Theory.

Algebra as a theory

Since antiquity, Geometry has been considered a prototype of theoretical systematization of mathematical knowledge, the archetype of what in modern terms is called an Axiomatic. On the contrary, Algebra found its systematization relatively late in history (…). Moreover there is no tradition of a theoretical approach to Algebra at the pre-university level, where the study of Algebra is often relegated to its operative aspects of "symbolic calculation", thus neglecting any relational interpretation of this new way of calculating, so that no suspect arises that this part of Algebra might be a theory.

In fact, symbolic calculation can be interpreted within an Algebra theory of equivalence that originates from the numerical context but achieves a new interpretation as soon as some basic equivalence relations are stated as axioms.

Within the numerical context, an equivalence relation can be stated between two numerical expressions. Such expressions can be defined as equivalent if and only if the respective computations yield the same result. On the basis of this relation the equivalence relation can be extended to the set of algebraic expressions in which all the possible numerical interpretations can be given to the letters and the computation of the derived numerical expressions.

Substituting any expression or sub-expression with an equivalent one is it possible to operate on symbolic expressions preserving the equivalence. Extending the original meaning of 'calculation' from the domain of numbers to that of algebraic expressions, what is usually called algebraic/ symbolic calculation consists in the transformation of an algebraic expression into a new one that is algebraically equivalent to it.

In the numerical context, the basic properties of operations – for instance the commutative property of addition or the associative property of multiplication - express the equivalence of two numerical expressions, conceived as computing procedures. In the numerical context these properties do not play any operative role, they state trivial truth, but in fact they are not directly employed to achieve the computation.

On the contrary, within the algebraic context, the operations' properties assume an operative role, they become *rules of transformation* and the chain of equivalence that originates from the subsequent application of these rules finally transforms any symbolic expression into an equivalent one.

In other terms, the set of equivalences, stating the basic properties of addition and multiplication, may function as an axiomatic system for a local Algebra theory, within which *symbolic calculation* can be interpreted as a *proving process*, validating the equivalence between two algebraic expressions (Cerulli, 2004)[4].

It is out of the scope of this paper to fully discuss the reasons why such a theoretical approach may contribute to developing an effective alternative to the traditional approach (Cerulli & Mariotti, 2002; Kieran & Drijvers, 2006) to Algebra. I would rather concentrate on explaining how we tried to design a microworld affording a semiotic potential with respect to the specific theoretical perspective that we aimed to develop for algebraic symbolic calculation.

Reconstructing the semiotic potential

Starting from the previous analysis, we planned to design a prototype microworld that could offer tools of semiotic mediation for developing an Algebra Theory. Without entering into too technical aspects, I will focus on the general principles inspiring the design. That is, I will explain how epistemological and cognitive analysis was used to identify key features of the microworld required to make semiotic mediation possible.

The complex relationship among the different mathematical meanings related to the notion of Algebra Theory was to be reproduced in a consistent way, so that acting on the objects in the microworld might generate meanings that could be related to the notion of Algebra Theory, and to Theory in general.

As previously discussed, a formula stating the equivalence between two algebraic expressions constitutes the generic statement of our Theory, while the substitution of an expression with an equivalent one constitutes the basic deduction rule, through which any equivalence may be derived from another according to the transitive property of equivalence.

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[4] As discussed in the current literature and using the *operational-structural* terminology used by Sfard (1994), one can say that the operational character transforming algebraic formula and expressions persist (ibd. p. 194), while the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397). On the contrary, a structural conception becomes crucial in order to grasp the meaning of “symbolic calculation”, in particular if one consider the change that the term 'calculation' has to achieve when passing from the numerical to the algebraic context.

Accordingly the microwolrd, L´Algebrista, was designed (Cerulli, 2002) having the basic elements were algebraic expressions and the basic actions on them could be accomplished through specific commands, represented by icons on a tool bar (Buttons) to be activated by a click of the mouse.

