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## **MATHEMATICS TEACHERS' KNOWLEDGE AND PRACTICES**

### INTRODUCTION

In education, the study of teachers and teaching has been an active field for a long time. In the 1980s, as PME was developing as an organization, new perspectives of teachers' knowledge had become prominent, notably those of Elbaz (1983), Shulman (1986), and Schön (1983), which influenced the direction of research on teachers. Elbaz (1983) focused on identifying what teachers know that others do not, which she called practical knowledge, and how teachers encapsulate that knowledge. She contended that this knowledge is based on first hand experience, covers knowledge of self, milieu, subject matter, curriculum development and instruction, and is represented in practice as rules, practical principles and images.

Shulman (1986) proposed seven categories of knowledge that make it possible for teachers to teach and deal with more than practical knowledge – knowledge of content, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge [PCK], knowledge of students, knowledge of educational contexts and knowledge of educational ends, purposes and values. He emphasized PCK as a key aspect to address in the study of teaching.

Schön's (1983) work distinguished between reflective practice and technical rationality, attributing the former to practitioners. When action is required, practitioners act on the basis of what they know, without separating the intellectual or formal knowledge from the practical. For a teacher, this means that reflecting-in-practice has to do with content and content-related pedagogical knowledge. It takes place when teachers deal with professional problems and therefore can be seen as a key part of their knowledge. In this sense, the teachers' knowledge is not only "knowing things" (facts, properties, if-then relationships...), but also knowing how to identify and solve professional problems, and, in more general terms, knowing how to construct knowledge. These perspectives of teachers' knowledge also include notions of teachers' beliefs and conceptions, which we consider to be relevant constructs to understand what teachers know.

The preceding notions of teachers' knowledge formed the theoretical background we considered to define the activity of the teacher, the focus of this chapter. In order to examine such activity, we assume that two main constructs are required: teacher knowledge and teacher practice. These constructs are not independent of each other, but we treat them separately to highlight their unique features. Our intent, then, is to identify and discuss studies reported to the PME community that focus on teacher knowledge and practice in terms of issues, perspectives, results and possible directions for future work.

In order to set a boundary on the studies we identified for this chapter, we also considered the different contexts in which the activity of teachers can be situated. These include: (i) *The classroom*. This may be considered as a natural setting, in which the teacher and students interact, when there is no external intervention (e.g., from a research project). It becomes a different setting with external intervention, such as teachers or researchers who act as observers. (ii) *The school*. Teachers are active participants of the school as an institution, which is another natural community setting. Their activities can be based on the school's own in-house projects, or its participation in wider projects that focus on curriculum innovation or action-research. (iii) *Inservice courses and preservice courses*. Teachers participate in formal preservice courses, when preparing to become teachers. Later, they may participate in formal inservice courses, in their schools, in a neighboring school, or in a teacher education institution. (iv) *Other professional settings*. Outside their schools, teachers can participate in formal or informal groups, associations and meetings. In all of these settings, the teacher acts, thinks and reflects. Thus, they offer opportunities to access teachers' knowledge and teachers' practices. But they also embody other elements of the activity of teachers, in particular, teacher development/education, which is outside the scope of this chapter. In this chapter, then, we focus on the activity of the teacher (interpreted in a broad sense to include preservice and inservice) by him/herself or working cooperatively with other teachers or researchers in all of these settings, but will include teacher education settings only when the focus of the study is on the teachers' knowledge or practice and not on the teacher education program.

Our review of research reports produced by the PME community revealed that, in the early years, researchers focused on students' learning with little attention on the teacher; only a few studies related the activity of the teacher to students' learning. However, beginning in the 1980's there was growing attention on the teacher. This provided us with a substantive list of studies on teacher knowledge and practice. Guided by the theoretical background we discussed earlier, we classified the papers based on the objectives of the studies. This produced four major categories: (i) teachers' mathematics knowledge; (ii) teachers' knowledge of mathematics teaching; (iii) teachers' beliefs and conceptions; and (iv) teachers' practice. Categories (i), (iii) and (iv) each has over 60 papers while category (ii) has 35. We consider categories (i) and (ii) to be significantly different as knowledge of mathematics has a referent in an academic discipline – mathematics, one of the most formalized and sophisticated fields of human thought – whereas knowledge of mathematics teaching is in the realm of professional knowledge, being highly dependent of evolving social and educational conditions and values, curriculum orientations and technological resources.

We used three periods – 1977-85, 1986-94, and 1995-2005 – as a basis to consider possible trends in quantitative terms (number of papers in each period) and qualitative terms (objects of study, theoretical emphases, methodological approaches, other issues). As we expected, in the first period, there are very few papers dealing with teachers' knowledge. In the second period, there are a great number of papers dealing with aspects of teachers' knowledge (mathematical and mathematics teaching), beliefs and conceptions. Studies on teachers' practices first appear in the second period and grew at an amazing rate in the third. Trends involving the qualitative parameters over the three periods will be integrated in the discussion of each of the four categories as appropriate.

For our discussion of the four categories of studies, we consider our guiding questions to be: (i) What do mathematics teachers know, believe, conceptualize, think and do in relation to mathematics and its teaching and learning? (ii) What methods, theoretical perspectives and assumptions about knowledge, mathematics and curriculum did researchers adopt in studying the teachers? These questions will be addressed by identifying themes based on theoretical and methodological foundations and findings of the studies and discussing particular studies in the context of these themes. Thus, while many of the papers presented at PME conferences may appear to be relevant to this chapter, only those that we select to exemplify each theme are included here. The remainder of the chapter discusses the four categories and ends with our reflection of them collectively.

#### TEACHERS' MATHEMATICS KNOWLEDGE

Mathematics knowledge is widely acknowledged as one of the critical attributes of mathematics teachers, thus, it is not surprising that studies of teachers' mathematics knowledge is a significant focus of the PME community. Beginning in 1980, studies in this area were reported in almost every year. However, there was less focus in the 1980s than later. These studies, directly or indirectly, dealt with a variety of mathematics topics with greater attention to geometry, functions, multiplication and division, fractions, and problem solving. The themes we identified to discuss this category of studies are based on the following questions: What are the deficiencies in teachers' mathematics knowledge? How do teachers hold their knowledge of mathematics? What are the implications for teaching mathematics and mathematics teacher education?

##### *Deficiencies in teachers' mathematics knowledge*

Most of the studies over the three decades of PME conferences, directly or indirectly, focused on the difficulties or deficiencies teachers exhibited for particular mathematics concepts or processes. For example, addressing knowledge about numbers and operations, Linchevsky and Vinner (1989) investigated the extent to which inservice and preservice elementary teachers were flexible when the canonical whole was replaced by another whole for fractions of continuous quantities. They found all of the expected misconceptions and confusions associated with canonical representations of fractions and the teachers' visual representations of fractions incomplete, unsatisfactory and not sufficient to form a complete concept of fractions. In a later paper, Llinares and Sánchez (1991) studied preservice elementary teachers' pedagogical content knowledge about fractions and found that many of the participants displayed incapacity to identify the unity, to represent some fractions with chips and to work with fractions bigger than one.

A few of the studies addressed teachers' knowledge in arithmetic in relation to a particular theoretical model. For example, Tirosh, Graeber and Glover (1986) explored preservice elementary teachers' choice of operations for solving multiplication and division word problems based on the notion of primitive models in which multiplication is seen as repeated addition (with a whole number operator) and division as partitive (with the divisor smaller than the dividend). The

findings indicated that the teachers were influenced by the primitive, behavioral models for multiplication and division. The teachers' errors increased when faced with problems that did not satisfy these models. Greer and Mangan (1986) used a similar notion of primitive models. Their study included preservice elementary teachers and focused on the results on single-operation verbal problems involving multiplication and division. They also found that primitive operations affected the participants' interpretation of multiplicative situations.

Another theoretical framework used in some of these studies was the distinction between "concept image", the total cognitive structure that is associated with a concept, and "concept definition", the form of words used to specify that concept (Vinner and Hershkowitz, 1980). For example, Pinto and Tall (1996) investigated seven secondary and primary mathematics teachers' conceptions of rational numbers. Findings indicated that three of the teachers gave formal definitions containing implicit distortions, three gave explicit distorted definitions and one was unable to recall a definition. None consistently used the definition as the source of meaning of the concept of rational number; instead they used their concept imagery developed over the years to produce conclusions, which were sometimes in agreement with deductions from the formal definition, but often were not. Whole numbers and fractions were often seen as "real world" concepts, while rationals, if not identified with fractions, were regarded as more technical concepts.

In geometry, Hershkowitz and Vinner (1984) reported on a study that included comparing elementary children's knowledge with that of preservice and inservice elementary teachers. They found that the teachers lacked basic geometrical knowledge, skills and analytical thinking ability. Using the van Hiele levels as theoretical framework, Braconne and Dionne (1987) investigated secondary school students and their teachers' understanding of proof and demonstration in geometry, and what kind of relationship could exist between the understanding of a demonstration and the van Hiele levels. Findings indicated that proof and demonstration were not synonymous for the teachers or for the students. Proofs belong to different modes of understanding but demonstration always pertains to the formal one, teachers emphasizing representation and wording. Furthermore, there was no obvious relationship between the understanding of a demonstration and the van Hiele levels.

Another topic studied extensively was teachers' knowledge about functions. Ponte (1985) investigated preservice elementary and secondary teachers' reasoning processes in handling numerical functions and interpreting Cartesian graphs. Findings indicated that many of the participants did not feel at ease processing geometrical information and had trouble making the connection between graphical and numerical data. Later, Even (1990) studied preservice secondary teachers' knowledge and understanding of inverse function. She found that many of the preservice teachers, when solving problems, ignored or overlooked the meaning of the inverse function. Their "naïve conception" resulted in mathematics difficulties, such as not being able to distinguish between an exponential function and a power function, and claiming that log and root are the same things. Besides, most of teachers did not seem to have a good understanding of the concepts (exponential, logarithmic, power, root functions). In another function study, Harel and Dubinsky (1991) investigated preservice secondary teachers in a discrete mathematics course for how far beyond an action conception and how much into process conceptions of function they were at the end of the instructional treatment framed in

constructivism. Findings indicated that the participants had starting points varying from primitive conceptions to action conceptions of function. How far they progressed depended on several factors such as (i) manipulation, quantity and continuity of a graph restrictions; (ii) severity of the restriction; (iii) ability to construct a process; and (iv) uniqueness to the right condition. Thomas (2003) investigated preservice secondary teachers' thinking about functions and its relationship to function representations and the formal concept. Findings indicated a wide range of differing perspectives on what constitutes a function, and that these perspectives were often representation dependant, with a strong emphasis in graphs. Similarly, Hansson (2005), in his study of middle school preservice teachers' conceptual understanding of function, found that their views of it contrast with a view where the function concept is a unifying concept in mathematics with a larger network of relations to other concepts.

