

**Doing \neq Construing
and
Doing + Discussing \neq Learning:
The Importance of the Structure Of Attention**

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ABSTRACT

I have observed that efforts to increase learner activity and engagement in classrooms have not made any significant difference in the large to the learning of mathematics. I am also confident that the current obsession in mathematics education with group work and discussion is not, and cannot be, a panacea. My aim in this paper is to suggest that in order to appreciate learners' experience of mathematics it is vital to become aware first of how my own attention and that of learners is differently and variously structured at different times when focusing on mathematical ideas, problems and tasks. To this end I follow the structure of my ICME10 regular Lectureⁱ on which this paper is based, with many more examples and elaboration of observations.

I begin with some remarks about the phenomena being addressed as signalled in the title, and some brief remarks about my methods of enquiry. I then adopt a musical form of prelude, theme, variations, and reprise. I offer a series of mathematical tasks from which to draw out aspects of the theme of structures of attention, before summarising what I have noticed in myself and amongst colleagues concerning the ways in which both what people attend to and how they attend subtly and rapidly shift during what I want to call 'mathematical thinking'. I then offer some further tasks as variations on the theme. Only then do I briefly summarise some of what has been learned about attention in the past which is similar to what I have noticed, and also which is different. I conclude with some conjectures about implications for teaching and then a brief summary of what I am calling the structure of (mathematical) attention.

My examples are necessarily text-based, but what I am offering applies to any mode or mixture of modes of presentation: symbols words, diagrams, animations, evocations onto mental screens, electronic screens including interactive animations and even thought experiments. The topics are intended to illustrate the role of attention in all levels of education from primary through tertiary education, and they apply in any mode of content-learner-teacher interaction and milieu, including learner-learner

discussion, mediation whether in electronic, material or virtual form, practical work, work on exercises and mathematical exploration.

1 CONTEXTUAL REMARKS

I became aware of attention as a significant factor in learning when, over many years, I found people in workshops asking me at the end “where is your accent from?”, from which I deduced that for much of the session they had been attending not to what I was saying but to my accent. This led me to the question of what it is that learners are attending to in a lecture, seminar discussion, workshop, laboratory or other class, and even while reading or thinking. I soon discovered that it is no use asking people what they are attending to, for the very nature of attention is that we are not usually aware of it as such, because it constitutes our world, our present moment. Furthermore, the question itself diverts attention onto the fact of the question and away from whatever had been being attended to. So I began to study my own attention, and then to develop some conjectures which I wish to put forward here.

1.1 PHENOMENA BEING ADDRESSED

This is a slight elaboration of the original abstract of the lecture. Part of my method of enquiry is to articulate the phenomena being addressed so that my conjectures and suggestions are seen in an appropriate context. More details about my methods of enquiry are elaborated at the end of this section.

Doing, Construing, and Discussing

Getting learners to do tasks in mathematics lessons is not sufficient to ensure that they make mathematical sense of what they are doing: it is often possible to follow a worked example as a template without having the faintest idea what it is all about, as many learners can attest. This fact is captured in Guy Brousseau’s notion of the *didactic contract* (Brousseau 1997) and the associated *didactic tension*. The contract includes the fact that learners expect that if they do the tasks they are set by the teacher, then somehow the required learning will take place. The endemic tension is that the more clearly and explicitly the teacher specifies the behaviour being sought, the easier it is for learners to display that behaviour without actually generating it meaningfully from themselves, without actually comprehending, understanding, or learning.

Getting learners to do tasks and also to discuss what they are doing is not enough to ensure learning: it is often the case that talking obscures doing, and that both doing and talking leave no lasting impression. This is acknowledged in the framework Do-Talk-Record (Floyd *et al* 1981; see also Mason and Johnston-Wilder 2004a) which can act as a reminder when

preparing for and when conducting lessons that each aspect contributes to the others, that it is wise to give learners tasks which involve them in trying to articulate what they are doing and what they are noticing when they are doing it, and that attempting to record too quickly can obstruct rather than support progress. Collaborative and cooperative working between learners is no panacea. There is no guarantee that meaningful learning of the intended subject matter is taking place. That said, many teachers are surprised to learn how 'on-task' most learner discussion is when tasks have been designed to provoke and promote mathematical sense-making. For example, Leon Festinger (1957) formulated the principle of *cognitive dissonance* to account for the impulse to make intentional sense so that learning can take place at all. Earlier, Theodoret Cook (1914) suggested that it is largely by

investigating the deviations that 'knowledge grows from more to more' (p12).

Deviations from expectation create disturbance which can lead to new sense-making. Without disturbance, there is no need to accommodate or assimilate.

Despite the desires of curriculum designers, textbook authors and teachers, there is no way to guarantee, ensure or force learning. Trying to 'cause' learning more often leads to tears than to success: personal commitment to mathematics and to learners, and the mathematical and social *being* (in the sense of Heidegger) are important factors in the socio-cultural and cognitive milieu of learners. Marjorie Hourd (1972) makes a strong case for teaching as a caring profession involving evident care for learners. Nel Noddings (1992) goes further in advocating recasting the curriculum in terms of opportunities for teaching young people to care and to be cared for, from caring for themselves physically, emotionally and mentally, to caring for plants, animals, people and disciplined modes of enquiry such as mathematics.

Put another way, cause-and-effect is too simplistic a mechanism to account for the functioning of the human psyche. One reason for this is that human beings have the power to direct their attention, yet do not always exercise that control; they have the power to harness their energies, but do not always exercise that power. An important aspect of schooling is to provide conditions and experiences through which and in which learners discover that they can make choices, control the focus of their attention, harness their energies, and develop personal discipline.

One manifestation of these observations is that task and activity are not the same thing (Christansen and Walther 1986): tasks are what teachers set learners to do; activity is what happens when learners engage in tasks. But

the task the learners 'do' is not always the task envisioned by the author, nor always the task intended by the teacher, since people tend either to do what they can from what they construe (Brown & van Lehn 1980) or else to wait until they are told exactly what to do. What matters is what learners are attending to, and how.

What is Attention?

Both ancient and modern writers usually connect attention with will, for attention is often, though not always, the subject, perhaps even the manifestation of will. However, I do not always exercise control over my attention: a sharp or loud sound, say of someone coming into the room, is enough to cause me to look up, presumably as an evolutionary residue from hunting and gathering. The fact that attention can be directed but is also subject to habit is what makes it crucial for learning and hence for teaching.

Attention is not a thing, at least in the sense of 'some thing' to which you can point. Nor is attention detectable by machine, at least as yet. Eye movements are one way of tracking some aspects of attention, but these break down when someone shifts into gazing, because the eyes tend to appear to focus, either on an object at hand or at infinity, or even to go out of focus altogether. I see attention not just as what puts me in touch with the world of my experience, but what creates and maintains that world. The totality of what I experience *is* my attention. This is meant to include things of which I am subliminally or covertly aware, sometimes through body awareness, sometimes through social awareness, sometimes through emotional resonance, and sometimes through cognitive awareness. None of these need be conscious. For example, the psychoanalytic literature contains many attempts to describe or indicate how it is that an analyst listens through the words offered by the patient to something behind or between those words (see for example Bollas 1987). Conscious analysis is displaced by placing oneself in a sympathetic mode, in the sense of sympathetic vibrations in physics, as when humming near a grand piano makes some of the strings vibrate.

The old adage

if you want to know about water, don't ask a fish

is highly pertinent, since in a strong sense

I am where my attention is,

or even

I am my attention.

Indeed, the curious phenomenon signalled by the word 'I' may usefully be thought of as a label for the subject-object of attention.

William James (1890 p402) expressed it as '*my experience is what I agree to attend to*' (his emphasis), although this implies voluntary agreement, which may not always be the case. At each moment, as my attention shifts, I am the totality of that attention. Thus it is not very helpful to ask direct questions about attention. The next task underlines this problem of approaching attention directly.

Task: Where is Your Attention Now?

Alright then, where is your attention? Can you reconstruct where it was in its entirety just before you read this question?

Comment

The question of 'where' is of course ambiguous: I meant, *where is it directed?*, but you could also ask where it actually is, implying some ontological commitment to attention being a 'thing' that can 'be somewhere'.