The mode of use (utilization scheme, in Rabardel (1995) terminology) of any Button is very simple: after a formula is selected the click on the Button results in the substitution of that formula with the corresponding one. Each Button corresponds to one of the basic equivalences, stating the properties of addition and multiplication, constituting the basic set of *Axioms* to start with[5]. As a consequence, the microworld offers elements referring to *any single axiom* and to the application of the basic *deduction rule.*

Furthermore, transforming any algebraic expression into another one by using the Buttons, corresponds to proving the equivalence of expressions within an Algebra theory, so that any transformation chain refers to a proof and, once proved, any equivalence refers to a Theorem.

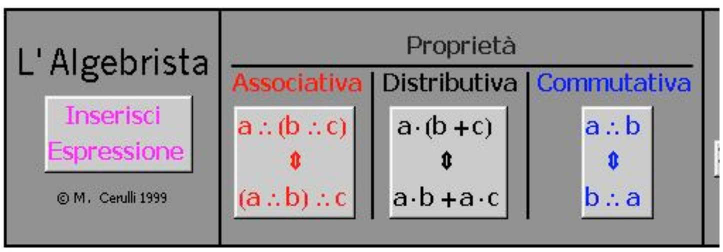

Figure 1. The icons representing the properties are active Buttons, each of them corresponding to the statement of an axiom and its functioning as tool according the basic deductive rule.

The specificity of acting within a theoretical domain is explicitly represented by the action of "entering" the microworld by the command *Insert Expression* (Ita. Inserisci Espressione). This command has the effect of initializing the application. The status of the selected expression changes: from being a string of characters it becomes an object of the microworld that can be acted on through the available Buttons. It is worth to remark that when an expression is *inserted*, its new instance comes out with some changes: every multiplication is represented with a dot ("•"), so either stars ("2*3") or spaces ("a 2") are substituted with a dot ("2•3+a •2"); every subtraction is transformed into sum, so expressions like "2-3" are substituted with "2+(-3)"; analogously every division is transformed into multiplication. L'Algebrista does not know subtraction, and division: this follows from a precise didactical choice because we wanted pupils to work in a "commutative environment".

[5] For instance the statement a+b=b+a. For brevity reasons, I will not enter into details in the description of axioms and definitions of the Theory, I would rather concentrate on the meta theoretical aspects, which I am interested in.

Figure 2 shows an example of a transformation procedure. Once introduced into the microworld, the expression is transformed using the Button of Commutative property of addition and the Button of Distributive property. This corresponds to the transformation according to the corresponding Axioms in the Algebra Theory.

Acting in the microworld requires the user to become aware of the property that is to be used at each single step, hence it offers the opportunity of becoming aware of the basic deductive rule, usually implicit, that leads one to transform one expression into an equivalent one. An active experience of the axioms and the deductive rules that are in play when a symbolic calculation is performed generates a rich system of meanings referring to algebraic calculation as a deduction chain within a theory. On the one hand, the constraints defined by the Buttons available in the tool bar correspond to the constraints defined by the axioms available in the theory. On the other hand, the effect of the commands on the expressions corresponds to the effect of the deductive rule of substitution. A trace of the deductive steps is displayed on the screen, as long as the commands are activated and the sequence of transformation progresses step by step, as shown in figure 2.