Other topics and mathematics processes and understandings addressed include the following examples. Van Dooren, Verschaffel and Onghena (2001) investigated the arithmetic and algebraic word–problem-solving skills and strategies of preservice elementary and secondary school teachers both at the beginning and at the end of their teacher training. Results showed that the secondary teachers clearly preferred algebra, even for solving very easy problems for which arithmetic is appropriate. About half of the elementary teachers adaptively switched between arithmetic and algebra, while the other half experienced serious difficulties with algebra. Barkai et al. (2002) examined inservice elementary teachers' justification in number-theoretical propositions and existence propositions, some of which are true while others are false. Findings indicated that a substantial number of the teachers applied inadequate methods to validate or refute the proposition and many of them were uncertain about the status of the justification they gave. Shriki and David (2001) examined the ability of inservice and preservice high school mathematics teachers to deal with various definitions connected with the concept of parabola. Findings indicated that both groups shared similar difficulties and misconceptions. Only a few participants possessed a full concept image concerning the parabola and were capable of perceiving the parabola in its algebraic as well as in its geometrical contexts or to identify links between them. Finally, Mastorides and Zachariades (2004), in their study of preservice secondary school mathematics teachers aimed to explore their understanding and reasoning about the concepts of limit and continuity, found that the teachers exhibited disturbing gaps in their conceptualization of these concepts. Most had difficulties in understanding multiquantified statements or failed to comprehend the modification of such statements brought about by changes in the order of the quantifiers.

#### *How teachers hold knowledge*

A few of the studies explicitly dealt with how the teachers held their knowledge, in particular, in terms of conceptual and procedural knowledge such as defined in Hiebert (1986). For example, Simon (1990) investigated preservice elementary teachers' knowledge of division. Findings indicated that they had adequate procedural knowledge, but inadequate conceptual knowledge and sparse connections between the two. Weak and missing connections were identified as well as aspects of individual conceptual differences. In general, they exhibited

serious shortcomings in their knowledge of division, in particular connectedness of that knowledge.

Philippou and Christou (1994) investigated the conceptual and procedural knowledge of fractions of preservice elementary teachers enrolled in first semester of their studies. Findings indicated that they had a narrow understanding of the ideas underlying the conceptual knowledge of fractions. Participants had greater success on addition and subtraction and lesser on multiplication and treated multiplication and division as unrelated operations. The poorer results were on items measuring their ability to connect real-world situations and symbolic computation.

In another study, Zazkis and Campbell (1994) investigated preservice elementary teachers' understanding of concepts related to the multiplicative structure of whole numbers: divisibility, factorisation and prime decomposition. Findings indicated strong dependence upon procedures. Such procedural attachments appeared to compromise and inhibit development of more refined and more meaningful structures of conceptual understanding.

Chazan, Larriva and Sandow (1999) explored a preservice secondary teacher's conceptual and procedural knowledge related to teaching the solving of equation. They found that the participant had a conceptual understanding of the topic, but this was not clear-cut in relation to how it was used to support her teaching. The authors wondered about using descriptions like conceptual and procedural understanding for an examination of teachers' substantive knowledge of mathematics. They explained, "Perhaps the difficulty is that conceptual understanding is not an 'achievement,' that is, something that one either has or does not have. Instead, maybe one can have conceptual understandings of different kinds, including partial, or confused, conceptual understanding" (p. 199).

Finally, Presmeg and Nenduradu (2005) conducted a case study of one middle school teacher, investigating his use of, and movement amongst, various modes of representing exponential relationships. They found that his facility in moving amongst representational registers was not matched by conceptual understanding of the underlying mathematical ideas as the teacher attempted to solve algebraic problems involving exponential relationships. They concluded that his case casts doubt on the theoretical assumption that students who can move fluently amongst various inscriptions representing the same concept have of necessity attained conceptual knowledge of the relationships involved.

#### *Implications for teaching and teacher education*

Some of the papers mentioned in the preceding sections explicitly point out implications for teaching based on the findings of the studies. For example, comparing teachers' knowledge with students learning, Hershkowitz and Vinner (1984) investigated the processes of concept formation in children and of the factors that affect their acquisition by studying the same concepts in elementary school teachers. Findings indicated that, on the surface, the children's concept reflected the teachers' conception.

In another study, van Dooren, Verschaffel and Onghena (2001) found that the problem-solving behavior of the preservice teachers was strongly related to their evaluations of students' solutions. They doubted whether the subgroup of primary school teachers experiencing great problems with algebra will have the proper

disposition to prepare their students for the transition to algebra, but also whether the future secondary school teachers will be emphatic towards students coming straight from primary school and bringing with them a strong arithmetic background.

Sánchez and Llinares (1992), in their study of fractions, concluded that the elementary teachers' understanding of subject matter influences presentation and formulation as well as the instructional representations that the teacher uses to make it understandable to students. Shriki and David (2001), based on their participants' lack of depth in their understanding of parabola, raised concerns about teachers' ability to implement reform recommendations.

Based on their findings of the studies, a number of papers also explicitly point out implications for teacher education. For example, Tirosh, Graeber and Glover (1986) suggest that efficient strategies must be developed for training teachers to monitor and control the impact that misconception and primitive models of multiplication and division have on their own thinking and their students'. Simon (1990), based on his findings of the teachers' way of holding their knowledge of division, concludes that it is important to facilitate cognitive connections much more than imparting additional information in teacher education. Philippou and Christou (1994) suggest that teachers need to be exposed to connecting conceptual and procedural knowledge of fractions in education programs. Sánchez and Llinares (1992) suggested that preservice teacher education should concentrate on the prospective teachers' knowledge about the relationship between mathematical processes and modeling such processes as referents. Finally, Barkai et al. (2002), based on their findings of inservice elementary teachers' justification in number-theoretical propositions and existence propositions, called for more attention of this topic in professional development programs.

The preceding account of studies in this category of teachers' knowledge indicates a range of findings based on a range of theoretical perspectives about mathematics and mathematics knowledge. This suggests the need to pursue the theorization of teachers' mathematics knowledge, framing appropriate concepts to describe its features and processes, and to establish clear criteria of levels of proficiency of mathematics teachers and instruments to assess it. We further reflect on this category later in the discussion section of the chapter.

#### TEACHERS' KNOWLEDGE OF MATHEMATICS TEACHING

In this section we address studies of teachers' knowledge of mathematics teaching. Using the themes of studies in the early years of PME conferences, studies of pedagogical content knowledge, other studies framed in cognitive psychology and studies dealing with theoretical issues of teachers' professional knowledge, we consider issues such as: What are important elements of teachers' knowledge of mathematics teaching? What is the nature of this knowledge and how does it develop?

##### *Early studies*

In the 1980s, only a few papers addressed teachers' knowledge of mathematics teaching. For example, Andelfinger (1981) presented a survey method to get information about everyday teaching and gave an extended example of its use in

surveying teachers about the role of fractions versus decimals in mathematics teaching. He indicated that teachers regard fractions and decimals as separated topics, with no problems and difficulties in common and little relationship to other topics.

Also using questionnaires, Brissiaud et al. (1982) investigated the relationship between the perceptions of elementary teachers and pupils regarding what is a problem. The results strongly supported the hypothesis that the pupil's perception of problems is modelled on the teacher's perception. In another paper, Rees (1982) reported on several studies using the notion of diagnostic teaching based in two main assumptions: (i) teachers' awareness of learners' misconceptions is critical for effective and efficient teaching and (ii) for this a general structure as a diagnostic outcome to which teachers can relate to is essential. She indicated that these studies "suggest strongly that explicit teaching of the concepts underlying the diagnostic tasks does result in more effective learning" (p. 96).

#### *Pedagogical content knowledge studies*

In the 1990s, the notion of teachers' pedagogical content knowledge [PCK] (Shulman, 1986) was one of the theoretical constructs introduced in studying the teachers' knowledge of mathematics teaching. Based in this notion, Even, Markovits (1991) studied junior high school teachers' PCK regarding teachers' responses to students' questions, remarks, and hypotheses on the topic of functions. They indicated that teachers often are not aware of students' difficulties: "some teachers ignore students' ways of thinking and their sources. Instead, they evaluate students' work only as either right or wrong" (p. 43). The authors stressed what the teachers could have considered but often did not, such as students' misconceptions, ritual versus meaning orientation, teacher versus student centeredness, and richness of responses. They summarized their findings saying that the teachers "recognized the central role (...) of understanding students' thinking" but most of them did not "recognize the importance of teachers' reaction" (p. 46).

In later studies, the notion of PCK was often combined with other theoretical ideas. For example, Klein and Tirosh (1997) evaluated preservice and inservice elementary teachers' knowledge of common difficulties that children's experience with division and multiplication word problems involving rational numbers and their possible sources. The authors indicated that most inservice teachers provided correct expressions for the multiplication and division word problems, what did not happen with prospective teachers. They summarized the findings saying that "most prospective teachers exhibited dull knowledge" of the difficulties that children's experience with word problems involving rational numbers and their possible sources, whereas "most in-service teachers were aware of students incorrect responses, but not of their possible sources" (p. 144). The researchers suggested that direct instruction related to students' common ways of thinking could enhance both preservice and inservice teachers' PCK.

More recent studies involving PCK show an effort to establish a critical perspective regarding it, either by reformulating it or complementing it with other theoretical notions. For example, Rossouw and Smith (1998) reported a study on elementary teachers' PCK in geometry, two years after they completed an in-service course. The authors emphasized the perspective of knowledge in action (Schön, 1983) and discussed the need to enlarge Shulman's (1986) notion of PCK.

They presented a model with four categories (knowledge of geometry, learning geometry, teacher representations, and the environmental context of teaching) and identified three main orientations of the teachers, described as “life skills”, “investigative” and “mastery”. The authors also indicated that even though the teachers had the same learning experience in the inservice course that they attended two years earlier, their PCK of geometry showed marked differences and concluded that “teachers eventually develop their own pedagogical content knowledge which is shaped by their own experiences and perceptions” (p. 64).

