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# EPISTEMIC CONFIGURATIONS ASSOCIATED TO THE NOTION OF EQUALITY IN REAL NUMBERS[1]

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RESUME

Les objets émergents des systèmes de pratiques mathématiques dans les différents contextes d'utilisation sont structurés par des *configurations épistémiques*. La détermination et la description des configurations épistémiques, associées à la notion d'égalité des nombres réels, nous permet d'introduire la notion d'*holo-signifié* d'une notion mathématique. La notion d'holo-signifié est constituée par l'interaction des différents modèles mathématiques associés à cette notion. Les notions d'holo-signifié et de modèle constituent un cadre pour la sélection des signifiés à enseigner par rapport au curriculum et pour la recherche de situations fondamentales dans un projet global d'enseignement.

ABSTRACT

The emergent objects of systems of mathematical practices, in different contexts of use, are structured through *epistemic networks*. The determination and the description of the epistemic networks associated to the notion of equality in real numbers are used to establish the idea of *holistic-meaning,* which is formed by the interaction of different mathematical *models* associated to a mathematical notion. The notions of model and holistic-meaning establish the referential framework in the selection of curricular meanings used in teaching, as well as in the search for fundamental situations within a global educational project.

RESUMEN

Los objetos emergentes de los sistemas de prácticas matemáticas en los distintos contextos de uso se estructuran formando *configuraciones epistémicas*. La determinación y la descripción de las configuraciones epistémicas asociadas a la noción de igualdad de números reales nos permite introducir la noción de *holo-significado* de una noción matemática, constituido por la interacción de distintos *modelos* matemáticos asociados a dicha noción. Las nociones de holo-significado y de modelo constituyen un marco para la selección de los significados curriculares que se pretenden enseñar y para la búsqueda de situaciones fundamentales dentro de un proyecto global de enseñanza.

**Key Words**: definition, model, meaning, system of practices, curriculum, praxeology, fundamental situation.

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## 1. Theoretical MOTIVAtioN and general PLAN

One of the major problems that research in mathematics education confronts today, if not the most important, is the analysis of the processes of construction and communication of mathematical knowledge by individuals and institutions. In fact, in a generic way, it can be said that the "didactics of mathematics" as a scientific discipline represents "a fundamental theory of communication of mathematical knowledge" (Brousseau, 1998, p.358). This objective brings with it the need to differentiate and describe the mathematical notions, processes and meanings that must be taught. In particular, it is necessary to determine the meanings associated to mathematical objects in different contexts within academic institutions, and organize them as a complex and coherent totality.

Godino and Batanero (1994) introduce the notion of "system of operative and discursive practices associated to a class of problems in which a mathematical object is put into play" as the primary focus of attention when describing the institutional and personal meaning of such mathematical objects. In this work we are especially interested in determining and describing the relation between the systems of practices, the emergent objects of these systems, and the relations that are established between the objects (which should be taken into account in the analysis of the meaning of the mathematical notions).

Godino (2002) identifies the "system of practices" with the content that an institution assigns to a mathematical object, establishing in this manner a correspondence between the system of practices (the systemic meaning) and the expression of the mathematical object. In this work, the description of the meaning of a mathematical object is presented through a list of specific objects that are classified in six categories: problems, procedures, languages, notions, properties and arguments.

We consider that this description of a system of practices is insufficient, for several reasons. In the first place, the categories that are mentioned are emergent objects of the system of practices in which the mathematical object is put into play and, for this reason, the objects refer explicitly to the institutional meaning.

> "The meaning begins by being pragmatic, relative to the context, but there exist types of uses that allow the orientation of the processes of the teaching and learning of mathematics. These types of uses are objectified by means of language, and they are the referents of the institutional lexicon." (Godino, 2003, 38).

In second place, both the systems of practices and the emergent objects are related amongst themselves, forming epistemic networks or configurations; the description of these networks should be the objective of the epistemological analysis of a mathematical notion, from the perspective of the teaching and learning of mathematics. In third place, the global teaching project can be divided in subsystems of practices linked to specific types of problems; to elaborate curriculum and construct teaching projects it is necessary to identify and describe both the subsystems of practices and the emergent objects of any project in question.