Attention is of course closely related to consciousness, which has been particularly exercising philosophers, psychologists and neuro-psychologists recently.

Task: Multiple Foci

How many different things can you attend to at once?

For example, in a lecture, can you attend to the speaker's voice, use of display, clothes, and content of what they are saying?

Can you at the same time as reading this imagine yourself going and getting something to drink, and being aware of some background music or other sounds?

Comment

You can attend to things physically present and also to things not physically present; you can 'gaze' while pondering, without seeing, at least until there is rapid movement within your field of vision or some other sensory impulse which attracts your overt attention; you can concentrate very specifically on some small detail.

There are deep questions about whether you actually attend to several things at once, or rapidly cycle through a variety of foci, just as computers have been constructed to do. Although of technical interest, this does not affect the rest of the analysis.

Before going further, it is important to say that anyone who has followed the development of these ideas in my writing will find that I see attention structured in two different ways. In Mason (1991, 1998), Mason & Davis (1989), and Mason & Monterio (1996), attention was seen as having different forms, different macro structures: focused and diffuse, localised

and global, single and multiple. This has not changed. But as I have tried to understand what is going on during exchanges between teacher and learners, whether face-to-face or at a distance via text and e-screens, I have also become aware of rapid changes in both what is attended to and how that attention is structured locally (Mason 1998, 2001, 2003a, 2003b) which closely relates to aspects of mathematical thinking and to the nature, role and use of mental imagery, both life-long interests of mine. What I am concentrating on here is the micro structure of attention, but it is informed by and is most usefully seen in the context of, the macro structure of attention described previously. There is also the socio-emotional aspect of attention manifested in desire and aversion to receiving attention which may dominate classroom interactions.

1.2 METHOD OF ENQUIRY

Because my method of enquiry is perhaps somewhat unusual, at least when stated explicitly and conducted in public, a brief mention of the principal components may be helpful. The data I am offering does not consist of observations of learners in classroom, nor of teachers teaching; it does not even consist of other people's reported observations of their experience. The data I am offering consists of your own experience. In particular there will be experiences which resonate with and are brought to mind through immediate experience carrying out a task, and also experiences which challenge through the proposal of possible counter-examples or disbelief. To me this is the primary data arising from any qualitative and even from many quantitative studies: not what is presented, but what is stimulated in the reader who finds their past experience brought to mind in resonance with current experience, enabling future action to be informed (Mason 1998, 2001).

There are some similarities with the experimental phenomenology adumbrated by Don Ihde (1977), but also some differences, for whereas his focus is on method, my focus is on the structure of attention. Furthermore, he, and both Husserl and Heidegger see experience as necessarily 'experience of something', with phenomenological theorists struggling to resolve philosophical issues concerning the 'I' which is fully involved and the 'I' which emerges later during reflection. Whereas they see a 'reflexive move' as being necessary (Ihde 1977 p47-49) after 'the event', this seems to me to be a result of the desire to express that experience whether articulated in words or other forms of discourse. The lived experience is usually of immersion, so that articulation in a language like English based on subject-predicate-object constructions requires a reflexive move. However, the lived experience can also be of simultaneous immersion and separation. It is possible to develop an inner witness or monitor which

observes without involvement while the predicated 'I' is immersed, fully involved and caught up in action. This is much more than post-event introspection or reflection. Its origins are ancient. For example, it is hinted at by the image of the two birds found in the ancient text the Rg Veda (Samhita 1.164.20):

Two birds, close-yoked companions,
both clasp the self-same tree;
one eats of the sweet fruit,
the other looks on without eating.

But then phenomenology wishes to avoid metaphysical stances (Ihde 1977 p50) which I am perfectly content to take. I see initial experience of something new as being at first monadic and undifferentiated (Bennett 1956-66), and only on subsequent exposure might discernment begin to be possible and hence the object of some subject's action, thus emerging from the hermeneutic circle (Ihde 1977 p41).

Some of the rigour of my approach lies in the discipline advocated by phenomenologists such as Edmund Husserl of *bracketing*, also known as *epoché* from the Greek *εποχή* for 'cessation', taken here to refer to judgement. It is nicely captured in a phrase attributed to Ludwig Wittgenstein as 'describe, don't explain', and it plays a central role in the *discipline of noticing* (Mason 2002). By offering *accounts of* experiences (whether as a reflexive move or as reported by the inner witness) which remove as much as possible of the inferred, the interpreted and the explanatory, as well as justifications, judgements, and investments, accounts of incidents provide the basis for fruitful analysis. When readers cannot distinguish between the account of the incident itself and implicit judgements and explanations which account for the incident, they cannot test out the articulation against their own experience. They cannot therefore decide whether what is proposed has any validity.

What colleagues offer each other are distinctions which they find useful, and which they think others may find useful. Distinctions permit discerning of differences against a background of invariance or sameness, and lead to noticing in the future. Papers present evidence of various kinds as justification for the efficacy of proposed distinctions, but the test of any distinction is whether it illuminates past experience by helping to make sense of it, and-or if it informs future practice. A distinction coming to mind at some future moment may open the way to choosing to act in a some way which is fresh and non-habitual, informed by the distinction.

If a distinction does not illuminate past experience, if it does not fit with or challenge past experience, then it may be that the distinction is either not relevant to you at the present time. It may mean that your attention is

focused elsewhere, and it may mean that the tasks used as vehicles for generating experiential data need honing or replacing. It may also be that the distinctions being made are idiosyncratic and self-delusion. But then I maintain that this is the case for most educational research!

2 DATA GENERATION

Having indicated the phenomena of interest and concern, it is vital to present some data. The data I am offering consists of your past experiences brought to the surface by present experiences through engaging in the task-exercises in this section. Some but not all of these were presented in the ICME10 lecture. The format is a musical prelude, theme, variations, and reprise, where I will try to draw threads together.

2.1 PRELUDE

Since my approach both to research and to reporting research is experiential, I begin with some experiences.

Task: Selectivity

Read the message in **this type** starting with the word **Attention**.

*There **Attention** are is two partly sentences under intertwined your in control this but paragraph partly. Yet not. it In is reading possible mathematics to attention read can one be without attracted being but severely it distracted can by also the be other, blocked even out when by there unfamiliar are or familiar overly trigger complicated words formulae. like Reading mathematics. mathematical Even diagrams numbers and like symbols 2004 requires or that formulae attention like be $2\pi r^2$ differently can structured be at selected different against times as as long I as hope there to is demonstrate a in distinguishing this feature paper.*

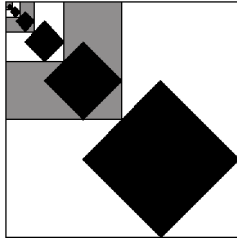
Comment

Did you try reading the italic non-bold text as well?

It is possible to select between bold ordinary and non-bold italic script with only the occasional distraction, and you can get better at it with practice. I had the sense in the lecture that most people were able to 'read' the two intertwined texts with only a few deviations such as when the word 'mathematics' appears twice in succession, once in each text, or when a number of formula appeared in the other text. Many experiments have been done to explore the dimensions of possible variation influencing selectivity of attention. This particular type of experiment is reported in Lechner (webref 2004).

Most general psychology texts treat attention *as* selectivity. The aim of this paper is to show that selectivity is only one of the dimensions of the structure of attention when it comes to mathematics.

Task: Arithmetic Composition One



What do you see when you look at this picture?
 What do you look at?
 What questions come to mind?

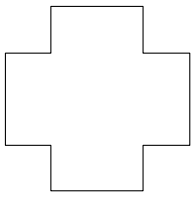
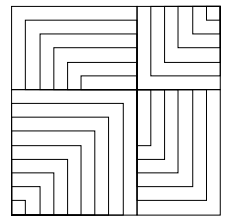
Comment

The picture is a version of Arithmetic Composition 1 (van Doesburg 1930: see Walter 2001, Pimm 2001, Sharp 2002).

What did you find yourself doing? Did you attend to the black squares first, or the grey-shaded squares? Did you wonder about the relative sizes of the black and the other squares, or how you might actually draw the figure? In the original, the alternating sequence of grey and white squares with edges parallel to the outer border each had edges $1/2$ the edge length of the previous one. There is however an implicit invitation to treat this as a dimension of possible variation, and so to generalise the original. What is the relation between the sizes of the black squares and the alternating sequence of grey and white squares? What fraction of the largest square is shaded black (assuming the construction continues indefinitely into the upper left hand corner)?