### *Cognitive psychology studies*

Some of the studies show a strong influence of information-processing theory, a major strand in cognitive psychology. In the 1990s, many of the studies framed in this paradigm used an expert-novice contrast to identify different ways of teacher thinking and decision-making. For example, Robinson, Even and Tirosh (1992), dealing with junior and high school teachers, examined the differences between two novice teachers and an expert teacher (reputed as an “excellent teacher”) in presenting mathematical material in a connected manner. The authors concluded that experts largely differ from novices in the role that connectedness plays in both their planning and teaching of lessons in algebra, as well as in their reflection on their own lessons. The expert teacher considered the issue of connectedness to be very important and used both vertical and horizontal connections to guide her teaching. In contrast, the novice teachers did not emphasize connectedness in their lesson plans and teaching. They also tended to stick to their plans regardless of what happened and drew conclusions that suited their plans but bore little connections with what really went on in the classrooms. In a later study, Robinson, Even and Tirosh (1994) investigated 7<sup>th</sup> grade teachers’ knowledge of issues related to the incomplete nature of algebraic expressions. The authors reported the customary differences that these studies tend to provide: while experienced teachers were aware of the existence of the difficulty and its possible sources, novice teachers attributed difficulties to other reasons, such as notations.

Also in the 1990s, a new construct grounded in cognitive psychology emerged in the PME community: Cognitively Guided Instruction (CGI), based in the work of Carpenter and Fennema (1989). The main idea is that understanding the knowledge of students’ cognition in mathematics is one important component of the knowledge of mathematics teachers. In one study, Bright, Bowman and Vacc (1997) sought to identify frameworks in primary teachers’ analysis of children’s solutions to mathematical problems and monitored changes in the framework use across the first year of implementing CGI. Five frameworks were identified in the findings: developmental, taxonomic, problem solving, curriculum, and deficiency. The curriculum framework was used most often, followed by the problem solving and deficiency frameworks.

In another study, Gal and Vinner (1997) addressed the difficulties experienced by students in understanding the concept of perpendicular lines and the difficulties shown by teachers when trying to explain it. The authors corroborated the main assumptions of CGI concluding that the teachers’ lack of tools which would help them to understand students’ difficulties makes them unable of providing adequate teaching. In addition, Gal (1998) addressed aspects of junior high school teachers’ knowledge concerning teaching special segments in the triangle. The aim was to

draw teachers' attention to the possibility that some difficulty is hidden (or not-hidden) behind the students' answers, to introduce an opening towards understanding the difficulties, and to increase their motivation to look for solutions. Findings showed that the teachers became aware of their cognitive processes and used them as a "didactic lever".

### *The nature and development of teachers' knowledge*

Some papers gave special attention to theoretical issues regarding teachers' knowledge of mathematics teaching. In one, Ponte (1994) presented several cases to illustrate aspects of this knowledge regarding problem solving and to discuss its nature. Based in the ideas of Schön (1983) and Elbaz (1983), he presented the notion of "professional knowledge" as essentially knowing in action, grounded on experience, reflection on experience and theoretical knowledge. In his view, this knowledge is different from academic and common sense knowledge and ought to be studied on its own right, and not just regarded as "deficient" academic knowledge. Discussing the cases of three middle and secondary school teachers, he analysed possible reasons for different views and practices regarding problem solving, suggesting that specific know how and confidence may interfere with general agreements of curriculum priorities and ways of acting in the classroom. He presented four elements of professional knowledge: (i) teachers' views and personal relationships with mathematics; (ii) teachers' knowledge and personal relationship to students; (iii) teachers' knowledge and attitude regarding the curriculum; and (iv) teachers' way of living the profession. These elements are shaped by past experience and influenced by the social and institutional contexts.

Chapman (2004) used the construct of "practical knowledge" to describe knowledge that guides actual teacher actions. In her perspective, such knowledge "corresponds with positions teachers take" and is "experiential, procedural, situational, particularistic, and implicit" (p. 192). The author emphasized the procedural aspect of practical knowledge and noted that it can be used either for adapting, shaping, or selecting elements in real life situations. Based in this framework, she presented aspects of the practical knowledge of high school teachers who consistently engaged students in peer interactions in their teaching of mathematics, which included teachers' conceptions that support a social perspective of learning, students' behaviours and outcomes in peer interactions, learning activities, and teacher's behaviours that support peer interactions.

Finally, Simon (1991) addressed the initial development of prospective elementary teachers' conceptions of mathematics pedagogy. This study, framed in terms of a constructivist perspective applied to the construction of teachers' knowledge, aimed to describe the conceptions of prospective elementary teachers early in their preparation to teach. The author sought to identify "what ideas are readily developed or changed and which are not developed or are resistant to development" (p. 271). Focusing on one participant, the author argued that prospective teachers do not have a well-developed model of student learning and that teaching strategies such as questioning and the use of manipulatives "are more easily learned than are new models of students' mathematical learning" (p. 275). However, in his view, prospective teachers have difficulty in decentring from their own thinking to focus on student thinking.

The preceding account of studies in this category of teachers' knowledge and practice depicts a picture of both weaknesses and strengths in teachers' knowledge of mathematics teaching. It suggests the need to pursue a global theory about the specific knowledge involved in the teaching of mathematics and how it relates to knowledge about mathematics, learning, curriculum, and the organization of instruction. We further reflect on this category later in the discussion section of the chapter.

#### TEACHERS' BELIEFS AND CONCEPTIONS

Beliefs and conceptions have played a prominent role as a basis of studying mathematics teachers and their teaching. While there was little attention on these constructs in papers presented at PME conferences in the 1980s (cf. Hoyles, 1992), there has been a significant increase in their use from the early 1990s. Thompson's work (e.g., 1992) and Ernest's work (e.g., 1991) seem to be important influences in fuelling this and were cited by many of the studies. In this category of studies we consider the following questions: How are beliefs/conceptions defined? What theoretical models or methods are used to access beliefs/conceptions? What aspects of teachers' beliefs are studied and how is the nature of these beliefs regarded? What is the relationship between beliefs/conceptions and practice?

##### *Concept of beliefs/conceptions*

For most of the studies, what is meant by beliefs seems to go unnoticed, or considered to not be an issue, by the researchers. The term belief is treated as a taken-for-granted in that an explicit discussion is not provided. In general, terms such as beliefs, conceptions, views, perspectives, perceptions, personal constructs, belief systems, and images are used synonymously or interchangeably. Our category of beliefs and conceptions, then, can be described by the contracted form 'beliefs/conceptions' and should be understood in a broad sense.

Three exceptions to the general trend of not explicitly defining beliefs/conceptions are the papers of Hoyles (1992), Ponte (1994) and Gates (2001). Ponte (1994) drew distinction between knowledge, beliefs and conceptions in his study of teachers' conceptions and practices regarding mathematical problem solving. He explained, "I take *knowledge* to refer to a wide network of concepts, images, and intelligent abilities possessed by human beings. *Beliefs* are the incontrovertible personal "truths" held by everyone, deriving from experience or from fantasy, having a strong affective and evaluative component. ... *Conceptions* are the underlying organizing frames of concepts, having essentially a cognitive nature. Both beliefs and conceptions are part of knowledge" (p. 199). Based in the perspective of Pajares (1992), the author regards beliefs as a part of relatively less elaborated of knowledge, not confronted with empirical reality, and that does not require internal consistency. Conceptions, on the other hand, are seen as organizing constructs, frame the way we tackle tasks and play an essential role in thinking and acting.

Hoyles (1992), who explained how the contention that teachers reconstruct their beliefs while interacting with an innovation, based on her work involving teachers' interactions with computer activities and the ways they incorporated them into their practice, led her to propose the notion of situated beliefs, i.e., all beliefs are, to a

certain extent, constructed in settings. They are “dialectical constructions, products of activity, context and culture” (p. 280). This notion challenges the separation of what is believed from how beliefs emerged. “Once the embedded nature of beliefs is recognized, it is self evident that any individual can hold multiple (even contradictory) beliefs and ‘mismatch’, ‘transfer’ and ‘inconsistency’ are irrelevant considerations and replaced by notions of constraints and scaffolding within settings” (p. 280). This perspective requires focusing on understanding “beliefs-in-practice”.

Another paper that deals with conceptual aspects was presented by Gates (2001). He discussed concepts that provide a sociological perspective of how belief systems are constructed upon teachers’ ideological foundations. In his view, much of the literature on teachers’ beliefs and conceptions and their effect on the teaching of mathematics fails to locate the sources of beliefs in the social world, treating them as if they existed in a social and political vacuum. One of the concepts he offered is *habitus*, the cognitive embodiment of social structure, that form the generative principles that organize our social practices leading to social action and provides systems of dispositions that force us (or allow us) to act characteristically in different situations. The mathematics teacher’s *habitus* will be at the root of the way in which teachers conceptualize themselves in relation to others; how they enact and embody dominant social ideas as well as how they transform and adapt them. The other concept offered by Gates is ideology, which addresses the relationships between ideas, society and individuals. It relates to matters of powers and social structure, as well as ideas and activity to the wider socio-cultural context and resides in language forms used and social imagery adopted. Relating this to teachers, ideological underpinnings appear as ideas and assumptions about human nature, about learning and educational difference, the role of education, the role of the teacher and ideas about priorities for teacher professional development.

#### *Theoretical models for accessing beliefs/conceptions*

Most of the studies employed conventional qualitative and quantitative methods to access beliefs/ conceptions. While many of the studies were case studies and used interviews and classroom observations, some used questionnaires and quantitative analysis. A few studies adopted specific theoretical models, in particular the Perry scheme and Ernest models, as a basis of data collection and/or data analysis as in the following examples.

In the 1980s, Perry’s (1981) theory was used in framing two studies. Oprea and Stonewater (1987) explored the relationship between 13 secondary school mathematics teachers’ cognitive development and their belief systems. Findings indicated that five teachers rated as relativistic, five as late multiplistic, and three as early multiplistic. The data did not support the authors’ hypotheses that there is (i) a positive correlation between the teachers’ Perry position and the view that mathematics is useful and (ii) a negative correlation between that position and the view that mathematics is closed. The authors resulting hypothesis is that Perry level might be different with regard to how teachers think about mathematics and its teaching. Owens (1987) reported on two case studies of preservice secondary school teachers’ personal constructs of mathematics and its teaching. Perry’s ideas provided a framework for the analysis of the experiential, mathematical and

pedagogical perspectives through which the participants interpreted their teacher preparation program and anticipated their roles as teachers. Findings indicated that constructs related to teaching roles tended to focus on personal, non-intellectual qualities. Constructs relating to mathematics were affected by prior success with pre-college mathematics and anticipated uses of mathematics in teaching roles and were often discordant with perception of subject matter preparation at the college level.