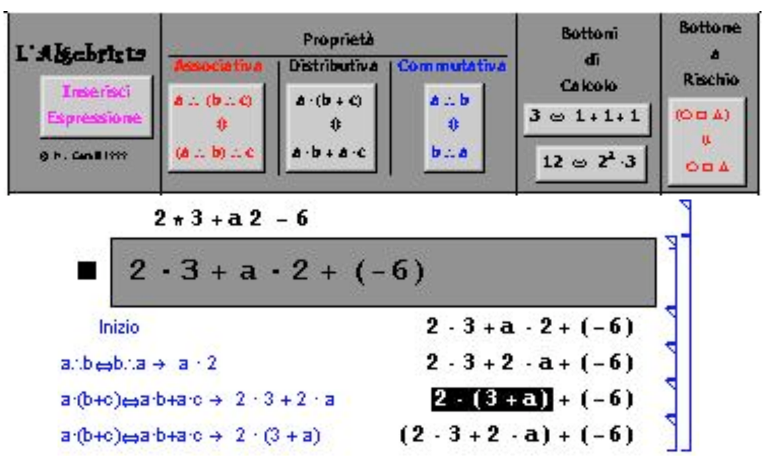

Figure 2. The user writes the expression to work with («2 * 3 + a 2 - 6» in the example), then after selecting it the Button Inserisci Espressione is clicked. Thus L'Algebrista creates a new working area where the Buttons are active. The trace of the Buttons used is displayed in blue .

In summary, according to this specific analysis the semiotic potential of the designed microworld is based on the following interpretation of some of the elements of the microworld in terms of mathematical meanings:

expressions in L'Algebrista refer to *algebraic expressions;*

Buttons/icons refer to *axiom statements* and *definition statements of an Algebra Theory*;

The functioning of Buttons commands refers to *the application of axioms according the basic deductive rule;*

Transforming an expression, *using the available commands* refers to *proving within the stated Algebra Theory.*

Consistently a set of tasks to be accomplished in the microworld was designed, in order to make students' use of the artefact and personal meanings emerge. The main task consisted in the comparison of two or more expressions: the student was asked to establish whether the expressions were equivalent or not and in any case prove her statement. The mathematical meaning of proving an equivalence relation by a sequence of applications of the axioms is collectively elaborated on the basis of the functioning of the microwolrd. As long as the corresponding Algebra Theory is collectively built, symbolic calculations enlarge its meaning, including that of proving process for an equivalence relation between two expressions.

Figure 3 shows an exemple of students' solution to a comparison task. In order to give an idea of how much the meaning of proving has become detatched from the microworld, I selected an example concerning a task given in paper and pencil context. The student, asked to evaluate the equivalence between different expressions, firstly checks the equivalence using his computing skills, once he made

I think the 1st and the 3rd are equivalent, but not the 2nd, because applying the properties they become equal, while the 2nd does not.

I applied the distributive property.

I applied the distributive property on these two pieces.

I added the two equal terms **–a*b –a*b** and I cancelled its result with its opposite obtaining **0** for the 1st theorem.

I cancelled also **+b*b** with its opposite and as it was **–2b*b** I obtained **–b*b.**

At this point the 3rd expression is equal to the 1st expression.