We ask ourselves, based on these considerations:

- Is it possible to structure, in a coherent system, the different definitions of a mathematical notion that emerge in the midst of different subsystems of practices in specific contexts?
- What does it mean to understand a mathematical notion?
- Does the description of a notion such as "totality" have consequences in the development of curriculum and, in particular, can the analysis of applications of educational proposals in relation to such a notion be carried out?

As a response to these questions, and to center the ideas on the notion of equality[2], we introduce the concepts of *model* and of *holistic-meaning* of a mathematical notion. Briefly, the *model* of a mathematical notion represents the structured complex of a system of practices in a specific *context of use*[3] and the objects that

[2] In this text, the term equality will be employed as a synonym of "equality of real numbers".

[3] In a first approximation, the contexts of use can be identified with the notion of *framework* introduced by Douady (1986, 10). However, this approximation does not take into account the characterization of a framework according to

emerge within those systems (including definitions); the *holistic-meaning* of a mathematical notion represents the expression of the diversity of models associated to that notion (understood as a single system). In the same way, the notions of holistic-meaning and model allow us to analyze the notion of *praxeology* from the Anthropological Theory of Didactics (TAD) (Chevallard, 1997) in relation to mathematical practice, and frame the search for *fundamental situations* (Brousseau, 1998) in a global teaching project; concretely, we will show that it is good for a fundamental situation to include a representative sample of the models that make up the holistic-meaning (although frequently that representativity will have to be restricted to some models associated to the mathematical notion introduced or developed)

In relation to the notion of equality, the objective of this paper is to show how the different contexts of use delimit specific meanings, which are synthesized in different definitions of these notions, without it being possible to privilege any of them. Thus, in section 2 we introduce different definitions of the notion of equality; illustrating with the proof of the proposition $\sqrt{2} = \frac{2}{\sqrt{2}}$, we indicate how these definitions condition the mathematical practices in the different contexts of use[4]. In section 3, after comparing the modelling of mathematical activity in the Anthropological Theory of Didactics (by means of the notion of "praxeology") and in the Onto-Semiotic approach (by means of the notion of "operative and discursive systems of practices"), we briefly describe the primary entities and the type of language associated with the notion of equality in different contexts of use.

In section 4 we make the structuring of the models and the meanings associated to the notion of equality, explicit. En section 5 the notion of holistic-meaning is introduced and the holistic-meaning of the notion of equality is described. Then some curricular implications of the generic notion of holistic meaning are analyzed (section 6). Finally, in section 7, some implications are highlighted and classified by their *macro-didactical* nature (referent to the evolution of the fundamental questions about the institutional, social, and cultural state of mathematical objects), *micro-didactical* nature (where the singularity of the mathematical objects and the individuality of the subjects prevail) and *theoretical* nature (related to the tools and techniques introduced together with the didactical notions as accepted within the scientific community).

## 2. DEFINITIONS OF THE NOTION OF EQUALITY

From the strictly *formal* and *official* viewpoint (Brown, 1998) it is accepted that the definition of a mathematical object forms its meaning, given that the definition points to the object's unmistakable situation in the universe of mathematical objects in which it is introduced. "Every definition is a classification. It separates the objects that satisfy the definition and those that do not, and it situates them in two different classes" (Poincaré, in Lorenzo, 1974, 58). Then, to introduce the notion of equality, it is sufficient to state: the sign '=' (equals) indicates that what is found to the left of this sign, the first member of the equality, and what is

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the type of objects (notions, processes and meanings) that are genuinely representative of the framework. This characterization requires the consideration of a reference institution given that, for example, it is not possible to associate the same problems, notions, properties, arguments, procedures and language to "elementary algebra in schools" and to "formal algebra in the university". For this reason, in a second approximation, the contexts of use can be considered as *frameworks of the organization of mathematical notions and propositions that determine the type of arguments, procedures and language that is admissible and pertinent to put into action in a specific institution, referent to a type of problem.*