Young children are often given discrimination tasks, prompting them to learn to discern features which are important in the culture (such as arrays of dots representing numbers). Here is a discrimination task for adults which gives a flavour of what children might experience.

Task: Hidden Shape

	<p>Is the shape shown on the left hidden in the figure on the right, or not?</p>	
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Comment

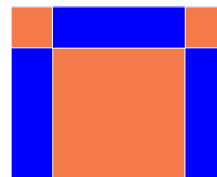
What is valuable in the task is to watch *how* you go about it. How do you control and direct your attention? What makes your attention jump from one feature or aspect to another? Did you find yourself going back to the left to refresh your image? Did you work systematically or did you expect the shape to emerge? How can you be sure it is not there, if that is your conjecture?

What is it about the shapes here which makes the task relatively difficult? In other words, what are the properties which could be used to construct even more challenging variations of this task?

Such tasks have been around for a long time. Educators have recognised the importance of promoting the development of control and direction over attention with young children, but, except for Dina van Hiele-Geldof and Pierre van Hiele, perhaps not fully appreciated it as something which develops with use, and which plays a central role in learning and doing mathematics at all ages. For example,

Task: Counting Squares and Rectangles

The light-shaded shapes are squares.
 How many squares can you count in the figure?
 There are more than 3.
 How many rectangles can you count in the figure?
 There are more than 10.



Revision puzzle 1933 (source mislaid!)

Comment

What did you do with your attention in order to locate more than 3 squares, and more than 10 rectangles? How did you convince yourself that you had them all?

Finding a way to be systematic so that you can be confident you have them all involves breaking the task into sub-problems (how many rectangles in the top 'row'?) and seeing that it will be the same as the number in the bottom 'row' and in the two rows combined. Paying attention to how you see often reveals a generality: seeing it in terms of rows enables you, with care, to answer the question for three and more rows.

It is well known that in order to make sense of geometrical figures you need to discern details, to locate relationships, and to perceive relationships as properties that other objects might share, as you did in the rectangle counting. The next task suggests that arrays of numbers can act as a similar bridge into the domain of number.

Task: Odd Table

What relationships can you detect in this array? Don't be satisfied with just one or two!

1	3	7	13	21	31	43	57	...
	5	9	15	23	33	45	...	
		11	17	25	35	47	...	
			19	27	37	49	...	

				29	39	51	...	
					41	53	...	
						55	...	

Comment

You probably noticed the triangularity, and the sequence of odd numbers. Did you notice the locations of the square numbers (including the virtual presence of the even squares in the middle of every other column)? Did you notice that the columns add up to cubes (and that there are easy and hard ways to add up the columns!)? Or that in the first row 3 divides the second, fifth and eighth terms, while in the second row, 5 divides the first and sixth terms? Did you find yourself wondering if any of the things you noticed carry on as the sequence is extended.

This table of odd numbers is also known as the Fibonacci triangle (Ollerton & Shannon 2003).

Mention of the word Fibonacci probably sent you back to the table to try to see what it has to do with the Fibonacci sequence. What does that say about your attention?

2.2 STATEMENT OF THE THEME

It is perhaps time now to present the central idea, the main theme of this composition, first through a task, then in words in the commentary, and then an articulation independent of particular tasks in the following subsection.

Task: Blinded By Symbols

How does your attention alter as you contemplate the following expression?

$$\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}_A + \frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)}_B + \frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)}_C + \frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}_D$$

Comment

You might find yourself overwhelmed by all the letters and complexity. You might then become aware of the horizontal bars, the plus signs, and the capital letters. This gives access to the fractions each of which is a product involving x in the numerator and some lower case letters in the denominator. Using the power to focus selectively, you might be drawn to, say, the first fraction, and to observe things that are the same about numerator and denominator, and also things that are different. This could lead to noticing that where in the numerator there is an x , in the denominator there is an a . To do this requires a matching process which can be holistic or can be detailed, term by term. Note that controlling visual attention in this way is not as easy for some people as for others, particularly those with strong forms of dyslexia (see later). Once the relationship between the role of x and the role of a is detected, the first fraction is seen to have a property: when x is

a , the fraction simplifies to 1. But to think in these terms already requires sensitivity to the importance of the number 1, and to experience with substitution. Awareness of 1 may trigger thoughts about 0, and the property that when x is b , c , or d , the fraction value is 0. Interest is aroused. You turn to the other fractions and see that they are similar. So this whole expression has the value A when x is a , B when x is b , C when x is c , and D when x is d . Remarkable!

In fact, this is the expression for a cubic polynomial going through four specified points (a, A) , (b, B) , (c, C) , and (d, D) . Although it is an example of what is called *Lagrange Interpolation Polynomials*, they seem to have been first published by Edward Waring in 1779, rediscovered by Leonard Euler in 1783, and published by Joseph-Louis Lagrange in 1795 as the appropriate polynomial for interpolating values of a function when given only a few sample points (here, four points) (Jeffreys & Jeffreys 1988).

Until you discern details, which involves focusing attention, you cannot do anything but be aware of the whole. This 'whole' may shift from the entire expression to one of the four terms added together, but the experience is of 'stuff', of 'a mess of symbols', as yet undifferentiated. Some learners are so taken by all the detailed symbols that they find it difficult to gaze, to 'fuzz' some details in order to get a sense of the overall structure, here, of four terms added together. It is possible to become 'fixated' by the occurrence of certain symbols.

As you focus on a detail, discerning it from what is around it (foregrounding against a background, which requires stressing some features and consequently ignoring others) you become aware of relationships. Here, there are relationships within a part such as one of the fractions (comparing numerator and denominator, seeing the distinct role of one of the lower case letters and the corresponding capital), and also between the parts (each has a capital letter and a fraction with four terms in both numerator and denominator, ...). Human beings naturally seek out similarities and recognise relationships. That is how sense is made of what our senses bring us. Note however that there is tremendous selectivity long before there is any cognitive processing: you have probably not paid attention to the font used, nor to the density of the black against the white on the page, nor the feel of the paper on your fingers.

Having analysed each of the terms (fraction with capital) and located some similarities between them, it is possible to change the way you are thinking and to ask, for example, whether all the possibilities are present. This is a step towards property-making or 'proposing properties': isolating a relationship or a similarity and then looking to see if other objects have that same relationship. For example, you might think to ask what the corresponding formula would look like for just two points, or for three or five points. To write them down you would be using incipient properties.

Articulating those properties turns them into a something which can be used elsewhere. For two points, there would be two values A and B , with corresponding a and b . If you see the fractions as constructed so as to have the value 1 when x is the repeated value in the denominator, then you can use this property to manufacture a fraction for any number of values:

Errore. for one of the two terms needed for two points, and

and $\frac{(x-b)(x-c)}{(a-b)(a-c)}A$ for one of the three terms needed for three points, with

clear generalization to other numbers of points. Thinking in terms of properties makes it easier to make deductions.

Definitions in mathematics arise when a shift is made from objects being tested to see if they satisfy properties, to deductions being made starting just from knowing that an object has those properties. Thus if there is an expression which takes the value 0 at $x = b, c$ and d , and the value 1 at $x = a$, then versions of it can be assembled to make a formula for a polynomial passing through four points.

As another example, take a grid of number facts presented to learners, though the same applies to any set of exercises or to any geometrical diagram.