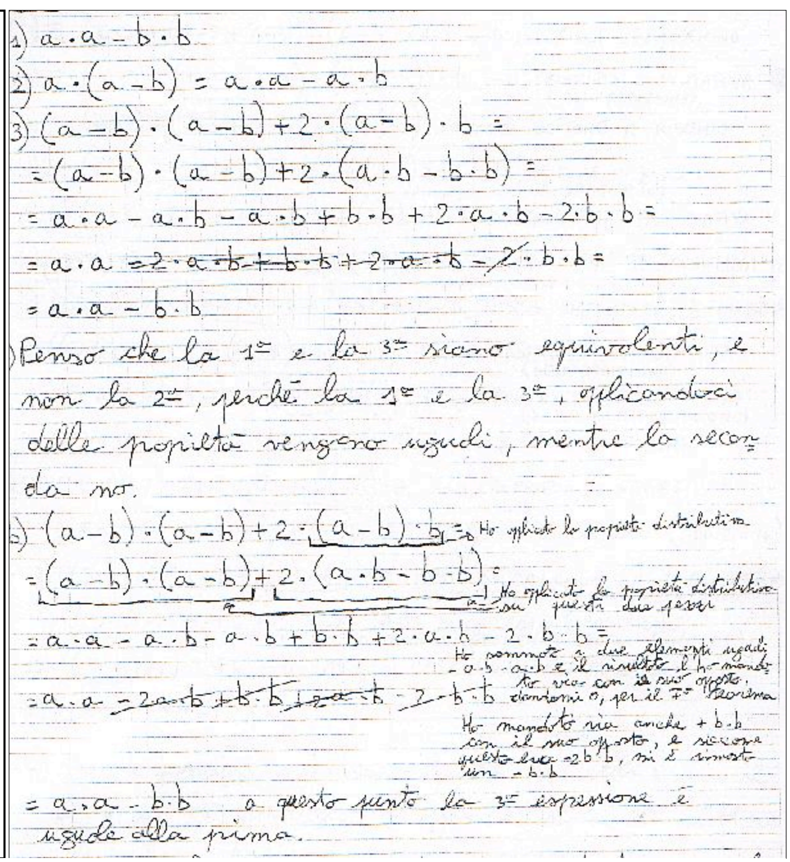

Figure 3. Exemplar of student's solution of a comparison task. A translation is reported on the left.

his conjecture he provides a proof. Such proof is based on the properties of the operations (i.e. Axioms) and a theorem.

Although not directly mentioned, the artefact in which the meanings are rooted is clearly evoked in the text produced, in particular the use of underlining the part of the expression that has to be transformed is an evident derivation of the selection mark used in the microworld, while the subsequent application of the axioms and theorems is described step by step as it is done in the microwolrd. More examples from specific teaching experiments based on the use of L'Algebrista can be found in (Mariotti & Cerulli, 2001; Cerulli, 2002; Cerulli & Mariotti, 2002; Cerulli & Mariotti, 2003).

The protocol clearly shows the distinction between the phase of conjecturing and the phase of proving. Whilst in the argument supporting the conjecture the student generically refers to the properties, in the proving process the single properties used are made explicit as well the application of a theorem (the 1st Theorem, as the student writes). The distinction between the role played by axioms and Theorems of the Theory, and theorems constitutes a crucial point in the evolution of a theoretical perspective, and it is what we are going to address in the following section discussing the semiotic potential of the designed artefact in this respect.

Development of the theory

As said, the exploitation of the semiotic potential is based on exploiting the correspondence between the microworld's activities and their counterpart in the Algebra Theory. When a new equivalence is produced by transforming an expression into another one through subsequent applications of the Buttons available, this fact can be interpreted as the fact that a new statement about the equivalence of two expressions was proved in the Algebra Theory. That means that the Theory now includes a new Theorem that can be used to prove new statements.

The act of enlarging of the theory by assuming new means of proving constitutes a delicate point in the development of a theoretical perspective: as soon as a statement is proved, its new status within the elements of the theory has to be recognized as well the fact that the new statement can be applied in the same manner as the axioms are.

Coping with this delicate issue suggested that we could design an extension of the microworld providing semiotic potentialities with respect to the mathematical meaning of *change of theoretical status for a statement*. The new environment was conceived and its management menu was named by the clearly evocative name of Meta menu.

The Meta menu

The new environment was designed to offers tools to be used to act on the set of the available Buttons, that is on the set of Buttons corresponding to the Theory itself[6].

The first tool is called **Theorem maker** (Ita. ***Il Teorematore***), it activates an environment where the user can create new Buttons. Once created, a new Button can be used to transform an expression into another. Any new Button can be selected and used, in addition to the others, to transform expressions in L'Algebrista, according to the basic substitution rule.

Figura 4. The Meta menu

Moreover, the a second tool was designed in the Meta menu, called **Palette personalizing** (Ita. ***Personalizza Palette***). It allows a new Button to be included in a separate menu (called Palette) that will appear next to the main menu of the microworld.

Both the Theorem maker and the Palette personalizing were designed after an a priori analysis of the semiotic potential that their use should unfold. In the following this analysis is presented.

The Theorem maker

Consistently with the previous analysis, the design of the tools of the Meta menu aimed to provide a counterpart to the mathematical notions of *changing of status of a statement* and *the enlarging of the theory*.

The change of status of a statement finds a reference in the functioning Theorem maker: when a new Button is to be created, the user has to enter the Theorem maker environment and re-write the statement according to specific formatting constraints.