[4] The mathematical definitions and propositions are the most visible part of the anthropological and cultural reality of mathematics (that which is susceptible of being explicitly reconstructed and communicated). The mathematical definitions are a part of the systems of mathematical practices, a discursive component of such practices that, from the ontogenic point of view, are dependent and "posterior" to the operative practices. The definitions interact in a complex and recursive manner with the problems, the previously established propositions, the type of arguments and language relative to the operational action and the discourse (in relation to a specific mathematical notion). This interaction gives way to new questions and new systems of practices; in this sense, we can affirm that "the definitions condition the mathematical practices in the different contexts of use".

found to the right of this sign, called the second member of the equality, are two ways of designating the same object, or two different ways of writing the same thing. This description of the notion of equality does not reference explicitly any system of mathematical practices or mathematical context of use. The academic institutions usually accept (irreflexively) that students possess the capacity to adapt this formal definition of the notion of equality to different contexts of use, producing the phenomenon called the *illusion of transparency*.

When students are not told that the "=" sign can have different meanings, the presence of didactical transposition can be ascertained; In fact, this can be explained by the desire to simplify: the student is led, implicitly to believe that the "=" sign always has the same status. However, in the different academic curricula, explicit study about the different statuses of the "=" sign is not carried out. Often those who teach may not be really conscious of the different functions of the "=" sign. This might explain the reason why de-transposition is not carried out when it is necessary. Indeed, we insist, to de-transpose a notion it is essential to have understood what the initial transposition of the notion consisted of (Antibi and Brousseau, 2000, 31)[5].

> "Although quantitative sameness is conventionally encoded in the equal symbol, it had not been necessarily so interpreted by the students. Hence, it was clear that the children needed to experience a variety of numerical equalities to continue their progressive understanding of the meanings of the equal sign." (Sáenz-Ludlow & Walgamuth, 1998, 182).

The definitions of equality represent the emergent objects of the systems of practices associated with the different contexts of use; they are not, in any case, a finished product of the meaning attributed to this notion. To justify that $a$ and $b$ represent the same number, it is necessary to make a context of use explicit: numerical, arithmetic, algebraic, analytic or topological. This way, according to the context of use, the equality between two numbers $a$ and $b$ ($a = b$) is determined by the specific relations that are given in these contexts.

In this section we will give the definitions of the notion of equality according to the mathematical context, as well as to certain properties of real numbers; we will carry out a brief discussion of the properties, which contribute to their correct interpretation; finally, we will show how the given definitions condition the operative and discursive practices.

### 2.1. Definitions

The definition of equality as equivalence classifies a set (the real numbers) in classes $\mathbf{R}/=$; in other words, the representation of the number is not important, what counts is the value that the number takes on: ($\frac{2}{4} \equiv \frac{1}{2}; 1 \equiv 0.\overline{9}$; etc.); we are not interested in determining how the classes became defined (Cauchy sequences, Dedekind cuts, etc.). They just are defined as:

> *Definition 1 (Equality as equivalence)* Two real numbers $a$ y $b$ are equal, denoted as $a = b$, if they represent the same class; that is:

$$a = b \Leftrightarrow a \equiv b$$

The equality of two real numbers $a$ y $b$ can also be established by a double inequality; $\mathbf{R}$, equipped with the operations sum (+) and product (·) and with the relation less than or equal to (≤), is an ordered field. It is defined:

[5] *In French in the original*: "Le fait de ne pas signaler aux élèves de Collège que le signe «=» peut avoir des significations différentes, relève de la transposition didactique; en effet, ceci peut s'expliquer par un souci de simplification: on laisse ainsi croire à l'élève, implicitement, que le signe «=» a toujours le même statut. Or, dans le cursus scolaire d'un élève, aucune étude explicite sur les différents status du signe «=» n'est effectuée. Les professeurs eux-même peuvent alors, souvent, ne pas avoir vraiment conscience des différents rôles du signe «=». Ceci peut expliquer que la dé-transposition n'est pas effectuée lorsqu'elle devient nécessaire. En effet, répétons-le, pour pouvoir dé-transposer convenablement une notion, il faut avoir bien compris en quoi consistait la transposition initiale de cette notion." (Antibi et Brousseau, 2000, 31).