Task: Sequential Generalisation

$2(2 \cdot 3 \cdot 3 + 3 + 3) + 1$ $= (2 \cdot 3 + 1) \cdot (2 \cdot 3 + 1)$	$2(2 \oplus 4 \oplus 3 + 4 + 3) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 3 + 1)$	$2(2 \oplus 5 \oplus 3 + 5 + 3) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 3 + 1)$
$2(2 \oplus 3 \oplus 4 + 3 + 4) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 4 + 1)$	$2(2 \oplus 4 \oplus 4 + 4 + 4) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 4 + 1)$	$2(2 \oplus 5 \oplus 4 + 5 + 4) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 4 + 1)$
$2(2 \oplus 3 \oplus 5 + 3 + 5) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 5 + 1)$	$2(2 \oplus 4 \oplus 5 + 4 + 5) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 5 + 1)$	$2(2 \oplus 5 \oplus 5 + 5 + 5) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 5 + 1)$
$2(2 \oplus 3 \oplus 6 + 3 + 6) + 1$ $= (2 \oplus 3 + 1) \oplus (2 \oplus 6 + 1)$	$2(2 \oplus 4 \oplus 6 + 3 + 6) + 1$ $= (2 \oplus 4 + 1) \oplus (2 \oplus 6 + 1)$	$2(2 \oplus 5 \oplus 6 + 5 + 6) + 1$ $= (2 \oplus 5 + 1) \oplus (2 \oplus 6 + 1)$

Are these statements true? Did you check?

What entry would you expect to see in the fifth column and sixth row if the patterns were to continued? Generalise.

Comment

At first it seems like a mass of numbers, signs for times and plus, and 2s. The profusion of 3s in the upper left cell may be off-putting if not confusing, making it hard to discern any detail.

Almost immediately the equals sign is seen to separate two expressions, one on each line (sub-objects discerned). It becomes possible then to focus on one expression and to calculate its value. Or what might be striking is the similarity between lines, in different cells in each column. Focusing on what is changing line

by line might suggest a way to extend beyond the grid shown, and having continued one or two more cells, awareness of structure might lead to a sense of properties or format of each line in each cell. This property-awareness not only facilitates but constitutes expressing a generality. Asking what else might change, or deciding where you would expect to find

$$2(2 \times 7 \times 9 + 7 + 9) + 1 = (2 \times 7 + 1) \times (2 \times 9 + 1)$$

is likely to expand the sense of form of a cell, to broaden the sense of relationships, to suggest a more general property, and to emerge as an expression of generality. Validating conjectures requires reasoning solely on the basis of arithmetical properties.

The structure of this task can be used in many different topics and settings. At a more sophisticated level, consider the following

$2(2ab + a + b) + 1$ $= (2a + 1)(2b + 1)$		$2(2ab + 3(a + b)) + 9$ $= (2a + 3)(2b + 3)$	
$3(3ab + a + b) + 1$ $= (3a + 1)(3b + 1)$	$3(3ab + 2(a + b)) + 4$ $= (3a + 2)(3b + 2)$		
$4(4ab + a + b) + 1$ $= (4a + 1)(4b + 1)$		$4(4ab + 3(a + b)) + 9$ $= (4a + 3)(4b + 3)$	

Again the profusion of symbols may be off-putting at first, but after a moment or two discerning of leading coefficients in the first column, perhaps, or discerning the presence of the equals signs, leads to recognition of form or structure. The presence of blank cells suggests an invitation to fill them in, and to do so requires explicit reproduction of the sensed structure, in the form of relationships as properties. Using the first column, perhaps with a strategy such as getting learners to “say what you see”, or to “watch what you do” as you copy what is invariant and change what is changing, provides more explicit access to structure. Moving to other cells is already signalling an implicit sense of property. Expressing a generalisation about the contents of *any* cell leads to seeing the structure as a starting point for making deductions. For example, to test whether a given number can be expressed in the form $mab + n(a + b)$ for integer values of a and b , instead of lots of try and improve, you multiply the given number by m , add n^2 , then factor it in all possible ways, looking for two factors which are both of the form $mx + n$. Here the reasoning has to be solely on the basis of the form of the numbers, not on any particular numbers.

Task: Magic Square Facts

Imagine someone has constructed a 3×3 magic square, but you don't know the entries.

However, you do know that, for example, the sum of any two entries in the same row is equal to the sum of the two other entries in the column corresponding to the third entry of that row. What other such relationships can you find? Try the same for 4 x 4 magic squares and larger.

Comment

Apart from the work needed to interpret the claim, the main point is that you can reason on the basis of the one property of magic squares: that the row, column and main diagonal sums are all equal. From this alone you can deduce all sorts of other sums which must be the same. There is no need to have an example to hand on which to 'check' results, which might in fact get in the way of the reasoning. Everything follows solely from known properties. The idea for this came from Johnny Baker (private communication) doing the same with magic hexagons.

This task provides quintessential experience of reasoning on the basis of properties alone, without recourse to particular examples for inspiration or checking. Indeed, particular instances are more likely to intrude and confuse, to direct attention away from the reasoning rather than supporting and inspiring it. In most mathematical topics however, learners have recourse to particulars and to several properties, so that it is difficult to isolate the defining properties and to use them alone as the basis for reasoning. Perhaps this is one of the reasons why teaching proof is so difficult in mathematics. There are evident difficulties for some learners simply with reasoning. More problematic still is the shift away from accumulating knowledge of various properties a specific object has or might have, to isolating properties which define and hence characterise those objects. To reason on the basis of specified or defining properties alone requires backgrounding all other information, facts and properties of which you are aware so that reasoning can proceed 'axiomatically'.

2.3 THEME ELABORATED

The basic idea is that attention shifts rapidly between holistic encompassing (gazing), discerning distinctions (stressing and ignoring, foregrounding and backgrounding), recognising relationships amongst discerned features, perceiving properties that objects in general may possess, and reasoning based on deducing from definitions (selected properties) and axioms. Whether attention is the subjective experience of physiological functioning, as Théodule Ribot (1890) would have it, or the engine for physiological response to environment, as William James (1890) suggests, reflection on the experience of working on tasks such as those presented earlier suggest to me quite distinctive if subtly different forms of attention:

 Holding Wholes (gazing)

 Discerning Details (features & attributes)

Recognising Relationships (part-part, part-whole)

Perceiving Properties (leading to generalisation)

Deducing from Definitions (axioms and definitions stated independently of particular objects)

Shifts between these are rapid, often subtle, but vital in order to engage in mathematical thinking. While gazing, some sudden movement, perhaps even apparent motion produced from circadian eye movement can suddenly switch to awareness of details amongst a mass of other, undiscerned detail. As details are detected and discriminated, the mind automatically looks for relationships: differences and samenesses. To do this requires something being relatively invariant as a background against which to detect change. Recognising relationships tends to focus on particulars, whereas perceiving properties is a move to the more general, to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for reasoning.

If learner attention and teacher attention are significantly differently structured then confusion is a most likely outcome. More particularly,

If some learners are attending holistically when the teacher is discerning specific details;

if some learners are discerning details when the teacher is talking about relationships amongst details;

if some learners are recognising relationships amongst discerned elements when the teacher is talking about properties of objects in general;

or if some learners are thinking about properties when the teacher is deducing from properties;

then here is likely to be a mismatch, a failure of communication.

Colette Laborde (ref) observed in the context of dynamic geometry that

It is a matter of getting pupils to understand that in geometry they have to rely on what they see to get ideas on how to solve a problem, but they do not have the right to use that when it is a matter of reasoning 'rigorously'; they must then restrict themselves to the theoretical plane.

This is the difficult move, the move which is uncommon outside of mathematics, the move which marks out the natural mathematician from others who have to struggle to make it.

Reprise Da Capo

Distinctions such as those being proposed between different forms or structures of attention are only useful if they help make sense of past experience, and-or inform future practice (Mason 2002). The following notes suggest some ways in which the structure of attention might have been experienced in the earlier tasks. The remainder of the paper builds up to the final section proposing ways in which awareness of the structure of attention can inform future practice.

Selectivity Task: your perceptual mechanisms were able to discern the bold type and to foreground it, pushing the italic text into the background, with more or less success. Foregrounding and backgrounding is a form of selectivity, which is what psychologists see as the main function of attention.

Arithmetic Composition Task: perceptual sensitivities immediately discern and register the black shapes and the alternately shaded backgrounds; geometrical sensitivities were evoked for recognising stilted squares (though many young learners want to distinguish 'diamonds' from 'squares'), and to ask yourself how the diagram could be constructed or what the relative sizes of the squares might be. At the same time you were probably aware of the (implied) infinite sequence, without necessarily actually focusing attention on it. This form of attention is sometimes referred to as *gazing*. As well as discerning shapes through perceiving familiar properties to hold, you may have looked for relationships between the sizes of successive squares, and to do this you will have reasoned using properties of squares and of squares tilted at an angle of 45° .