Attaining the status of **theorem** corresponds to the statement overcoming the context in which it was produced. In other terms, the equivalence relationship has to acquire the role of a transformation scheme to be applied through the substitution rule. The move from the standard environment, where expressions are treated, to the Meta menu environment, where Buttons are created, represents the move

[6] Recall that with *theory* we mean set of axioms, definition and theorems that have a counterpart in the collection of Buttons available in the microvorld.

from interpreting a formula as an equivalence between algebraic expressions to interpreting a formula as a new transformation scheme.

Gaining further generality that allows the use of a formula according the substitution rule is a very delicate point. Actually the achievement of such generality requires that the domain of interpretation of a letter be extended from the domain of numbers to the whole domain of algebraic expressions. The need of different levels of interpretation for an algebraic expression finds a counterpart in the features and in the functioning constraints of the editor of the Theorem maker environment.

The editor offers different fonts for the editing of a new Button (see fig. 5), each font corresponds to a different level of generality to be assigned to the formula when it will be used after activating the Button. If the standard font is used, the formula will be used as it is. That means the substitution will be possible not only if the structure of the formula is recognizable, but also if it contains the same single letters. On the contrary, if the special font is used, the formula will be interpreted in its highest generality. In other terms, in the *Theorem maker* environment, the very act of assigning such generality to a formula is represented by the use of special editing Buttons: the selection of a font corresponds to the choice of the level of generality that a letter is assigned in the new Bottom/Theorem.

Figure 5. The Theorem maker environment

The Palette personalizing

The act of enlarging the theory, that is the change in the set of the "theoretical means" available for proving, has a counterpart in the Palette personalizing (see figure 7 below). In this environment the user can define a new menu that will appear next to the main menu active in any pre-defined Theory ("Teoria #"). The user can group and place new Buttons within a particular Palette to which it is possible to assign a name. The user can create different Palettes, for instance a single Palette for each Button, but no new Button can be added in any pre-defined Theory. The separation between the bar of commands corresponding to the set of Axioms and any new Palette was designed with the aim of expressing the difference between basic assumptions and new acquisitions[7].

Figure 6. A palette of theorem Buttons, as created by our pupils during a teaching experiment.

the location of a Button in the pre-defined menu of a Theory represents its status of being an Axiom, while its location in a Palette represents its status of being a theorem. Thus, such organization of the palettes of Buttons and the constraints of its functioning corresponds to a classification of the statements according to their status in an algebraic theory.

The mediation of the meta menu

The case of the first theorem

As said the Palette of a predefined Theory does not contain any rule of symbolic calculation except those related to the basic properties of the sum and the multiplication. One of the first activities proposed to the pupils was designed in order to make pupils face the need of summing monomials and, in particular, the need of cancelling two "opposite monomials", when in fact there were no corresponding buttons in L'Algebrista. The following example is drawn from data collected during the teaching experiments carried out after the realization of the prototype of the microworld, L'Algebrista (more details on this example and the teaching experiments can be found in Cerulli, 2002, p. 120 ff.)

Computations were carried out on paper, but during the collective discussion the impossibility of the realization of the corresponding transformation within L'Algebrista emerged and consequently the impossibility of accepting such computation as a proof. At this point the teacher suggested that the students enter the microworld and look for a proof. Correct chains of transformation were obtained, one of them is shown in fig. 7, where the numbers of the lines are introduced for the reader's convenience. Marco (9th grade) entred the microworld, through the command *Insert expression, as a consequence,* the expression was re-written as "b+(-1)•b==0". Then Marco applied the available Buttons until the

[7] Consistently, because of their reference to specific sets of axioms, any pre-defined Thoery cannot be modified.

last line (7) presented the identity 0==0. At this point the transformation process stops and it can be stated that the initially questioned equivalence actually holds. The following collective discussion allowed pupils to share different chains of transformations leading to this equivalence and finally agree that the new proven statement could be utilized as a step in a chain of transformation. For expressing this particular status of the new equivalence the teacher introduced the mathematical term "Theorem". Because of the importance of this Theorem the class decided to make it available as a Button of the microworld, using the Meta menu.