Beginning in the 1990s, Ernest's work was used to frame many of the studies as in the following examples. Carrillo and Contreras (1994) reported a study that tested a framework for the analysis of the teachers' conceptions of mathematics and its teaching. The framework included a model for conception of mathematics teaching with four "didactic tendencies" – traditional, technological, spontaneous and investigative and six categories of 35 descriptors (many of which coincided with those in Ernest, 1991) for each tendency – methodology, subject significance, learning conception, student's role, teacher's role and assessment. The framework also had a model for conception of mathematics with three tendencies proposed by Ernest – instrumentalist, platonic, and problem solving –, and three categories of 21 descriptors for each tendency – type of knowledge, aims of mathematical knowledge and means of development of mathematics. Case studies for six inservice secondary teachers were conducted illustrating the use of this analytical tool.

Valero and Gómez (1996) explored the effects on the belief system of a university teacher who was involved in a curricular innovation centered on graphic calculators. They considered five elements of teachers' belief systems: view about mathematics, about the aims of mathematics teaching, about learning, about teaching and about role of instructional materials. They also considered Ernest's (1991) categories of teachers. During the first semester, the teacher was identified as "industrial trainer" with beliefs of mathematics as a set of unquestionable, accepted truths and that mathematics education aims at the mechanization of basic skills. In semester 2, her behaviour changed to "public educator" but her belief system still reflected "industrial trainer", i.e., the calculator influenced change in behaviour in interacting with students but not her beliefs in general.

Charalambous, Philippou and Kyriakides (2002) in their study of teachers' philosophical beliefs, adopted Ernest's (1991) model of Platonist, instrumentalist, and experimental-constructivist views of mathematics. Their goal was to collect empirical data to examine the efficiency of this model in describing primary and secondary teachers' beliefs about mathematics, the factors influencing the development of these beliefs, and their relation to teachers' beliefs and practices about teaching and learning mathematics. Findings revealed a five-factor model representing a combination of the dimensions of Ernest's model but the data failed to verify this model.

Finally, two studies focused on alternative research approaches to take context into consideration. Critiquing Likert scales, Ambrose et al. (2003) developed an alternative instrument, intended to capture qualitative data that are quantified to provide a common metric for measuring change in individuals and for comparing individuals to one another in relation to their beliefs, e.g., about mathematics, knowing and/or learning mathematics, and students learning mathematics. The instrument uses video clips and learning episodes to create contexts to which users respond in their own words rather than choose from one of several options.

Findings from two administrations of the instruments with preservice elementary teachers suggest that it is an effective tool for assessing belief change. Chapman (1999) discussed a humanistic approach for researching teacher thinking, defined to include beliefs and conceptions. The approach is related to the work of Connelly and Clandinin (1990) that promotes narratives as a way of capturing and studying lived experience, e.g., the teaching of mathematics. Thus, with it, beliefs can be accessed in a situated way in the form of stories of experience. Teachers can be asked to tell stories of their choice, stories prompted by the researcher, and stories to support claims they make during interviews. Case studies of high school mathematics teachers using this approach suggested that it is effective in capturing their thinking about mathematics and its teaching from their perspective.

#### *Aspects of teachers' beliefs/conceptions*

As reflected from the sample of studies discussed in the preceding sections, investigating teachers' beliefs or conceptions about the nature of mathematics and the teaching of mathematics was a key focus of many studies. However, some of the studies focused on describing teachers' beliefs or conceptions in relation to a particular aspect of teaching and learning mathematics, e.g., problem solving, students' mathematics errors, technology, and gender differences. Of these, problem solving and technology were dealt with in multiple studies and we highlight some examples here.

Grouws, Good and Dougherty (1990) interviewed junior high teachers to determine their conceptions about problem solving and its instruction. They found four categories of conceptions of problem solving in which the teachers could be grouped: problem solving is (i) word problem, i.e., the mode of presentation of the problem situation was the determining factor; (ii) solving problems, i.e. anytime students found an answer to a mathematical problem they were doing problem solving; (iii) solving practical problems, i.e., what teachers perceived as real-life situations; and (iv) solving thinking problems, i.e., need to incorporate *ideas* in the solution process. The first three focus on the nature of a problem and its computational aspects while the last one is primarily concerned with processes involved in finding a solution.

Chapman (1994) reported on a case study of an elementary teacher's perspective of problem solving and its teaching. Findings indicated that the teacher viewed problem solving as both a cognitive and a social endeavour. She made no distinction between problem and problem solving, i.e., the problem solving is the problem, one does not have a problem until one starts to experience and deal with a barrier in a situation one is curious about or has an interest in. She viewed teaching of problem solving in terms of a three-stage process, i.e., preparation, collaboration, and presentation, as a basis to organize her teaching.

For technology, Ponte (1990) investigated the conceptions and attitudes of secondary and middle school mathematics teachers involved in an innovative inservice program. Findings indicated that the participants had a major concern with the dynamics that the computer could bring to their classroom and some of them were also interested in using this instrument for interdisciplinary activities. Bottino and Furinghetti (1994) focused on the reaction of teachers facing curricular innovations involving the use of computers. They reported on five case studies of inservice secondary teachers' beliefs on the use of computers in teaching

mathematics. Findings indicated that the teachers' beliefs on the role of computers were mainly a projection of their beliefs on mathematics teaching. If teaching of mathematics was interpreted as a transmission of knowledge, without real participation of students, the use of computers appeared of little relevance. The teachers interested in constructing knowledge found in computers answers to their needs. Beliefs on the nature of mathematics were less influential in the acceptance or refusal of computers but played a role in the choice of the type of software tools used. Kynigos and Argyris (1999) investigated two elementary teachers' practices and beliefs constructed after eight years of innovative practice involving small cooperative groups of students in a computer-based mathematics classroom. Findings focused more on practice and indicated that one teacher had confidence with mathematics and appreciation of encouraging reflection, but interventions were infrequent and often lacking in mathematics content. The other teacher expressed uneasiness with mathematics, but was very directive and mathematically explicit in her interventions. Finally, Valero and Gómez (1996), discussed in a previous section, focused on a teacher who was involved in a curricular innovation centered on graphic calculators and found that technology by itself does not promote change in beliefs but could play a role in destabilization of the teacher's beliefs.

#### *Relationship between beliefs/conceptions and practice*

Many of the studies focused on this relationship directly or indirectly. However, overall, the findings were mixed, with a few researchers offering some explanation for reported inconsistencies in the relationship. For example, Hoyles (1992), in her review of research on teacher beliefs, noted that these studies threw up evidence of inconsistencies between beliefs and beliefs-in-practice. She argued that this mismatch was thrown into relief when teachers were faced with an innovation, particularly when the innovation involved computers – a point brought home by the metaphor of the computer as window and a mirror on beliefs. Grouws, Good and Dougherty (1990) found that while some relationships were evident between the conception of problem solving and the reported instructional practices of their sample of junior high school teachers, other aspects of instruction were heavily influenced by external factors such as textbooks, district expectations, and standardized testing and were similar across all teacher responses. Fernandes and Vale (1994) found that their two participants revealed very similar conceptions of mathematics and problem solving as preservice middle school teachers but as beginning teachers their practices differed quite substantially. In one case mathematical problem solving was integrated in the curriculum and there was a consistency between her claimed ideas and intentions and her practice. In the other case contradictions emerged between what the participant claimed to be his ideas and intentions and what actually happened in his classrooms. The authors suggested their teacher education program, official mathematics curriculum, and pedagogic school culture as possible factors to explain their situations, however, given that both teachers encountered these factors but behaved differently leave the issue unanswered. Finally, Beswick (2004) reported on a case study of one secondary school teacher that focused on what specific teacher beliefs are relevant to teachers' classroom practice in various classroom contexts. She found that the teacher held beliefs that were consistent with the aims of the mathematics

education reform movement but there were significant differences in his practice in regard to the various classes. For example, his belief in relation to older students of average ability had a significant impact on his practice in their lessons and limited the extent to which some students in this class were likely to engage in mathematical thinking.

One study that focused on explaining the basis of the inconsistencies was reported by Skott (1999) who investigated the relationship between the teacher's images of mathematics and its teaching and learning and his or her classroom practices. Based on a case study of an inservice elementary teacher, the author explained that certain moments of the teacher's decision making are characterized by the simultaneous existence of multiple motives of his/her activity. These motives may be experienced as incompatible and lead the teacher into situations with apparent conflict between beliefs and practice. Rather than examples of inconsistencies these may be conceived as situations in which the teacher's school mathematical priorities are dominated by other motives of his/her educational activity, motives that may not be immediately related to school mathematics. Thus, apparent inconsistencies may be understood as situations in which the teacher's motive of facilitating mathematical learning is dominated by other and equally legitimate motives of, for instance, ensuring the student a space in the classroom community or developing his or her self-confidence.

The preceding account of studies in this category of teachers' knowledge and practice shows that beliefs and conceptions are important to understand what teachers do and why they do it. We further reflect on this category later in the discussion section of the chapter.

#### TEACHERS' PRACTICES

This section addresses studies of teachers' practices in mathematics teaching. Guiding questions include: What is the nature of this practice? What are important factors that shape and support development of teacher's practices? We organize this section in terms of the following themes: cognitive psychology studies, classroom interaction studies, socio-cultural studies, curriculum-based studies, teachers' biographical and collaborative studies, and views of practice. Studies carried out in the early years of PME conferences did not directly deal with teachers' practices, which explains their absence in this category.

##### *Cognitive psychology studies*

As could be expected in the PME community, some studies were strongly based in psychological frameworks. One of the first such studies was reported by Dougherty (1990) who investigated cognitive levels of eleven elementary teachers and their relationships with their problem solving instructional practices. The indicators of teachers' practices were classroom variables such as "amount of time spent on lesson development, types of problems selected for examples during development, teaching techniques used for problem solving instruction, teacher use and types of questioning, teacher modelling, lesson format, and so on" (p. 121). The results support the theory that cognitive structures are related to instructional practices and conceptions about mathematics and problem solving. That is, teachers at lower cognitive level used teacher-directed lessons, rigidly adhering to lesson objectives

and teachers at high cognitive level valued students' opinions, encouraged creativity, and appreciated divergent thoughts and individual differences.