Hidden Shape Task: discerning is not enough, because there are relationships which characterise the shape being sought, and these have to be maintained and imposed on a background of similar shapes. Asking yourself what it is that makes the task difficult signals a move to properties, and to construct your own would require working from those properties alone to construct some other figures. In the process you would probably find yourself re-specifying and refining those properties.

Counting Squares Task: discerning squares requires maintaining the relationships which characterise squares as properties, but at the same time or perhaps rapidly interpolated, attention needs to hold onto some shapes already counted while looking for ones not yet counted. Finding a systematic way to locate all the squares essentially involves reasoning on the basis of properties of squares in order not to double count. The same happens in a slightly more complex fashion for counting rectangles.

Odd Table Task: having discerned some simple facts (triangularity, odd numbers) it is natural to assume that these relationships will continue as

the array is extended. In other words, you naturally turn relationships into properties. To verify other relationships which are conjectured to continue as properties requires reasoning on the basis of specified properties (odd numbers and triangularity) alone.

The conjecture being advanced is that these various structures in the form of attention are not restricted to sophisticated adults, but rather form the experience of the youngest of children. In mathematics lessons, apparent failure to communicate or to 'be heard', apparent failure to comprehend what is being said and done, may be due to momentary differences in the structure of attention rather than to 'abilities'. Since learners' self-image and self-confidence are often delicate, mismatches in what is being attended to and how may have serious consequences. Cultural expectation and peer pressure are additional forces, and together these can open a chasm between teacher attention and learner attention which may take considerable effort to re-bridge. There are evident connections with van Hiele levels, with the Pire-Kieren onion model of understanding, and with the SOLO taxonomy which are elaborated in section 3.

2.4 VARIATIONS

In this section I offer a number of further examples in which the fluid nature of the structure of attention can be experienced. The variety is intended to demonstrate that sensitivity to the structure of attention is relevant to all aspects of teaching mathematics, at every level.

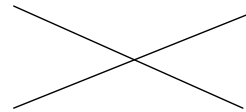
A Classroom Example

My wife, Anne Watson, drew my attention to a phenomenon observed in many classrooms concerning 'vertically opposite angles' (when two straight lines meet at a point, there are two pairs of equal angles, called 'vertically opposite'). When the teacher says 'vertically opposite angles', learners are often able to complete the sentence with 'are equal'. But it turns out that sometimes this is only a mantram that has been learned. When given a diagram such as the ones displayed here,

some learners see only the configuration of two lines crossing;

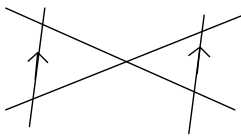
some learners discern at most one pair of angles (usually the ones 'at the top and the bottom' of the intersection), but do not see them as being related (as equal);

some learners discern one pair of opposite angles and see them as equal, perhaps through seeing them as generated by the rotation of one line about the point of intersection;

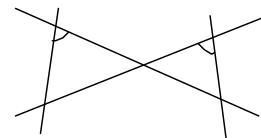


some learners are aware of the property of 'being vertically opposite' as synonymous with 'being equal', but don't use these equivalences to carry out reasoning;

some learners are able to reason about the equality of angles because of being vertically opposite, in situations such as the following.



Show that the two triangles are similar (have equal angles).



A good way to prompt learners to reveal the extent of their awareness and the structure of their attention is to ask them to construct diagrams which display a pair of vertically opposite angles, or which use a pair of vertically opposite angles to reason about the equality of a pair of apparently unrelated angles. Asking for another example, and then another example often invokes creativity and variations of which learners are becoming aware (Watson & Mason 2002, 2004).

One difficulty may be that the words 'vertically opposite' bring meaning which is not actually wanted mathematically. Some learners may even be reinforced in their view that mathematics has no meaning and consists simply of techniques for getting answers. This illustrates perfectly the effects of teacher and learner attention being differently structured.

Pedagogic Strategies

There are some useful pedagogic strategies which could enhance your experience of the remaining tasks if you use them intentionally.

Say What You See; Watch What You Do

One of the most powerful strategies for directing learner attention is to develop in them the habit of 'saying what they see', that is, of directing their attention around the diagram, figure, expression, and even around a set of exercises and describing some feature, some detail that they discern. As different people offer different details, each participant comes to see more fully, as well as learning to pause before diving in and 'doing' the first calculation that comes to mind. 'Say What You See' is a useful strategy for getting groups of learners to improve their 'thinking time' before diving into computations. It usually helps to get each person to say one thing that they can see, without trying to be clever, for often others have not been attending to things that some people see right away.

When looking for patterns and regularities, it often helps to produce a particular case for yourself, not just by copying and calculating, but by watching how your body sets about making the copy, because often your

body detects regularity and invariance which your cognition has not articulated. There was an opportunity to use this implicitly if not explicitly in the *counting rectangles* task, where by attending to how you count can reveal a general way both to count systematically and to be confident that you have indeed counted them all.

Same & Different

Since discerning detail involves making distinctions, and since seeking relationships involves becoming aware of both what is changing and what remains (relatively) invariant in the midst of that change, a useful strategy is to develop in learners the habit of asking themselves ‘what is the same and what different?’ about two or more objects. You might have done this implicitly if not explicitly in most of the tasks.

Invariance in the Midst of Change

Closely related to ‘same and different’ is looking for what remains invariant, or at least relatively invariant’ as other features are changed. For example, in the *arithmetic composition* task, the orientation of the squares is invariant as the sequence gets smaller and smaller, or at least relatively invariant as it rotates through 45° each time. The *sequential generalisation* task is principally about looking not only for what is the same and what different about nearby entries, but seeing the rows and columns as indicating a systematic change, and the generalisation is about denoting what can change by letters and expressing an invariant relationship amongst entries.

What are Learners Attending to?

I hope that these experiences have raised the question not only of what learners are actually attending to in lessons, but how they are attending to it. That is, how can you detect when a learner is attending holistically, discerning details, recognising relationships, perceiving properties, or ready to reason on the basis of properties alone? Since these types of attention are often fleeting, it is not so much a matter of capturing one, but developing strategies for directing or focusing attention in a way which is pertinent to the learning, while at the same time being sensitive to different learners’ needs to dwell in a particular form of attention before being rushed on to another.

Task: What Whole?



What fraction of each individual triangle is shaded? What fraction of each pair of triangles is shaded? What fraction of all three triangles is shaded? What do you have to attend to in order to see the diagram as depicting $1/2 + 1/3 + 1/6$?

Comment

In order to become aware of a part-part or part-whole relationship it is necessary to have a sense of a whole, with related parts, and then to shift attention to the relationship rather than to the component parts.

Here, the usual question asked is ‘what fraction of a whole triangle is made up from the three shaded pieces’.

Task: Rationality

On a calculator, $53/83$ is given as 0.63855421687. Is the quotient rational?

Comment

Of course it is rational, as it is the quotient of two whole numbers. But this is only evident if you see $53/83$ as a number rather than as an as-yet-unperformed operation, and if you are not dwelling in the property-as-definition of rational number, but rather focusing on the decimal ‘number’

Learners (including novice elementary school teachers) do not always find their attention held by the fraction and by the property which characterises or defines rational numbers. Rather, their attention is sometimes held by the decimal, supported perhaps by seeing $53/83$ as an operation not a number so the only ‘number’ presented is the decimal approximation (Zazkis & Sirotic 2004). Rina Zazkis and Natasa Sirotic also point out that to someone who sees the fraction as a number, the notation is transparent concerning rationality, whereas the decimal notation is opaque. So the structure and focus of your attention can assist or block access to resolution of a question (see also Zazkis & Gadowsky 2001).

Task: Re-cognising

What do you make of this table?	2	3	4	5	6	7	8	9	10
Could you continue the table to the right, downwards, or to the left or upwards?	3	5	7	9	11	13	15	17	19
	4	7	10	13	16	19	22	25	28
What is it about the structure of the table which means that the properties you perceive actually continue indefinitely?	5	9	13	17	21	25	29	33	37
	6	11	16	21	26	31	36	41	46
	7	13	19	25	31	37	43	49	55
	8	15	22	29	36	43	50	57	64
	9	17	25	33	41	49	57	65	71
	10	19	28	37	46	55	64	73	82

Comment

How does your attention change in order to spot, or now to check for, symmetry?