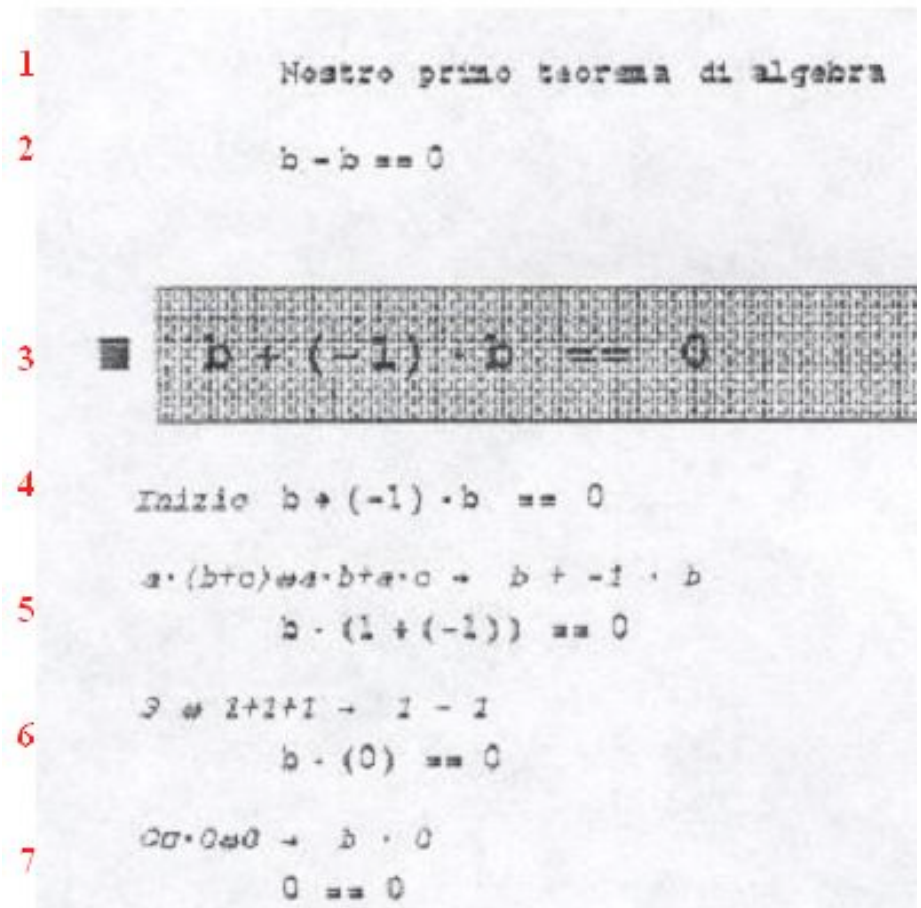

Figura 7. Marco (9th grade) the proof of the first Theorem

Hence a new Button was created and added to the set of Buttons available.

The long discussion, as well pathos created around the proof of this equivalence, led pupils to feel as a collective endevour, this is the reason why pupils often refer to it as to "our first theorem".

The following protocol (Figure 8) shows how Marta (9^{th} grade) uses this new theorem. Marta wants to prove the equivalence between the expression of line 1 and the expression of line 3. As she clearly explains, the first step is achieved by using an axiom whilst the second step is achieved by using the first theorem, that she calls "our theorem " (Ita.: "nostro teorema).