Within the cognitive tradition, another important line of research is information-processing. For example, Escudero and Sánchez (1999) addressed the relationship between professional knowledge and teaching practice for secondary school mathematics teachers, focusing in the teaching of similarity. Schema is a key theoretical concept and practice is (explicitly) regarded as the work that the teacher faces when performing his/her professional tasks. The authors stressed the interrelation between the structure of the lesson and the teachers' understanding of the mathematical content and point that "some characteristics of the integration of the different domains [of teachers' knowledge] are noted in decision making" (p. 311). In a later paper, Escudero and Sánchez (2002) addressed similar issues, now concerning the teaching of Thales theorem. Discussing the cases of two secondary school teachers, they concluded that PCK and subject matter knowledge were integrated in their decisions. They claimed, however, that the two teachers used different structures and that the teachers' initial decisions regarding the structures to adopt were linked to different characteristics of the domains of knowledge, which these teachers integrated in a different way.

#### *Classroom interaction studies*

Another line of interest of teachers' practices stems from the work on classroom interactions and focus on classroom discourse. A study by Khisty, McLeod and Bertilson (1990) addressed the linguistic factors involved in the acquisition of mathematical knowledge by elementary school students with limited English proficiency. The teachers' practices included the language used, the nature of the classroom discourse and the tasks proposed to students. Observing four elementary classrooms, the authors found that Spanish was seldom used to develop mathematical understanding, even by bilingual teachers. They also noted that, similarly to what usually happens in mainstream classrooms, students often work individually and in silence, and little contextualized mathematics activity took place.

In another study, Wood (1996) took mathematical argumentation as a fundamental activity in mathematics. She examined the processes of teaching when elementary school students engage in the resolution of disagreement or confusion in their mathematical thinking. The results, the author noted, "reveal the intricate ways the teacher sustains the interaction to allow children's reasoning to prevail, while restricting her own instructive contributions" (p. 427). The author also stressed the central importance that teachers play in classroom discussions, as active listeners as well as in establishing social norms in the classroom. In a later paper, Wood (1998) proposed a framework with descriptive categories that addressed the work of the elementary teacher in "reformed curricula" (teaching in ways that promote students' thinking and reasoning about mathematics), and discussed the challenges that the teachers faced in putting such perspective into practice. A set of patterns of interaction referred to the teachers' activity in interacting with students, asking questions at different levels of demand. Another dimension concerned the norms established regarding teachers' expectations for students' participation, most notable for students as listeners.

In a more recent study, based in the examples of a Year 1 lesson in Japan and a Year 7 lesson in Australia, Groves and Doig (2004) discussed the nature of “progressive discourse” and examined critical features of teacher actions that contribute to mathematics classrooms working as inquiry communities. They concluded that this is promoted when the teacher (i) focuses on the conceptual elements of the curriculum and uses complex, challenging tasks, (ii) orchestrates classroom interventions to allow all students to contribute towards solving the problem; and (iii) focuses on “seeking, recognizing, and drawing attention to mathematical reasoning and justification, and using it as a basis for learning” (p. 501).

### *Sociocultural studies*

In the 1990s, sociocultural theory, based in the work of Vygotsky, became prominent in the PME community and evolved as one of the more productive lines of work regarding teachers’ practices. For example, in a theoretical paper, Adler (1996) suggested the combination of Lave and Wenger’s (1991) social practice theory with sociocultural theory for a full and effective elaboration of knowing, learning and teaching mathematics in school. In another paper, Adler (1995) applied this framework to the analysis of the complexities of teaching mathematics embracing democratic ideals in multilingual classrooms, focusing on the nature of the teacher intervention. The paper considered events on a 6<sup>th</sup> grade classroom and discussed whether inquiry participative pedagogy can “turn on itself”, reducing the development of mathematical knowledge. The author concluded that in some cases the teacher must withdraw as a reference point for students, enabling a participatory classroom culture, but in other cases the teacher mediation is essential to improve the substance of communication about mathematics; therefore, finding a proper balance is a continuous professional challenge.

In some papers, the notion of “scaffolding”, a sociocultural construct, was used to examine teacher intervention. For example, in a theoretical paper, Anghileri (2002) characterised some classroom practices that can be identified as scaffolding, combining original classifications with further strategies that she identified. In her framework, scaffolding can be done at different levels, beginning with the creation of the physical learning environment, interacting at a low level of cognitive demand, and moving forward to more sophisticated forms of interaction such as making connections, developing representational tools, and generating conceptual discourse. This notion of scaffolding was taken up by Tanner, Jones (1999) who identified several styles of teaching metacognitive skills in a project for pupils aged 11 to 13, the two most successful being the dynamic scaffolders and the reflective scaffolders.

Also from a sociocultural perspective, Khisty (2001) conducted a study on the processes of instruction that contribute to positive student achievement in mathematics. She regarded sociocultural activity as the context in which children participate and from which they appropriate tool use and cultural thinking and studied five teachers of elementary and middle schools with Latino second language learners. The author concluded that writing mathematics is a process that can support and advance student thinking and indicated that “effective teachers” share characteristics such as: (i) encourage mutual support among students; (ii) formulate high expectations, (iii) are skilful at conceptualising of mathematical

situations; and (iv) use probing questions and statements, both oral and written, as tools for learning.

Combining a sociocultural orientation with perspectives from situated cognition, post-structuralism and psychoanalysis, Mendick (2002) focused on the practices through which teachers, explicitly and implicitly, answer the students' question, "Why are we doing this?" The author presented a case study of a secondary school class in which preparing for examination, competition among students, and procedural work were prominent features. She argued that the practices in which students and teachers engage, the meanings they give to them, and the possibilities these make available for the development of their identity are critical to understand students' success and failure in mathematics. The paper draws attention to developing a sense of purpose for learning mathematics as a key issue to understand students' achievements. However, one is left with the question of how much of that sense depends on the teacher and how much is framed by the constraints of the social and educational contexts to which teacher and students belong.

#### *Teachers' biographical and collaborative studies*

Some studies of teachers' practices involve teachers researching themselves or working in collaboration with researchers. One such study is reported by Tzur (2002), a mathematics educator who taught a 3<sup>rd</sup> grade classroom for four months. He examined how useful to guide practice is a theoretical model of mathematics teaching and learning, paying special attention to the activities and its effects. He regarded teaching as a cycle of four main activities: (i) inferring learners' conceptions; (ii) hypothesizing a learning trajectory; (iii) designing and engaging learners in activities; and (iv) orienting learners' reflections and inferring their new conceptions, etc. The author concluded that this general model is useful to orient teaching and can be combined with content-specific models of students' thinking. Working collaboratively, Rota and Leikin (2002) carried out a study in the first author's 1<sup>st</sup> grade classroom, addressing her development of proficiency in orchestrating discussion in an inquiry-based learning environment. They suggested elements of a structure for leading discussions and indicate a possible evolution of teacher proficiency.

Based in the study of actual secondary teachers' practices, Jaworski (1991) developed a framework, the "teaching triad", to model the role of the teacher, taking into account the complexity of the classroom. The teaching triad encompasses: (i) Management of learning that describes the teacher's role in the constitution of the classroom learning environment by teacher and students and includes the classroom groupings, planning of tasks and activity, establishment of norms. (ii) Sensitivity to students that concerns the teachers' knowledge of students, attention to their needs and the ways in which the teacher interacts with students and guides group interactions. (iii) Mathematical challenge that refers to the challenges offered to students to engender mathematical thinking and activity and includes tasks set, questions posed and emphasis on meta-cognitive processing. Jaworski and Potari (1998), working in partnership with two secondary school mathematics teachers, used the teaching triad as an analytical device. They saw its three domains as closely interlinked and interdependent and conjectured that teaching is most effective when the three categories are most harmonious.

### *Curriculum-based studies*

Other studies of teachers' practices are strongly based in curriculum issues. For example, some are based in a curriculum perspective that emphasizes one or more aspects regarding the aims and processes of mathematics education. One such study was reported by Askew et al. (1997) who studied elementary school teachers' practices in a numeracy framework. They indicated that transmission and discovery-oriented teachers may provide challenge to the higher attaining pupils but structured the mathematics curriculum differently for lower attaining pupils. In contrast, the connectionist-orientated teachers placed strong emphasis on challenging all pupils, did not see that pupils should have learnt a skill in advance of being able to apply it, and considered that the challenge of application could result in learning.

In another paper, McDonough and Clarke (2003) tried to capture main aspects of the practice of effective teachers of mathematics in the very early years of elementary school. Interviews with the teachers revealed that they had a clear vision of the mathematical experiences needed, were able to engage the students, and were prepared to probe students' thinking and understanding. The authors proposed a detailed framework to describe teachers' practices that includes mathematical focus, features of tasks, materials, tools and representations, adaptations/connections/links, organisational style and teaching approaches, learning community and classroom interaction, expectations, reflection, and assessment methods and suggested that many of the 25 teacher behaviours and characteristics in the list are applicable to other grade levels.

Some of the studies on teachers' practices are framed in current mathematics education reform efforts. For example, Manouchehri (2003) investigated mathematics teachers involved in reform mathematics curriculum to identify common traits and factors that could have influenced their positive disposition towards innovation. The findings indicated that the participants had strong confidence in their ability to control student learning and a detailed view of the type of teaching that could promote it. They also had strong philosophic views on the role of education and mathematics education, seeing teaching as moral and ethical act and seeing themselves as social change agents.

Saxe (1999) reported on a study that evaluated the extent to which upper elementary school teaching practices were aligned with reform principles, focusing on "integrated assessment", the degree to which classroom practices elicit and build upon students' thinking, and "conceptual issues", the extent to which conceptual ideas are addressed in problem solving activities involving fractions. Results showed that, in classrooms low on alignment (whether using traditional or reform curricula), students with poor understanding of fractions had little basis on which to structure mathematical goals except using whole number or procedural strategies. However, at higher levels of alignment, students' posttest scores were related to alignment and these scores increased sharply.

Finally, Boaler (2003) discussed three contrasting teaching and learning environments in reform classes. The author framed her discussion in terms of the *locus* of authority. In two cases, there was adherence to reform principles from the teachers but the classroom culture did not provide the kind of learning experiences that these principles suggest. A more productive situation was found in the third teacher's class where students had more authority than those in traditional settings. This teacher deflected her authority to the discipline of mathematics, implicitly

saying, “Don’t ask me – consider the authority of the discipline – the norms and activities that constitute mathematical work” (p. 8). Boaler proposed that progress in mathematics requires an interchange of human agency and “agency of the discipline”, and stressed that the teacher needs to create conditions so that students engage in a “dance of agency” between these two instances.