Did you work successively, comparing adjacent cell entries? Did you think to, or consider a global relationship as in the multiplication table previously?

To extend the table you need to detect and formulate some relationship, and then to act as if that relationship is continued (remains invariant), in other words, to perceive that relationship as a property of cells that are present and of cells as-yet un-manifested. Are you sure that if you continue the table downward using the properties of columns you perceive, the rows will continue to have the appropriate properties?

Task: Collecting Like Terms

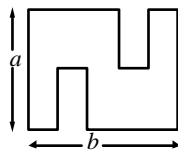
What does a learner have to do with their attention in order to collect like terms in the following expression:

$$3xyxyz + 2xzy^2x + zyxyx$$

Comment

You need to ignore the coefficients and discern that each of the terms has the same number of x s, same number of y s and same number of z s, and no other letters, so all three are in fact 'like terms'; then you have to ignore the letters and stress the coefficients, recognising a coefficient of 1 in the third term. Finally you add up the coefficients and then write the letters in a succinct form as $6x^2y^2z$.

Collecting like terms only becomes an action (as distinct from a task) when learners develop an implicit 'sense' of how terms can be rearranged (putting letters in alphabetic order), and attention shifts spontaneously and naturally between letter sequences and coefficients. This is part of what Eddie Gray & David Tall (1994) refer to as the development of a procept: when the process becomes internalised as an object. Anna Sfard (1991, 1994) uses the term *reification* and Hans Freudenthal (1978) used *condensation* for very similar awarenesses, and other writers use yet other language.

Task: Perimeters of Rectilinear Shapes

The perimeter of the shape shown depends only on the overall outside measurements of the shape. Why? What is it about this shape that makes it have that property? How complicated an example can you draw with this same property? Bring to articulation what it is that makes the perimeter depend only on the two outside measurements.

Characterise rectilinear shapes which are bounded by an a by b rectangle and have a perimeter of, say, $2a + 4b$ by describing how to draw them.

Comment

Notice the movement from seeking relationships between edges of the particular figure and discovering that the perimeter depends only on the outside measurement, to seeing 'having perimeter depend only on outside measurement' as a property which many different figures can have. It requires a small but subtle shift in thinking.

Finding the area or perimeter of a compound shape is a calculation, a 'doing', which involves discerning details and recognising relationships. Switching to considering shapes with a given perimeter, a given area, or a given perimeter and area, involves a shift to thinking about properties. Reasoning begins when properties are taken as characterising, as 'all that is known' about the objects under consideration.

Task: Factorings

$x^2 + 5x + 6 = (x + 3)(x + 2)$	$x^2 + 5x - 6 = (x + 6)(x - 1)$
$x^2 + 13x + 30 = (x + 10)(x + 3)$	$x^2 + 13x - 30 = (x + 15)(x - 2)$
$x^2 + 25x + 84 = (x + 21)(x + 4)$	$x^2 + 25x - 84 = (x + 28)(x - 3)$
$x^2 + 41x + 180 = (x + 36)(x + 5)$	$x^2 + 41x - 180 = (x + 45)(x - 4)$

Say What You See in the table of factorings: what is the same, what is different, what is changing, what is invariant, whether in rows or in columns.

Comment

Did you catch yourself discerning details and then relating these to other details elsewhere? Did you spend any time dwelling on the invariance of the x^2 , the x s, the +, =, and - signs, and the presence of brackets? Even if you did not spend any time on them, they present an invariance against which changes can be detected.

Did you find yourself naturally wondering whether there were any more examples? Examples of what? Even to frame such a question already presupposes properties which 'other examples' would have to share.

In order to make sense, it is necessary to probe beneath the surface of 'quadratic expressions' and 'factors', by looking at the coefficients and relating them to each other. Were you aware if only briefly of thinking in terms of 'properties of all the factorings'? Did you notice yourself 'going with the grain' (Watson 2000) with sequences such as 2, 3, 4, 5, and -1, -1, -3, -4, ... and naturally extending these? Did you find that it took a little longer to find a relationship amongst the other sequences of coefficients?

The next task has similarities with the Lagrange polynomials in the *blinded by symbols* task.

Task: Some Sums

What do you make of $\sum_{k=\sum_{j=1}^{n-1} j}^{\left(\sum_{j=1}^n j\right)-1} (2k+1) = n^3?$

Comment

At first the three summation signs and all the little symbols may be off-putting, but once you set yourself to focus on one feature at a time, and perhaps using a particular case of n , you may recognise something. Indeed the $(2k + 1)$ may already suggest a sum of odd numbers, which may in turn trigger memory of a recent experience from a task earlier in these notes.

Seeking relationships is not always an active process. Sometimes it simply 'happens': something about the current phenomenon rings bells about, resonates with, past experience which then comes to mind. In this case, the symbols are expressing the conjecture about the sums of consecutive odd numbers in the task *Odd Table*. So there is more to discerning details than being immersed in the particular. Making links between different mathematical topics is something teachers would like learners to do much more than they do; making links explicit for learners may or may not support them in making links for themselves.

2.5 THEME (REPRISE)

The upshot of these examples is an illustration of how human attention shifts rapidly between different ways of attending and different foci of attention. Sometimes there are several of these going on either simultaneously, or in swift succession. Attention is the basis for ontological acts, that is for the creation of entities, of objects. There is

awareness of or focus on wholeness, a holistic encompassing, such as *gazing* (monadic attention): the identification of an object, which may be a part of some other object;

awareness of or focus on discerning details comprising that object, and so literally creating objects and sub-objects;

awareness of relationships or similarities between features comprising sub-objects at any level of detail;

awareness of or focus on properties as attributes that (sub-)objects might satisfy or possess;

awareness of or focus on reasoning solely on the basis of properties which requires specified properties to be seen as definitions or as axioms, not just as attributes.

The suggestion is that shifts between these states, that is, shifts in the structure of attention, are going on all the time, though the property-

making to some extent and the defining and axiomatising particularly are perhaps predominately mathematical activities. By being aware of these shifts in your own attention, you can be more alert to the possibility that learners are not making the same shifts with you, and that there are things you can do to try to prompt those shifts. For example, if you notice that you are thinking in terms of properties, you can pause and be explicit that that is what you are doing, or you can pause and give learners time to articulate the relationships they recognise, then offer other objects to see if these have the same relationships, before prompting a move to articulating the relationship as a property.

3 LEARNING FROM THE PAST

There are evident close similarities with van Hiele levels in particular but also notable contrasts with approaches taken to the nature of attention in the psychological literature. I begin this section with similarities before moving on to differences.

3.1 SIMILAR APPROACHES

Within mathematics education literature there is very little on attention as such. One notable exception is a paper by Kaye Owens & Ken Clements (1998). They found that obstacles encountered in problem solving could often be accounted for by selective focus of learners' attention on particular details. They arrive at the core role of attention as the basis for selectivity. Attention is seen as co-emergent with responsiveness to stimuli from the teacher and from fellow learners:

[attention] might be thought of as the engine room driving ... responsiveness, and as something arising from responsiveness (p210)

They conclude that

Expectations and intention are part of the inner visual system, and alter internal and external feedback. Social interactions frequently influence expectations and intentions. Attention is influenced by perceptions, expectations, intentions and internal feedback. Attention, then, influences changes that occur in memory and these influence responsiveness in the problem situation and the attaining of a solution. [p212]

They also found that

Changes in attention often heralded changes in understanding (p213),

where I would say that attention changes constitute changes in understanding. If the changes are temporary, so is the understanding; if

the changes inform or influence future behaviour then change in understanding is more robust.

Walter Whitely (private communication) has worked long and hard to convince people that geometry is not about proof but about working with mental imagery, and that geometry most naturally begins with topological awareness and three dimensional space, not Euclidean geometry and two dimensional space. He points out that the van Hiele levels as originally conceived were driven by a 2D Euclidean view rather than 3D topological experience. He also notes that 'students do not see what we see - unless they apprentice with ... images and ... reasoning', which is consonant with and explained by distinguishing different forms or structures of attention.

A compatible but non-mathematical approach to the subject of attention can be found in Pierre Lacout (1969) who writes that 'contemplative silence is a special form of attention' (p7), referring to holistic gazing, rather than analytic discernment, as a form of prayer.