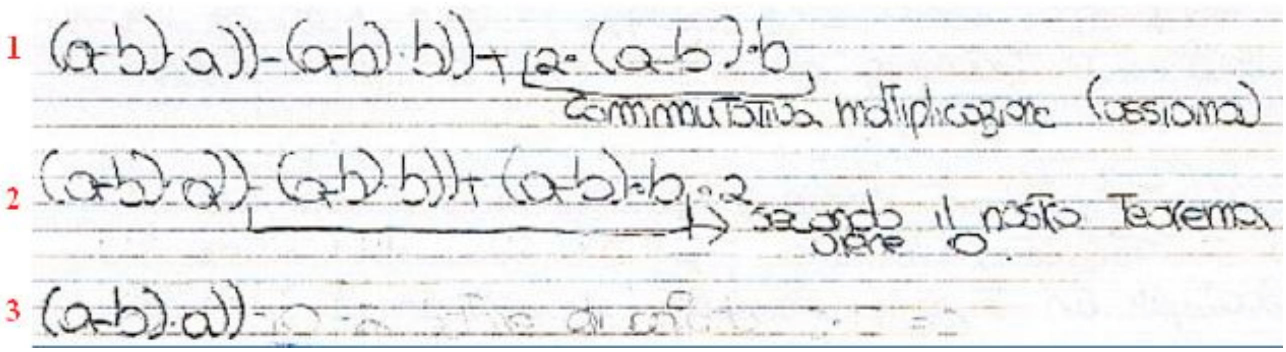

Figure 8. Marta's Proof shows the use of a proven and accepted equivalence as a means to transform an expression. Line 1, Marta writes "commutative [property] of multiplication (axiom)"; line 2, Marta writes "according to our theorem this becomes 0".

The insertion of a new button in L'Algebrista, corresponds to the enlargement of the Theory by a new theorem, that means not only that it is possible to use it to prove new equivalences, but also that it represents a shortcut in the proof. In order to give an idea of how this idea of enlarging the theory can emerge, consider the following example.

Pupils are asked to prove the equivalence between two expressions corresponding to the sum between two monomials. After an activity within L'Algebrista and the subsequent collective discussion, the theorem of the sum of monomials as well as the corresponding Button are added respectively to the Theory and to L'Algebrista. At this point pupils are asked to prove, in the paper and pencil environment, the following equivalence: $13 \bullet m + m \bullet 17 == 30 \bullet m$.

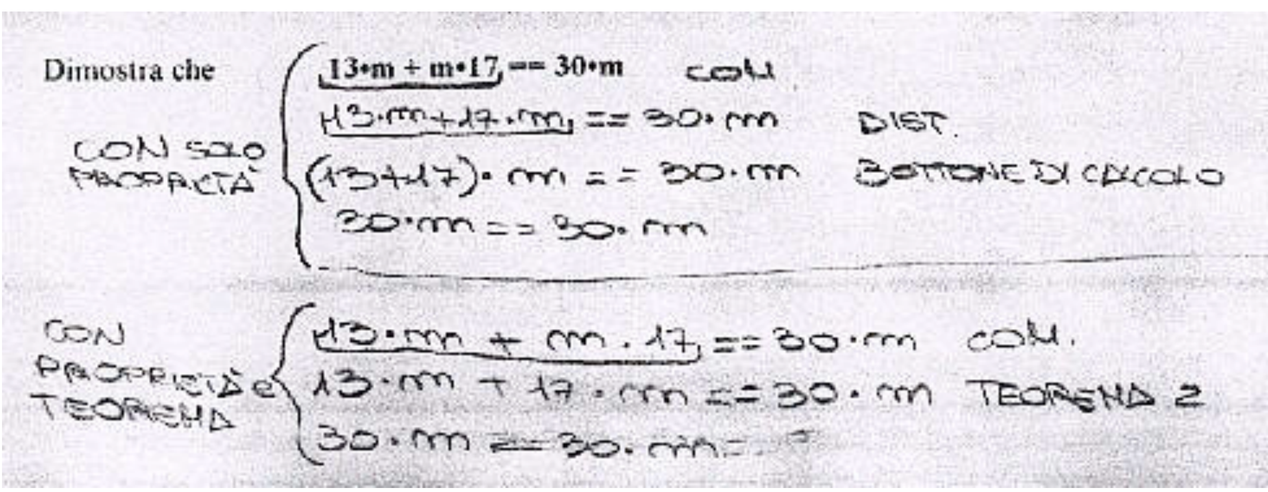

Figure 9. Elena's proofs. The pupil provides two proofs. The first only by properties (ita. "con solo proprietà"), the second is achieved through the use of properties and Theorem (ita. "con proprietà e Teorema") . At each step of the chain Elena indicates what axiom or theorem is used to transform the expression: "com" stands for commutative property; "dist" stands for distributive property; the "button of computation"(ita. "bottone di calcolo") is a command that calculate only sums of numbers.