*Definition 2 (Equality of order)* Two real numbers $a$ and $b$ are equal, denoted as $a = b$, if the order relation on **R** (≤) has the antisymmetric property, that is:

$$a = b \Leftrightarrow [\, a \leq b \wedge b \leq a]$$

Or equivalently:

$$a = b \Leftrightarrow (\mathrm{a} \in (-\infty; b] \wedge b \in (-\infty; a])$$

Absolute value equips the set of real numbers with a metric (standard). The distance between two numbers $a$ and $b$ is defined and denoted $d(a; b)$, as the absolute value of the difference ($|a - b|$). It is defined:

*Definition 3 (metric equality)* Two real numbers $a$ y $b$ are equal and denoted $a = b$, if the distance between them is null; that is:

$$a = b \Leftrightarrow d(a; b) = |a - b| = 0$$

The absolute value metric can be interpreted as a topology on **R**, in which case (**R**; $d$) is a topological space; in this context, to affirm that the distance between two points $a$ y $b$ is zero is equivalent to determining that the set $\{a; b\}$ is connected[6]. This is defined as:

*Definition 4 (connective equality)* Two real numbers $a$ y $b$ are equal, denoted as $a = b$, if the set $\{a; b\}$ is connected.

The algebraic definition supposed the determination of a number as the solution of an equation. We will denote by δ() the characteristic function that associates 1 to a true sentence and 0 to a false one; and by $E$() the relation associated to an equation $E$. This way, $\delta(E(a)) = 1$ means that the value $a$ verifies the relation $E$() or, in other words, $a$ is the solution to the equation $E$. In the same way, $\delta(E(a)) = 0$ means that the value $a$ does not verify the relation $E$(), that is, $a$ is not a solution to the equation $E$. It is defined:

*Definition 5 (Algebraic equality)* Two real numbers $a$ y $b$ are equal, denoted $a = b$ if, when $a$ is a solution to an equation $E$, $b$ also is a solution:

$$a = b \Leftrightarrow [\delta(E(a)) = 1 \Leftrightarrow \delta(E(b)) = 1]$$

Equality between real numbers can be defined as well by turning to the theory of functions. In effect, to determine if two real numbers are equal it is sufficient to determine if their images with respect to an injective function are equal. This is defined:

*Definition 6 (functional equality)* Let $F_i(D)$ be the set of real injective functions with domain $D$. Two real numbers $a$ and $b$ are equal, denoted as $a = b$, if their respective images with respect to an injective function are equal; that is:

$$a = b \Leftrightarrow \exists\, f \in F_i(D),\ \{\mathrm{a}; b\} \subseteq D, \text{ such that } f(a) = f(b)$$

In the previous definition, if $f$ is the identity function, a semantic tautology is established. The distinction is made between logical tautology and semantic tautology. A *logical tautology* is an affirmation of the type $a = \bar{a}$, where $a$ and $\bar{a}$ represent the same object, without their having the same ostensive representative (for example, $a > 0$, $\sqrt{a} = \frac{a}{\sqrt{a}}$). The *semantic tautology* is a logical tautology that demands the equivalence both of the object and its ostensive representative (for example, $a > 0$, $\sqrt{a} = \sqrt{a}$). A semantic tautology is self-evident; a logical tautology does not have to be. In practice, the proof of the equality of two numbers, given by different ostensive representatives, implies the determination of an injective function that is different from the identity function.

[6] "By topology we understand the study of those qualitative aspects of spatial forms, or of the laws of connectivity, of the mutual position and order of points, lines, surfaces, volumes, as well as their parts and unions, making abstraction of measure and magnitude" (Listing (1836), in Ayala et al. (1997, p. ix)).