### *Views of practice*

In early studies of teachers’ practices, practice was mostly regarded as “actions”, “acts” or “behaviours”. But this evolved in interesting ways over the years as suggested by the following examples. Simon and Tzur (1997) discussed practice as including what the teacher does, knows, believes and intends, adding: “we see the teacher’s practice as a conglomerate that cannot be understood looking at parts from the whole (i.e., looking only at beliefs, or questioning, or mathematical knowledge, etc.)” (p. 160). Skott (1999) underlined the importance of motives in the study of teachers’ practices. Saxe (1999) considered practices as “recurrent socially organized activities that permeate daily life” (p. 25). A key assumption is that there is a reflective relation between individual activities and practices, since the activities of the individual are constitutive of practices and, at the same time, practices give form and social meaning to the activities of the individual. Boaler (2003) described practices as “the recurrent activities and norms that develop in classrooms over time, in which teachers and students engage” (p. 3). Common to Boaler and Saxe is the notion of stability and recurrence of practices. However, Saxe emphasized their socially organized nature and Boaler considered not only activities but also norms.

If we regard the study of the practices of social actors in their natural contexts to be: the activities, the recurrence, the social setting and the knowledge, meanings and motives of the participants, then teachers’ practices can be viewed as the activities that they regularly conduct, taking into consideration their working context, and their meanings and intentions. This includes the social structure of the context and its many layers – classroom, school, community, professional structure and educational and social system. But this can be problematic, as noted by Even and Schwartz (2002) who discussed the issue of competing interpretations of teachers’ practice and its implications for research. They showed that any given theoretical framework tends to ask its own kind of questions and leads naturally to a different picture of the situation. They suggested that practice is too complex to be understood by only one perspective but pointed out that while combining several theoretical approaches may seem an appealing proposal, it may raise questions of legitimacy that must be addressed by researchers. However, they leave it as an open question to be addressed by researchers.

The preceding account of studies in this category of teachers’ knowledge and practice shows the significant growth of research concerning mathematics teachers’ practices, which suggests that it is the most salient aspect of research concerning the activity of the teacher in recent years. We further reflect on this category, along with the other three categories, in the next section of the paper.

## DISCUSSION AND CONCLUSIONS

In this section we reflect on each category and discuss them in terms of selected issues related to five themes: categories of studies, findings, theoretical perspectives, methodology and future directions.

### *Categories of studies*

We found the categories that we used helpful to think about the work reported at PME conferences during these 29 years concerning the activity of mathematics teachers. However, the space in this chapter did not allow us to include all aspects of the work that has been done. The large number of studies covered a broad range of variables and constructs, sometimes in multilayered ways that cut across these categories. For example, the category of teachers' knowledge of mathematics teaching has much fewer papers than the other categories, but this does not necessarily imply a lack of interest in this notion as many papers look at it through the lenses of beliefs and conceptions, which are covered in a separate category; in addition, in the last few years, there is a trend to look at teachers' knowledge of mathematics teaching increasingly in conjunction with teachers' practices. However, we noticed very few studies that implied other inter-category relationships, in particular between teachers' mathematics knowledge and teachers' knowledge of mathematics teaching (exceptions are, e.g., Llinares and Sánchez, 1991; Simmt et al., 2003; Tzur et al., 1998). On the other hand, for the categories of teachers' beliefs and conceptions and teachers' practices, from a scarcity of studies, there was a growth of interest in these themes, asking new and more interesting questions, using more elaborated theoretical frameworks and resorting to more diversified methodologies.

We identified other categories regarding the activity of the teacher but did not discuss them separately because of space restrictions and the relatively small number of papers; they were: "teachers' attitude and affective aspects"; "teachers' researching"; "teachers in community"; "university teachers"; "teacher thinking and metacognition"; and "teacher reflection and reflective practice." We were surprised that so few papers seemed to deal with teachers' actual thinking, reflective practice and metacognitive processes, given the attention these had at some point in cognitive psychology and teacher research in general.

### *Findings*

The papers we reviewed offer a wide range of findings that are insightful and meaningful to our understanding of the activity of the mathematics teacher. Many studies show that teachers' mathematics knowledge is generally problematic in terms of what teachers know, and how they hold this knowledge of mathematics concepts or processes, including fundamental concepts from the school mathematics curriculum. They do not always possess a deep, broad, and thorough understanding of the content they are to teach. We do not find it meaningful to provide summaries of these findings because they are based on a variety of theoretical and research perspectives and are not necessarily generalizable. The findings of teachers' knowledge of mathematics teaching also suggest problems with this knowledge. The studies of mathematics teachers' beliefs and conceptions

provide evidence that these constructs can be useful to understand the reasons behind teachers' decisions and classroom behaviours, however, how they develop and operate is still an open question. Overall, the findings in the mathematics knowledge and beliefs/conceptions categories tend to be about teachers' knowledge independent of consideration for its situatedness in practice.

Findings of teachers' practices varied depending on theoretical perspectives. For example, studies framed in cognitive psychology, findings tend to emphasize relation between practice and the need for a strong domain of mathematical knowledge and, in some cases, of PCK. These studies provide important information, but vary widely on which external variables (concerning teachers' content knowledge, PCK or cognitive level) could be a key element to understand teachers' practices. For studies framed in the sociocultural perspective, some offer examples of "good practices", others present the teacher with a critical eye, and still others address tensions that underlie the teachers' activity. The most common conclusion is that teachers need further learning to carry out "better" practices, more aligned with the researchers' espoused perspectives.

### *Theoretical perspectives*

The papers cover an interesting set of theoretical perspectives. Cognitive psychology is an important influence in many of the studies, but specific theoretical influences cited in the papers include beliefs/conceptions frameworks, constructivism, classroom interaction models, and activity theory/situated cognition/ social practice theory. On the surface, this seems to suggest progress in terms of the growth of new waves of research with new theoretical perspectives/frameworks. However, the lifespan of these frameworks varies and besides interest in new ideas, there is no clear justification for abandoning 'old ideas'. The concern is that in the rush to adopt new ideas we may disregard what we had too quickly, which may not help in increasing our understanding of the problems and issues.

The framework of cognitive psychology has been challenged, in the last few years, regarding their implicit conception of the nature of the knowledge of mathematics teaching, as just a matter of individual cognitive ability, "amount" of knowledge or "level of thinking". Alternative views have developed suggesting it is rather a matter of teachers' activity in professional contexts, schools and professional cultures, and social and contextual factors. However, it is difficult to explain individual agency and individual actions just in terms of social, cultural and contextual aspects. How may we combine the social and the individual levels of analysis? Not many papers have dealt explicitly with this issue. A specific related construct that has been subjected to many criticisms is Shulman's notion of PCK. Shulman himself has become a critic of his model, which, in his view (i) lacks emphasis in the level of action, (ii) posits the individual as the unit of analysis and misses the role of the community of teachers, (iii) does not consider affect, motivation or passion, and (iv) needs a starting point broader than just content knowledge, and including students, community, curriculum, etc. (See Boaler, 2003). In fact, the notion of PCK suggests a dominant conception of teacher's knowledge as declarative, rather than action-oriented or imbedded in practice. As a consequence, many of the studies that used this notion viewed the

mathematics teacher as a “deficient professional”, missing “important ideas” essential for their practice.

Other aspects of theoretical influences on the studies reported are the perspectives used for mathematics, curriculum, good teaching, the teacher and practice. In the case of teachers’ mathematics knowledge, although not explicitly stated, many papers seem to treat mathematics in a formalistic way and assume that the curriculum is mainly a collection of mathematics topics, concepts and procedures. However, in later studies, there was evidence of reform orientation of mathematics and curriculum being used. Also, in the case of teacher knowledge of mathematics teaching, there is a sharp contrast between the earlier and later studies regarding the curriculum orientation assumed. At first, the curriculum orientations are implicit, but with time, more and more of the studies assume a reform view about the curriculum, in many cases with explicit references to the NCTM documents and to constructivism. The papers framed in the PCK perspective espouse an implicit model of good teaching. In some cases, good teaching takes into account students thinking as modelled by research in the area; in other cases, it focuses on the very specifics of mathematics knowledge, taking into account research on cognitive processes of constructing such knowledge. Finally, in the case of teachers’ practices, some studies assume a descriptive view of practice, often regarded as “what teachers do”, whereas others undertake a more conceptual approach, problematizing the notion of practice and relating it to a theoretical framework. Most of these papers assume a “reform view” of curriculum, either focusing on a single main idea (such as problem solving or classrooms as inquiry communities) or on a global vision such as proposed by NCTM (1989, 1991). However, some researchers seem to take new curriculum orientations at face value, whereas others tend to problematize them, taking into account the culture and the conditions of the educational setting.

In all of the four categories of teachers’ knowledge and practice, the emergent image of the teacher is that of a professional with a deficient knowledge, in particular, of mathematics and mathematics teaching. Such studies stress what the teacher does not know, does not understand or does not do. Some of these studies are still being carried out today, either framed on a reformed curriculum perspective or on a more traditional view. The expert-novice contrast used in a few studies provides a more positive image, at least for experienced teachers. However, if we look at teachers as professionals, emphasizing the notion of professional knowledge, we may signal the complexity of their knowledge and its intimate relation to their practices. We also suggest that to study teachers’ professional knowledge we need to take into account the subject matter (mathematics), the participants (teacher and students), the explicit and implicit aims (curriculum, social values) and the working conditions (context, institutions).

### *Methodology*

The papers cover an interesting set of methodologies. In all of the four categories of teachers’ knowledge and practice, there was a clear emphasis on questionnaires in the early years, which later moved towards interviews and observations. Some more “intimate” methods of expression and communication also began to appear, for example, teacher diaries and joint reflections of teachers and researchers. While many of the papers focus on either quantitative or qualitative methods, some

researchers are trying to combine the two. Recently, some sophisticated collaborative studies and teacher research studies have been undertaken, combining insider and outsider views and, in some cases, with a heavy use of technology for data gathering. Other innovative ideas for meeting the challenge of studying teachers' knowledge include using concept maps (Leikin, Chazan and Yerushalmy, 2001) and therapy style "focus groups" (Vinner, 1996). Papers by Brown and Coles (1997) and Chapman (1999) also suggest the relevance of narratives in researching teachers. Even though research in this topic seems to have acquired unprecedented power, it still faces the challenge of addressing multisided social and institutional factors and relating them to actual teachers' and students' meanings and experiences.