Although the task did not explicitly ask to provide more than one solution, Elena (9^{th} grade) produces two different proofs of the given statement (fig. 9). The first proof is achieved by using axioms, as the pupil says " only with properties" (ita. "con solo proprietà"); the second proof is achieved by also using a theorem, that she calls "Teorema 2", accordingly to the social practice of the class of giving names to theorems reflecting the chronological sequence of their official introduction into the theory. At each step of the chain Elena indicates what axiom or theorem is used to transform the expression. As in the previous protocols, a clear trace of the reference to the actions in L'Algebrista is provided by the terms used and the practice of underlining the parts of the expressions to be transformed.

Conclusions

The examples discussed in this study concern two different phases of a long research study and are centred on the use of two artefacts conceived as tools of semiotic mediation (Bartolini Bussi & Mariotti, in press). The theoretical frame adopted considers an artefact in respect to its two relationships: on the one hand the relation to the meanings emerging from the use of the artefact in the solution of a task, on the other hand the relation to the mathematical meanings evoked by this use. We call this double semiotic relationship the semiotic potential of an artefact with respect to a particular mathematical knowledge, and we assume that this double link can be exploited by the teacher to achieve educational goals related to the evoked mathematical meanings

The construct of semiotic mediation, used in our study, showed its effectiveness in firmly framing our analysis to assure the integration between the epistemological, the cognitive, and the didactic perspectives.

The discussion presented in this paper shows in particular how to coordinate the epistemological and the cognitive perspective concerning the mathematical notion of proof and the specific artefact we intend to use in the classroom. Beside the rich source of suggestions coming from history, as also shown by the examples discussed by Bartolini (this volume), new technologies seem to provide powerful means to be shaped to fit specific purposes.

In the first example the artefact we considered was a particular DGE, Cabri. The analysis of the semiotic potential was carried out a posteriori. Then we discuss how the functionalities of Cabri tools in the solution of a construction task could be referred to the theoretical aspects of Geometrical Construction, and consequently offer a semiotic potential to introduce pupils to proof.

The second example addressed the issue of designing a particular artefact to become a tool of semiotic mediation with respect to the specific educational goal of developing the mathematical idea of Algebra Theory. In other terms, opposite to what had been done in the case of Cabri, the analysis of the semiotic potential is developed a priori, coordinating the identification of the key elements related to the mathematical meanings to be fostered, with identification of the key features of the microworld to be designed. As highlighted in the previous study on the Cabri environment, the possible correspondence between the use of a set of commands and the application of the axioms of a theory inspired the design of a symbolic manipulator the use of which might evoke the functioning of the Axioms of an Algebra Theory.

Hence in the design of L'Algebrista the parallel between commands and axioms was exploited, so that acting through commands in the microworld could have a direct counterpart in proving within a theory. In this sense the domain of "transformations of expressions in L'Algebrista" was conceived to provide a semantic domain for the notion of Theorem as the triplet (statement, proof, theory). Furthermore, the specific environment was designed to offer mediation tools for two crucial aspects of the functioning of a theory: the idea of theoretical status of a statement – axiom or theorem – and the idea of theory empowering. Both this elements belong to what is called the meta-theoretical level and, in spite of being so crucial for a genuine sense of theory, are quite hard to be accessed directly. On the contrary the use of specifically designed environments, such as that implemented in L'Algebrista, offers a semiotic potential that once exploited can effectively support the development of such delicate and crucial meanings.

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