In the context of mathematical analysis, the equality is substituted by the intersection of a whole uncountable class of inequalities or neighborhoods. It is defined as:

*Definition 7 (Equality as the process of taking the limit)* Two real numbers $a$ y $b$ are equal, denoted as $a = b$, if $a$ belongs to every open neighborhood centered at $b$ ($B(b; \varepsilon)$) or vice versa; that is:

$$a = b \Leftrightarrow \forall\ \varepsilon > 0,\ a \in B(b;\ \varepsilon) \Leftrightarrow \forall\ \varepsilon > 0,\ b \in B(a;\ \varepsilon)$$

Or, equivalently:

$$a = b \Leftrightarrow \forall\ \varepsilon > 0,\ |a - b| < \varepsilon$$

Finally, the numerical definition of equality presupposed the acceptance of a margin of error that depends on the nature of the problem, or is attributed to the instrument with which the calculations are made. The rupture with the previous definitions is radical from the formal point of view; its inclusion has *pragmatic* reasons (restrictions on measuring and computing, instruments to do calculations —calculators, computer program) and *epistemological* (notions like *sufficient approximation* and neighborhood, *monad*, etc. —non-standard analysis).

*Definition 8 (numerical equality)* Let $T > 0$ be the admitted error tolerance; two real numbers $a$ y $b$ are equal, denoted as $a = b$, if $a$ belongs to an open neighborhood centered at $b$ with radius less than or equal to $T$ ($B(b; t)$, $0 < t \leq T$) or vice versa; that is:

$$a = b \Leftrightarrow \forall\ t > 0,\ t \leq T,\ a \in\ B(b;\ t)$$

$$\Leftrightarrow \forall\ t > 0,\ t \leq T,\ b \in\ B(a;\ t)$$

Or, equivalently:

$$(a = b) \Leftrightarrow |a - b| \leq T$$

### 2.2. Brief analysis of the previous definitions

The objective of the following analysis is to clarify the previous definition in order for a correct interpretation. A concise confrontation of the different definitions will be carried out at the end of section 2.3.

The arithmetic definition takes us to the "identity of a name". In other words to show, basing oneself on the arithmetic definition, that two expressions represent the same number, transformations are made that preserve equality, until a *semantic tautology* is obtained, that is, the same ostensive representative for both number.

The metric and order definitions give criteria for procedures that show the equality of two real numbers; they represent an interpretation of the arithmetic definition in function of certain characteristics attributed to **R** (ordered field, metric space). This way, theoretically, a two step process is used to identify real numbers: **R** is given a property (order, metric) and, in terms of the property, the equality (or inequality) of two numbers is established.

The algebraic definition of equality is founded on what, traditionally, has been named *conditional equality*: the equality only is true for certain values of the variable. It is logical to associate to every equation the set of solutions or values that make the equality true and, in an indirect way, define a number as the solution of a class of equations. To talk of a "class of equations" is strictly necessary, given that infinite equations have a certain real number as a solution, and only the set distinguishes that certain real number. In fact, the definition results more operational if it is formulated as a negation:

$$a \neq b \Leftrightarrow \exists\ E \text{ tal que } [\delta(E(a)) = 1 \Leftrightarrow \delta(E(b)) = 0]$$

In other words, two numbers $a$ and $b$ are different if an equation $E$ is known such that $a$ is a solution and $b$ is not; and, vice versa, if $b$ is a solution and $a$ is not. An equation $E$ definitely exists for which $a$ and $b$ are not simultaneously solutions.

The definition as a process of taking the limit does not lead us to the identity of a name, but to a reasoning process by sufficient conditions and a controlled loss of information, through chains of inequalities. This fact determines a radical difference between the analytic proofs and the algebraic ones.

It was in this vein that Newton wrote in *Philosophiae Naturalis Principia Mathematica*, in an analytic passage: "Quantities, and the ratio of quantities, that in any finite interval of time converge continuously to equality and that, before the end of this time approximate each other more than any given difference, are finally equal.", quoted by Boyer (1969, 500).