#### *Future directions*

Research reported on mathematics teachers' activities at PME conferences has grown substantively and is likely to continue to have a significant presence in the future. However, although there has been significant progress in terms of the variety of research topics, perspectives and methodology covered in this area, we offer some ideas for consideration in shaping future directions of this work.

Most of the studies on teachers' mathematics knowledge focused on a particular mathematics fact, concept, or procedure in a way that does not give us a sense of the relationship to practice. While some authors made conjectures about the relationship between the teachers' mathematics knowledge and their ability to teach mathematics meaningfully, few addressed such conjectures. There was also no focus on looking at the knowledge of effective teachers to determine the depth of their knowledge in order to understand what this could look like and whether it reflects theory or conjectures of what is adequate knowledge for teaching. It seems, then, that future work should include a focus on understanding the knowledge the teachers hold in terms of their sense making and in relation to practice.

As we previously pointed out, many of the studies on teachers' beliefs and conceptions also tended to focus on describing the nature of these teacher characteristics as an end in itself with no real connections to other aspects of teachers' activities. This leads us to wonder about the meaningfulness of ongoing studies of teachers' mathematics knowledge and teachers' beliefs and conceptions in themselves, without relating them to other aspects of practice. While we feel that such research has passed its apogee point and is now declining, we also recognize the need for researchers who continue to work in this area to start to follow the lead of those who have shifted to combining these constructs with others related to practice in more creative ways.

Regarding teacher practice, there seems to be an increase in awareness that it is important to analyse the conditions that promote "good" practices, aligned with sound curriculum efforts, looking at the social and institutional conditions in which teachers work and paying special attention to exemplary instances of such practices already taking place. Although such awareness is clear in several authors, not many papers report on studies that do full justice to it, which suggests the need to consider the theoretical and methodological implications of such an undertaking. We need a better grasp regarding how educational, professional and institutional factors influence teachers' practices – noting that this may widely vary from country to country and system to system. For example, some studies report on a

school with a particular culture (e.g., Chazan, Ben-Chaim and Gormas, 1996), but what can we say about many other kinds of schools in which teachers collaborate at different levels? What factors promote such collaborations or weaken them? Also, professional cultures and other cultural and affective factors may strongly affect the teachers' way of being in the profession and, therefore, teachers' practices. How can such factors be studied? Another issue is that in most of the studies the value of practices has been implicitly or explicitly judged by their alignment with the researchers' values or reform curriculum principles. However, what about students' learning? Is this a secondary criterion to analyse teacher's practices, or, as Saxe (1999) seems to suggest, should be a main concern?

There are several other aspects of teachers' practices that would be worthwhile to study. One is mathematics teachers' practices in higher education, a level of education where there is strong failure in mathematics, but which is possible to study fruitfully (e.g., Jaworski, 2001). There is also a need for long-term studies about teachers' practices that could help to put things in perspective, as Sztajn (2002) suggests.

Our ideas for future directions in researching the activity of the teacher imply the need for reconsidering theoretical and methodological orientations. This, then, is also something that requires future attention, in particular, the development or adaptation of innovative research designs to deal with the complex relationships among various variables, situations or circumstances that define teachers' activities.

To conclude, research of teacher knowledge and practices has made extraordinary progress in these 29 years of PME conferences. We expect this progress to continue, but with some shifts in focus that could take us to new ways or levels of understanding the mathematics teacher and the teaching of mathematics.

#### REFERENCES

- Adler, J. (1995). Participatory inquiry pedagogy, communicative competence and mathematical knowledge in a multilingual classroom: A vignette. In L. Meira & D. Carraher (Eds), *Proceedings of the 19<sup>th</sup> PME International Conference*, 3, 208-215.
- Adler, J. (1996). Lave and Wenger's social practice theory and teaching and learning school mathematics. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference*, 2, 3-10.
- Ambrose, R.C., Philipp, R., Chauvot, J., & Clement, L. (2003). A web-based survey to assess prospective elementary school teachers' beliefs about mathematics and mathematics learning: An alternative to Likert scales. In N. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference*, 2, 33-40.
- Andelfinger, B. (1981). Provocative texts and spontaneous reactions of teachers: A method for recognizing teaching and learning of mathematics. In Equipe de Recherche Pédagogique (Eds.), *Proceedings of the 5<sup>th</sup> PME International Conference*, 1, 381-386.
- Anghileri, J. (2002). Scaffolding practices that enhance mathematics learning. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference*, 2, 49-56.
- Askew, M., Brown, M., Rodhes, V., William, D., & Johnson, D. (1997). Effective teachers of numeracy in UK primary schools: Teachers' beliefs, practices and pupils' learning. In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME International Conference*, 2, 25-32.
- Barkai, R., Tsamir, P., Tirosh, D., & Dreyfus, T. (2002). Proving or refuting arithmetic claims: The case of elementary school teachers. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference*, 2, 57-64.
- Beswick, K. (2004). The impact of teachers' perceptions of student characteristics on the enactment of their beliefs. In M.J. Hoines & A.B. Fuglestad (Eds.), *Proceedings of the 28<sup>th</sup> PME International Conference*, 2, 111-118.

- Boaler, J. (2003). Studying and capturing the complexity of practice: The case of the dance of agency. In N. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference, 1*, 3-16.
- Bottino, R., & Furinghetti, F. (1994). Teaching mathematics and using computers: Links between teachers beliefs in two different domains. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference, 2*, 112-119.
- Braconne, A., & Dionne, J.J. (1987). Secondary school students' and teachers' understanding of demonstration in geometry. In J. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11<sup>th</sup> PME International Conference, 3*, 109-116.
- Bright, G.W., Bowman, A.H., & Vacc, N.N. (1997). Teachers' frameworks for understanding children's mathematical thinking. In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME International Conference, 2*, 105-112.
- Brissiaud, R., Moreau, J.P., Perrot, G., Valentin, D., & Vaudy, J. (1982). Representation des problèmes par le maître et par l'élève à l'école élémentaire. In A. Vermandel (Ed.), *Proceedings of the 6<sup>th</sup> PME International Conference, 2*, 84-90.
- Brown, L., & Coles, A. (1997). The story of Sarah: Seeing the general in the particular? In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME International Conference, 2*, 113-120.
- Carpenter, T.P., & Fennema, E. (1989). Building on the knowledge of students and teachers. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of 13<sup>th</sup> PME International Conference, 1*, 34-45.
- Carrillo, J., & Contreras, L.C. (1994). The relationship between the teachers' conceptions of mathematics and of mathematics teaching: A model using categories and descriptors for their analysis. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference, 2*, 152-159.
- Chapman, O. (1994). Teaching problem solving: a teachers' perspective. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference, 2*, 168-175.
- Chapman, O. (1999). Researching mathematics teacher thinking. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference, 2*, 185-192.
- Chapman, O. (2004). Facilitating peer interactions in learning mathematics: Teachers' practical knowledge. In M.J. Høines & A.B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference, 2*, 191-198.
- Charalambous, C., Philippou, G., & Kyriakides, L. (2002). Towards understanding teachers' philosophical beliefs about mathematics. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference, 2*, 217-224.
- Chazan, D., Ben-Chaim, D., Gormas, J. (1996). Overcoming isolation in teaching. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference, 2*, 217-224.
- Chazan, D., Larriva, C., & Sandow, D. (1999). What kind of mathematical knowledge supports teaching for "conceptual understanding"? Preservice teachers and the solving of equations. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference, 2*, 193-200.
- Connelly, F. M., & Clandinin, D. J. (1990). Stories of experience and narrative inquiry. *Educational Researcher, 14*(5), 2-14.
- Dougherty, B.J. (1990). Influences of teacher cognitive/conceptual levels on problem-solving instruction. In G. Booker, P. Cobb & T. Mendicuti. (Eds.), *Proceedings of the 14<sup>th</sup> PME International Conference, 1*, 119-126.
- Elbaz, F. (1983) *Teacher thinking: A study of practical knowledge*. London: Croom Helm.
- Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer.
- Escudero, I., & Sánchez, V. (1999). The relationship between professional knowledge and teaching practice: The case of similarity. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference, 2*, 305-312.
- Escudero, I., & Sánchez, V. (2002). Integration of domains of knowledge in mathematics teachers' practice. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference, 4*, 177-185.
- Even, R. (1990). The two faces of the inverse function: Prospective teachers' use of undoing. In G. Booker, P. Cobb & T. Mendicuti (Eds), *Proceedings of the 14<sup>th</sup> PME International Conference, 1*, 37-44.
- Even, R., & Markovits, Z. (1991). Teachers' pedagogical knowledge: The case of functions. In F. Furinghetti (Ed.), *Proceedings of the 15<sup>th</sup> PME International Conference, 2*, 40-47.
- Even, R., & Schwarz, B.B. (2002). Implications of competing interpretations of practice to research and theory in mathematics education. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference, 2*, 337-344.
- Fernandes, D., & Vale, I. (1994). Two young teachers' conceptions and practices about problem solving. In J.P. Ponte, J.F. Matos (Eds.), *Proceedings of 18<sup>th</sup> PME International Conference, 2*, 328-