Attention and Understanding

Although attention does not feature prominently in mathematics education research currently, there is a great deal of writing about what constitutes understanding. From my perspective understanding, whatever it means, has to do with how attention is structured. Indeed, one of my long-standing conjectures is that each mathematical term signals that someone experienced a shift in the way they perceived, in what they noticed, in how they attended, and that in order to integrate that term and its use into your own thinking it is necessary to experience a corresponding shift in your own attention. I say 'corresponding' because differences in experience and culture may mean that the actual shift experienced is different. In this section I consider some evident similarities between the structures of attention I have identified and some other authors.

Anyone familiar with van Hiele levels (van Hiele-Geldof 1957, Burger & Shaunessy 1986) will be aware of close similarities with the structures of attention being proposed. Here is one version, to which I have appended a version cast in terms of what reasoning might look like:

- Level 1: Visualization (reasoning from direct perception)
- Level 2: Analysis (reasoning from component parts and attributes)
- Level 3: Abstraction (reasoning from necessary conditions)
- Level 4: Informal Deduction (reasoning by relating properties)
- Level 5: Formal Deduction (reasoning from axioms)

Pierre van Hiele generalised these beyond geometry (van Hiele 1986) but in the process made them even more abstract and for me harder to connect to moment-by-moment experience, which is where I think it is important to

focus in order to influence and improve learners' experiences of being taught mathematics. Although I was familiar with the van Hiele levels, I had found them difficult to work with, partly because I am unconvinced by attempts to specify distinct levels. I came to the structures of attention through an entirely different route based on Eastern philosophy (Bennett 1956-1966). As the structures reached refined articulation, I was influenced by the van Hiele language.

The difference between the structures identified here and the van Hiele levels lies precisely in the notion of levels. Rather than seeing these structures as levels or even as hierarchical qualities in the way researchers have developed the van Hiele ideas to date, I am proposing the radical stance that these so-called levels are actually descriptions of the way that people attend all the time, often with rapid shifts from one to another.

Susan Pirie & Tom Kieren developed an onion-skin model of understanding (Pirie & Kieren 1989, Kieren & Pirie 1992, Kieren 1994) which they have used to capture the way in which the manifested form of understanding (I would say the focus of attention), shifts back and forth between layers, as depicted below.

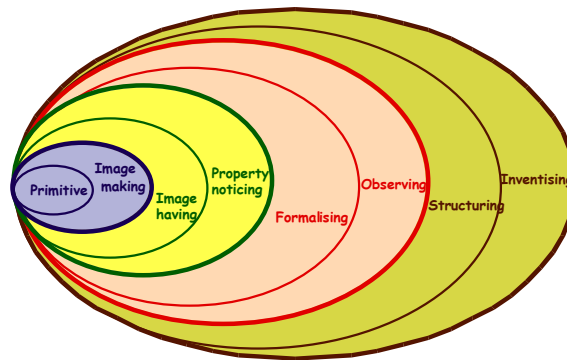


Image-making depends upon discerning and discriminating, and image-having contributes to seeking relationships. The slip from relationship between particulars to relationship as property is perhaps not always as immediate in mathematics classrooms as it is in ordinary life, perhaps because teachers tend to skip over it as being obvious and automatic. Furthermore, property-noticing and property-making is an essential step on the way to formalising, which requires separating a property as simply one attribute of an object, to property as the basis for further reasoning.

The Solo Taxonomy was devised and elaborated by John Biggs & Kevin Collis (1982) in an attempt to refine assessment so as to reveal different types of understanding, again seen as levels, across all subjects. A very succinct summary of their broad levels is as follows, cast in terms of observed behaviours as learners try to justify and reason:

Pre-structural (denial, bound to specifics)

Uni-structural (focus on & generalise one factor only; no need for consistency)

Multi-structural (several factors; different conclusions)

Relational (integrates several factors; generalisation within experience)

Extended Abstract (deductions, tries to resolve contradictions, accepts leaving as conjectures)

These behaviours could perhaps be re-interpreted as manifestations of how learners' attention is structured. Focus can be solely on the particular (pre-structural) as an isolated whole, although within that there is likely to be discernment of details and recognition of relationships confined to the particular. Focus might be placed sequentially on one aspect at a time. One feature, relationship or property is stressed, without trying to achieve consistency as attention shifts to another feature. Focus can include several aspects of features but isolated from each other, leading to different conclusions which are held side-by-side, a behaviour that most teachers will recognise in their learners. Focus can integrate and relate the influence of several features, which has much in common with recognising relationships. The fact that the SOLO taxonomy of reasoning behaviour both cuts across and mirrors the structures of attention adds force, in my view, to the importance of recognising the fleeting nature of subtle shifts in attention which contrast with theory of levels of thinking through which learners are traditionally expected to progress.

Willi Dörfler (2002), in discussing the complex matter of how mathematical objects come into being, builds on but also critiques Charles Saunders Peirce's view that mathematical objects receive their object-like qualities through becoming diagrammatic (in a technical sense). For Peirce, a diagram is an inscription together with a collection of operations which can be performed upon it. The manipulative rules of algebra provide a paradigmatic example. Peirce goes on to say that

deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (Peirce 1935-1966 3.363)

Thus working with Peircean diagrams involves aspects of what I am calling structures of attention.

Mathematical Ontology Through Discernment

Considerable attention has been given in the mathematical education literature to how it is that mathematical ‘objects’ come into being (Mason 1987, Sfard 1994, Gray & Tall 1994, Dörfler 2004). The conjecture arising from focusing on attention is that the creation-identification of ‘objects’ arises from discernment, from foregrounding and hence backgrounding, from stressing and so also ignoring as Gattegno (1987) describes. This is an act of attention. A fruitful area of enquiry could be to pursue William James’ questions about the relation between attention and intention (see for example Mason 2001).

Ference Marton, stresses that learning arises from discernment arising from being exposed over a short period of time to sufficient variation within which some systematicity is recognised (Johannson & Marton 1985, Marton & Booth 1997, Marton & Trigwell 2000, Marton, Runnesson & Tsui 2004). For this there is ancient pedigree, for example Anaxagoras writing in the 5th century BCE proposed that ‘all things were together; and mind separated them and put them in order’ (Fairbanks 189 p235), and in Aristotle (ref).

I choose to see becoming aware of possible variation as a shift in the structure of attention. Discernment may then unleash natural powers of recognising relationships, perceiving properties and reasoning on the basis of properties, but often it seems that learners do not naturally make use of their powers in this way, so learning also involves discerning the types of relationships worth attending to for mathematical purposes, the types of properties worth perceiving, and most particularly, making the relevant if subtle shifts of attention which these discernments enable so as to develop the habit of mind (Goldenberg 1996) to reason from (on the basis of) announced properties.

Basing learning fundamentally on discernment can be found in earlier philosophers, such as John Locke, whom William James quotes (1890 p483-484) as saying:

Another faculty we may take notice of in our minds is that of discerning and distinguishing between the several ideas it has. ... unless the mind had a distinct perception of different objects and their qualities, it would be capable of very little knowledge. ...

Judgement ...lies ... in separating carefully one from another ideas wherein can be found the least difference, thereby to avoid being misled by similitude and by affinity to take one thing for another.

[An Essay on Human Understanding II xi 1, 2]

Seeing discernment as one particular structure of attention amongst others broadens this view of learning to encompass other ways of attending such as recognising relationships (which stresses similarity amongst differences)

and perceiving properties (through isolating or stressing certain features and so ignoring or backgrounding others temporarily). Learning is then the opening up of possibilities to attend in different ways in different contexts, or put succinctly, to notice (Mason 2002).

3.2 DIFFERENT APPROACHES

In his comprehensive, and to my mind still unsurpassed treatise on *The Principles of Psychology*, William James (1890) devotes an entire chapter to the subject of attention. However, the difficulty of defining *attention*, and the shift to a behaviourist perspective which eschewed internal workings of the brain or psyche, meant that psychologists moved away from attention as a subject for enquiry for a time. It began to return in the 1960s as behaviour-dominated researchers studying animals began to realise that there was more to animal and human interaction than merely the triggering of specific behaviour patterns. A moment's recollection of what it is like when you are talking or listening to someone who never looks at you suggests that the admonition 'pay attention!' signals a truth about human and animal psyche: attention is something which people possess and which they can choose to direct, and that most people like to receive attention from others, at least in some circumstances. There has of course been considerable interest in attention-seeking behaviour by children and adolescents, which contributes a socio-psychological dimension to the structure of attention. More pertinently, it is possible not only to be aware of other people's attention, but even to become more sensitive to its nature and form.