The functional definition relates a genuinely analytic concept (function) and the solution of an equation. In effect, let $f$ be an injective function and consider the equation $f(x) = h$; then, if $a$ and $b$ are solutions to the equation, that is, the relations $f(a) = h$ and $f(b) = h$ are true, necessarily $a = b$. This way, the analysis of the equation $f(x) = h$ in terms of the properties of the associated function $f$ determine a sufficient condition for algebraic equality: it is not necessary to verify that, for every relation $E()$, "$\delta(E(a)) = 1 \Leftrightarrow \delta\,(E(b)) = 1$", it is enough to find a homogenous[7] relation $E^*()$, such that $y = E^*(x)$ is injective, where $E^*() \equiv f() - h$.

This way, the functional definition does not explicitly involve the limit notion, which is central in the model of analytic equality. However, if the function $f$ is continuous, it is possible to make that notion explicit. In effect, if $f$ is continuous at the point $a$, $f\,(a) = h$, then:

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ tal que si } |b - a| < \delta \Rightarrow |f\,(b) - f(a)| < \varepsilon$$

$$\Leftrightarrow \text{lím}_{b\to a} f(b) = f(a)$$

The numerical definition of equality can be understood as a "restriction" of the analytic definition, when understood as the process of taking the limit: it fixes orders of approximation or establishes admissible neighborhoods of inclusion. In other words, two analytically equal numbers are numerically equal for any margin of error. On the other hand, the numerical definition can be also be given in terms of solving equations; in effect, if we denote the characteristic function that associates 1 to a true sentence and 0 to a false one by $\delta()$, and by $E(,T)$ the relation associated to an equation $E$ with an order of approximation $T$, two numbers $a$ y $b$ are numerically equal (with an order of approximation $T$) if:

$$a = b \Leftrightarrow [\delta\,(E(a,\,T)) = 1 \Leftrightarrow \delta\,(E(b,\,T)) = 1]$$

This way, the numerical equality totters between the algebraic equality and analytic equality as taking the limit.

Finally, each definition of equality that has been introduced in section 2.1 is an emergent notion of a system of mathematical practices relative to a class of specific problems that include linguistic objects, notions and concrete operational techniques. These systems of practices are differentiated one from the other by their relative efficiency and generality in carrying out mathematical work, as will be exemplified in section 2.3.

### 2.3. Influence of the definitions in mathematical work: proof of the proposition "$\sqrt{2} = \frac{2}{\sqrt{2}}$"

The act of defining consists of the establishment of a set of necessary and sufficient conditions that allow the unmistakable differentiation of an object within a universe. In many circumstances, this differentiation is carried out by means of formalization; a formalization that has made certain authors assert that *to define in mathematics is to give a name* (Leikin & Winicki-Landman, 2000). This perspective supposes the assertion

[7] In this context an equation is homogeneous when one of its members is zero.

that the formal definition differentiates the mathematical object; it is its "measure". However, the same mathematical object can be defined by means of equivalent forms. Two definitions are equivalent if they designate the same object; it is not possible to privilege *a priori* any of them. The relevance in the use of a definition is measured by the level of adaptation in the context of applications. In particular, basing ourselves on the proof of the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$, we will show how the definitions of equality, that will be introduced, condition the operative and discursive practices. The example will also allow the observation of non-trivial relations between the given definitions and their associated practices.

### Proof according to the arithmetic definition as equivalence

We will carry out the arithmetic proof by transformation of one ostensive representative into another, using basic properties of the real numbers, that are presumed justified previously from the axiomatic definition of **R** (14 axioms organized in four groups: existence, algebraic, ordinal, and topological or continuous[8]).

$$\sqrt{2}\overset{(1)}{=}\sqrt{2}\cdot 1\overset{(2)}{=}\sqrt{2}\cdot\frac{\sqrt{2}}{\sqrt{2}}\overset{(3)}{=}\frac{\sqrt{2}}{1}\cdot\frac{\sqrt{2}}{\sqrt{2}}\overset{(4)}{=}\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}\overset{(5)}{=}\frac{\left(\sqrt{2}\right)^2}{\sqrt{2}}\overset{(6)}{=}\frac{2}{\sqrt{2}}$$

The equalities are justified in the following way:

$\overset{(1)}{=}$ Existence of an identity element in **R**.