- Gal, H. (1998). What do they really think? What students think about the median and bisector of an angle in the triangle, what they say and what their teachers know about it. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22<sup>nd</sup> PME International Conference*, 2, 321-328.
- Gal, H., & Vinner, S. (1997). Perpendicular lines: What is the problem? Pre-service teachers lack of knowledge on how to cope with students' difficulties. In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME International Conference*, 2, 281-288.
- Gates, P. (2001). Mathematics teacher belief systems: Exploring the social foundations. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> PME International Conference*, 3, 17-24.
- Greer, B., & Mangan, C. (1986). Choice of operations: From 10-year-olds to student teachers. In Univ. of London Inst. of Educ. (Eds.), *Proceedings of the 10<sup>th</sup> PME International Conference*, 1, 25-30.
- Grouws D.A., Good T.A., & Dougherty, B.J. (1990). Teacher conceptions about problem solving and problem solving instruction. In G. Booker, P. Cobb & T. Mendicuti (Eds.), *Proceedings of the 14<sup>th</sup> PME International Conference*, 1, 135-142.
- Groves, S., & Doig, B. (2004). Progressive discourse in mathematics classes? The task of the teacher. In M.J. Høines & A.B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference*, 2, 495-502.
- Hansson, O. (2005). Preservice teachers' view on  $y = x + 5$  and  $y = x^2$  expressed through the utilization of concept maps: A study of the concept of function. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29<sup>th</sup> PME International Conference*, 3, 97-104.
- Harel, G., & Dubinsky, E. (1991). The development of the concept of function by preservice secondary teachers: From action conception to process conception. In F. Furinghetti (Ed.), *Proceedings of the 15<sup>th</sup> PME International Conference*, 2, 133-140.
- Hershkowitz, R., & Vinner, S. (1984). Children's concepts in elementary geometry. A reflection of teacher's concepts? In B. Southwell (Eds.), *Proceedings of the 8<sup>th</sup> PME International Conference*, 63-69.
- Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.
- Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, 19, 333-355.
- Hoyle, C. (1992). Illuminations and reflections: Teachers, methodologies and mathematics. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16<sup>th</sup> PME International Conference*, 3, 263-286.
- Jaworski, B. (1991). Some implications of a constructivist philosophy for the teacher of mathematics. In F. Furinghetti (Ed.), *Proceedings of the 15<sup>th</sup> PME International Conference*, 2, 213-220.
- Jaworski, B. (2001). University mathematics teaching: Where is the challenge? In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> PME International Conference*, 3, 193-200.
- Jaworski, B., & Potari, D. (1998). Characterising mathematics teaching using the teaching triad. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22<sup>nd</sup> PME International Conference*, 3, 88-95.
- Khisty, L.L. (2001). Effective teachers of second language learners in mathematics. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> PME International Conference*, 3, 225-232.
- Khisty, L.L., McLeod, D., & Bertilson, K. (1990). Speaking mathematically in bilingual classrooms: An exploratory study of teacher discourse. In G. Booker, P. Cobb, & T. Mendicuti (Eds.), *Proceedings of the 14<sup>th</sup> PME International Conference*, 3, 105-112.
- Kynigos, C., & Argyris, M. (1999). Two teachers' beliefs and practices with computer based exploratory mathematics in the classroom. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference*, 3, 177-184.
- Klein, R., & Tirosh, D. (1997). Teachers' pedagogical content knowledge of multiplication and division of rational numbers. In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME International Conference*, 3, 144-152.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Leikin, R., Chazan, D., & Yerushalmy, M. (2001). Understanding teachers approaches to school algebra: Contributions of concept maps as part of clinical interviews. In M. van den V-Panhuizen (Ed.), *Proceedings of the 25th PME International Conference*, 3, 289-296.
- Llinares, S., & Sánchez, M.V. (1991). The knowledge about unity in fractions tasks of prospective elementary teachers. In F. Furinghetti (Ed.), *Proceedings of the 15<sup>th</sup> PME International Conference*, 2, 334-341.
- Linchevsky, L., & Vinner, S. (1989). Canonical representations of fractions as cognitive obstacles in elementary teachers. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of the 13<sup>th</sup> PME International Conference*, 2, 242-249.
- Manouchehri, A. (2003). Factors motivating reform: Learning from teachers' stories. In N. Pateman,

- B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference*, 3, 221-228.
- Mastorides, E., & Zachariades, T. (2004). Secondary mathematics teachers' knowledge concerning the concept of limit and continuity. In M.J. Hoines & A.B. Fuglestad (Eds.), *Proceedings of the 28<sup>th</sup> PME International Conference*, 4, 481-488.
- Mendick, H. (2002). 'Why are we doing this?': A case study of motivational practices in mathematics classes. In A.D.E. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference*, 3, 329-336.
- McDonough, A., & Clarke, D. (2003). Describing the practice of effective teachers of mathematics in the early years. In N. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference*, 3, 261-268.
- National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.
- Oprea, J.M., & Stonewater, J. (1987). Mathematics teachers' belief systems and teaching styles: Influences on curriculum reform. In J. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11<sup>th</sup> PME International Conference*, 1, 156-162.
- Owens, J.E. (1987). Personal constructs of mathematics and mathematics teaching. In J. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11<sup>th</sup> PME International Conference*, 1, 163-169.
- Pajares, M.F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, 62(3), 307-332.
- Perry, W.G. (1981). Cognitive and ethical growth: The making of meaning. In A. Chickering (Ed.), *The modern American College* (pp. 76-116). San Francisco, CA: Jossey-Bass.
- Philippou, G., & Christou, C. (1994). Prospective elementary teachers' conceptual and procedural knowledge of fractions. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference*, 4, 33-40.
- Pinto, M., & Tall, D. (1996). Student teachers' conceptions of the rational number. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference*, 4, 139-146.
- Ponte, J.P. (1985). Geometrical and numerical strategies in students' functional reasoning. In L. Streefland (Ed.), *Proceedings of the 9<sup>th</sup> PME International Conference*, 1, 413-418.
- Ponte, J.P. (1990). Teachers' perceived roles of the computer in mathematics education. In G. Booker, P. Cobb & T. Mendicuti (Eds.), *Proceedings of the 14<sup>th</sup> PME International Conference*, 1, 183-190.
- Ponte, J.P. (1994). Mathematics teachers' professional knowledge. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference*, 1, 195-210.
- Presmeg, N., & Nenduradu, R. (2005). An investigation of a preservice teacher's use of representations in solving algebraic problems involving exponential relationships. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29<sup>th</sup> PME International Conference*, 4, 105-112.
- Rees, R. (1982). The teacher and diagnosis: Too much Piaget? In A. Vermandel (Ed.), *Proceedings of the 6<sup>th</sup> PME International Conference*, 2, 91-96.
- Robinson, N., Even, R., & Tirosh, D. (1992). Connectedness in teaching algebra: A novice-expert contrast. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16<sup>th</sup> PME International Conference*, 2, 258-263.
- Robinson, N., Even, R., & Tirosh, D. (1994). How teachers deal with their students' conceptions of algebraic expressions as incomplete. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference*, 4, 129-136.
- Rossouw, L., & Smith, E. (1998). Teachers' pedagogical content knowledge of geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of 22<sup>nd</sup> PME International Conference*, 4, 57-63.
- Rota, S., & Leikin, R. (2002). Development of mathematics teachers' proficiency in discussion orchestrations. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of 26<sup>th</sup> PME International Conference*, 4, 137-145.
- Sánchez, V., & Llinares, S. (1992). Prospective elementary teachers pedagogical content knowledge about equivalent fractions. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16<sup>th</sup> PME International Conference*, 2, 274-275.
- Saxe, G.B. (1999). Professional development, classroom practices, and students' mathematics learning: A cultural perspective. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference*, 1, 25-39.
- Schön, D.A. (1983). *The reflective practioner: How professionals think in action*. Aldershot Hants: Avebury.
- Shriki, A., & David, H. (2001). How do mathematics teachers (inservice and preservice) perceive the concept of parabola? In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> PME International Conference*, 4, 169-176.

- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Simmt, E., Davis, B., Gordon, L., & Towers, J. (2003). Teachers' mathematics: Curious obligations. In N. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference*, 4, 175-182.
- Simon, M.A. (1990). Prospective elementary teachers' knowledge of division. In G. Booker, P. Cobb & T. Mendicuti (Eds.), *Proceedings of the 14<sup>th</sup> PME International Conference*, 3, 313-320.
- Simon, M.A. (1991). Initial development of prospective elementary teachers' conceptions of mathematics pedagogy. In F. Furinghetti (Ed.), *Proceedings of the 15<sup>th</sup> PME International Conference*, 2, 40-47.
- Simon, M., & Tzur, R. (1997). Generalizing theoretical accounts of mathematics teachers' practices. In H. Pekhonen (Ed.), *Proceedings of the 21<sup>st</sup> PME Conference*, 4, 160-167.
- Skott, J. (1999). The multiple motives of teacher activity and the roles of the teachers school mathematical images. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference*, 4, 209-216.
- Sztajn, P. (2002). Changes in mathematics teaching: Learning with an experienced elementary teacher. In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference*, 4, 257-264.
- Tanner, H., & Jones, S. (1999). Dynamic scaffolding and reflective discourse: The impact of teaching style on the development of mathematical thinking. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> PME International Conference*, 4, 257-264.
- Thomas, M. (2003). The role of representation in teacher understanding of function. In N. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27<sup>th</sup> PME International Conference*, 4, 291-298.
- Thompson, A.G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 127-146). New York, NY: Macmillan.
- Tirosh, D., Graeber A., & Glover R. (1986). Pre-service teachers' choice of operation for multiplication and division word problems. In Univ. of London Inst. of Educ. (Eds.), *Proceedings of the 10<sup>th</sup> PME International Conference*, 1, 57-62.
- Tzur, R. (2002). From theory to practice: Explaining successful and unsuccessful teaching activities (case of fractions). In A.D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26<sup>th</sup> PME International Conference*, 4, 297-304.
- Tzur, R., Simon, M.A., Heinz, K., & Kinzel, M. (1998). Meaningfully assembling mathematical pieces: An account of a teacher in transition. In A. Olivier & Newstead, (Eds.), *Proceedings of the 22<sup>nd</sup> PME International Conference*, 4, 145-152.
- van Dooren, W., Verschaffel, L., & Onghena, P. (2001). Arithmetic or algebra? Pre-service teachers' preferential strategies for solving arithmetic and algebra word problems. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> PME International Conference*, 4, 359-366.
- Valero, P., & Gómez, C. (1996). Precalculus and graphic calculators: The influence on teacher's beliefs. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference*, 4, 363-370.
- Vinner S., & Hershkowitz R. (1980): Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4<sup>th</sup> PME International Conference*, 1, 177-184.
- Vinner, S. (1996). Some psychological aspects of professional lives of secondary mathematics teachers: The humiliation, the frustration, the hope. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference*, 4, 403-410.
- Wood, T. (1996). Teaching to create discussion as mathematical argumentation. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> PME International Conference*, 4, 427-434.
- Wood, T. (1998). Differences in teaching for conceptual understanding of mathematics. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22<sup>nd</sup> PME International Conference*, 4, 193-200.
- Zazkis, R., & Campbell, S. (1994). Divisibility and division: Procedural attachments and conceptual understanding. In J.P. Ponte & J.F. Matos (Eds.), *Proceedings of the 18<sup>th</sup> PME International Conference*, 4, 423-430.

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