Attention as Physiological

Writing in Germany just before William James, Théodule Ribot (1890) devoted an entire book to the mechanism of attention, claiming that it was overlooked in the rush to study forms of attention. He proposed that attention is 'the subjective aspect of physical manifestation expressing it' (p. xx). After distinguishing spontaneous and voluntary attention he concludes that

Attention *acts* only upon muscles and *through muscles* (p. 51, his translator's emphasis)

Ribot remains ambivalent about whether attention causes physiological change, is caused by it, or even both. I cannot myself reconcile Ribot's claims with my own experience of being able to direct my attention inward as well as outward, virtually as well as actually. Although stressing one sense over another (for example hearing or taste) as a mechanism for attending to something can be seen as physiological, it doesn't really help account for how this can be done simultaneously but contradictorily in

both the virtual and the actual. For example, I can recall the smell of a rose at the same time as smelling a lemon, and I can recall a tone of voice even though I cannot actually reconstruct or reproduce that tone precisely.

Maurice Merleau-Ponty (1945 trans. 1962 p30) gives a challenging example of the structure of attention by pointing out how psychologists at first were unable to conceive that young children might not 'see' colours, and that 'seeing colours' requires a shift in the 'structure of consciousness'. Partly this is physiological, but partly a matter of attention. He also suggests that 'once the idea of an equation has been acquired, equal arithmetical quantities appear as varieties of the same equation' (p30). He sees attention as

The active constitution of a new object which makes explicit and articulate what was until then presented as no more than an indeterminate horizon (Merleau-Ponty 1945 p30).

This sense captures the movement from inattention to monadic attention as object creation, but does not fit very comfortably with other forms of attention during mathematical tasks identified in previous sections.

Attention and Construction

For William James, attention is what makes it possible to perceive, conceive, distinguish and remember, in short, the basis of all our psychological functioning (James 1890 p424). As might be expected, he deals with a number of important issues concerning attention in general. For example, he argues on the basis of Helmholtz's experiments that attention is not simply what the eyes are looking at, or indeed any other particular source of sense impressions (p438). He links attention to anticipative imagination (p439-411) as a prerequisite for discerning anything at all thus presenting a version of Michel de Montaigne's observation (Montaigne 1588 p600) that

Human eyes can only perceive things in accordance with such
Forms as they [already] know,

which Norwood Hanson (1958 p19) rephrased in his influential book on methods of enquiry as

There is a sense ... in which seeing is a 'theory-laden' undertaking .

In keeping with the subject of attention, this could be rephrased yet again as

you discern what you are predisposed, attuned or prepared to
discern.

James (p456-7) develops this theme of discernment, or discrimination, to make use of what he calls Helmholtz's law, that

we leave all impressions unnoticed which are valueless to us as signs by which to discriminate things (p456).

James then goes on to discuss pedagogic implications such as that it is useful for teachers to work with learners to strengthen and attract their attention in order to improve motivation, since people engage with what catches their attention (James 1890 p 446). To do this requires being aware of what in learners' previous experience can be used as a basis of previous attention-experience, what John Dewey referred to as *psychologising the subject matter* (Dewey 1902 p12).

Although convinced that attention is the cause rather than the result of sense impressions, James gives a sensitive exposition of the two sides of the argument (p448). He sees attention as a form of 'free energy', since when you make an effort to attend to something you can sustain it for only very short periods before attention wanders (p420) requiring a further expenditure of effort, but when attention is engaged it requires no energy expenditure at all for it to remain focused for long periods of time. This observation was picked up and exploited by P. D. Ouspensky (1950).

Where I differ with James is in his metaphor of attention or consciousness as like a flowing stream, for it seems to me that his own descriptions, as well as my observations, lead to the conclusion that attention is suddenly sharp and alert, and then slowly declines into absence of awareness until some fresh stimulus wakes it up again. The sense that we have of experience flowing by is actually much more episodic, as attempts to reconstruct recent and distant experiences demonstrates all too clearly.

Selection and Memory

Modern psychology seems to be concentrating on exploring links between the selectivity of attention and memory: 'where selective attention goes, long term memory follows' (Barsalou 1998 p9). Barsalou calls upon a variety of sources to back up his claim that selective attention serves not only to isolate but also to store:

Isolating (discerning) (Treisman 1993, Norman 1976, Shiffrin 1988)

Recognizing relationships (Treismann 1993)

He notes that 'while non-selected information may not be filtered out completely, there is no doubt that it is filtered to a significant extent (Garner 1974, 1978; Melara & Marks 1990)'.

The proposals made here suggest that attention is much more diverse and subtle in its form and in the roles that it plays than is indicated by the research that Barsalou summarises.

4 PREPARING FOR THE FUTURE

The only virtue in making distinctions is to inform future practice. That practice might be to do with analysing classroom incidents, but more positively, it can mean having more choices open up while preparing and in the midst of the moment by moment flow of interacting with learners. The conjecture being put forward is that distinguishing different forms of attention can open up a range of possibilities which might otherwise be invisible.

For example, the different structures of attention proposed offer opportunities to be sensitive to different ways in which learners are attending, making it possible to choose to dwell longer so that learners can gain confidence before being expected to attend differently, or to direct or attract attention suitably. See also Covey (1980).

Hans Freudenthal (1991) drew attention to the difficulty of getting learners to move away from 'it just is so' as justification for a conjecture, to 'it is so because ...' and reasoning on the basis of properties. Even the recognition that an assertion is about a relationship holding and constituting a property involves 'pre-reasoning', and is essential in order to manipulate properties.

A plausible conjecture is that reasoning involves a shift of attention away from discerning details, recognising relationships and perceiving properties, to isolating some of those properties as the basis for reasoning. To let go of 'all the things you know about something' and to start reasoning simply on the basis of some stated properties is quite sophisticated, certainly in historical terms, as well as in developmental terms. This is not to say that reasoning itself arises late in development. Very young children display moments of reasoning from a property to draw a conclusion. What is required mathematically is to undertake that shift intentionally and knowingly. This is most effectively achieved by drawing learners' attention to what they have done when they have done it spontaneously, in order to support them in choosing to do it another time in the future.

5 SUMMARY

The basic idea in this paper is that attention shifts between holistic encompassing, discerning distinctions (stressing and ignoring, foregrounding and backgrounding), recognising relationships amongst discerned features, perceiving properties that objects or elements may possess, and deducing from definitions and axioms. These shifts can be rapid, but can be blocked; they are often subtle, but always vital to learning. They have been illustrated through your experience of various task-exercises that I have offered, in the hope that you recognise something

of what I myself have noticed, and that you also recognise similar phenomena in other experiences of your own.

I am trying to understand deeply the nature of my own attention in the context of mathematics with a view to informing my own practice when working myself, and with others on mathematics. I am not offering a mechanism to try to ensure or cause a particular form of attention, which I consider would contradict the way learning takes place. Nor am I offering a 'model' of how attention can or should be structured.

The importance of being aware of different attention structures and shifts between them is to alert myself to situations where there is a mismatch, or where someone has become stuck in one structure. If some learners are attending holistically when the teacher is discerning specific details, if some learners are focusing on discerning details when the teacher is talking about relationships amongst details, if some learners are recognising relationships amongst discerned elements when the teacher is talking about relationships as properties, or if some learners are thinking about properties when the teacher is deducing from properties, there is likely to be a mismatch, a failure of communication, a less than maximally effective use of time and energy. If I am aware of ways in which attention can be differently structures, I am in a better position to be sensitive to learners' experience and to make informed choices while listening to learners as well as when directing their attention through what I say and do.

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ⁱ A copy of the presentation slides can be found following links from <http://cme.open.ac.uk/JHMFurthPartics.htm> to presentations. A shorter version of this paper will appear in the proceedings of ICME10 (CD).