$\overset{(2)}{=}$ $\forall\, a \in \mathbf{R}\setminus\{0\},\ 1=\frac{a}{a}$.

$\overset{(3)}{=}$ $\forall\, a \in \mathbf{R},\ a=\frac{a}{1}$.

$\overset{(4)}{=}$ Product in **R**.

$\overset{(5)}{=}$ exponentiation in **R**.

$\overset{(6)}{=}$ $\forall\, a \in [0;\infty),\ \left(\sqrt{a}\right)^2=a$.

The proposed proof is not unique. If $\sqrt{2}$ is the only positive real such that the square is equal to 2, it is enough to carry out the following calculation; $2=\left(\sqrt{2}\right)^2=\sqrt{2}\cdot\sqrt{2}$ and, dividing by $\sqrt{2}$, we have: $\frac{2}{\sqrt{2}}=\frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}=\sqrt{2}$. The way of proving "by successive equivalences" implicitly supposes the acceptance of certain privileged operative practices, naturalized in contemporary academic institutions (*know how, technique*) in relation to the notions of equality.

"The analysis of textbooks, as well as of the answers of students and teachers […], show that when faced with this type of problem [Show that A=B],it is almost always "better" to transform $A$ in a series of equal expressions $A_1,\dots,A_n$ such that $A = A_1 = \dots = A_n = B$, so that "we start with $A$ to get $B$". It is clear that $A$ and $B$ could be […] transformed together, using the type of reasoning "$A = C$ and $B = C$ then $A = B$" […] These

[8] These aspects (existence, ordinal, algebraic and topological) of the structure of **R** have abundant interdependence, some of which come from the axiomatization of **R**; for example, to postulate that (**R**, +, ·, ≤) is an *ordered and complete Arquimidean field*.

procedures are seldom used by students, and there are teacher that do not even accept them." (Antibi and Brousseau, 2000, 30)[9].

**Proof according to the arithmetic definition of order**

Let $A = (-\infty;\ \sqrt{2}\ )$ y $B = (\ \sqrt{2}\ ;\infty\ )$. By the trichotomy law:

$$\frac{2}{\sqrt{2}} \in A;\ \text{or}\ \frac{2}{\sqrt{2}} \in B;\ \text{or}\ \frac{2}{\sqrt{2}} = \sqrt{2}\ ;$$

Suppose that $\frac{2}{\sqrt{2}} \in A$, then: $\frac{2}{\sqrt{2}} < \sqrt{2}$ and, given that $\sqrt{2} > 0$, then $\frac{2}{\sqrt{2}} \cdot \sqrt{2} < \sqrt{2} \cdot \sqrt{2}$ hence, $2 < 2$; which is absurd. In the same way, it is proved that $\frac{2}{\sqrt{2}} \notin B$, which proves that $\frac{2}{\sqrt{2}} = \sqrt{2}$.

**Proof according to the metric definition**

Let $\varepsilon$ be the distance between $\sqrt{2}$ and $\frac{2}{\sqrt{2}}$:

$$\varepsilon = d\left(\sqrt{2}; \frac{2}{\sqrt{2}}\right) = \left|\sqrt{2} - \frac{2}{\sqrt{2}}\right|$$

Then $\varepsilon = 0$; in effect ($\forall\ x \in \mathbf{R}$, $|x| = \sqrt{x^2}$ ):

$$\varepsilon = \sqrt{\left(\sqrt{2} - \frac{2}{\sqrt{2}}\right)^2} = \sqrt{\left(\sqrt{2}\right)^2 - 2 \cdot \sqrt{2} \cdot \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{2}}\right)^2}$$
$$= \sqrt{2 - 4 + 2} = \sqrt{0} = 0$$

In conclusion, $\varepsilon = 0$ and, therefore, $\sqrt{2} = \frac{2}{\sqrt{2}}$.

**Proof according to the connective definition**

In the set of real numbers, the notions of connected and convex sets are equivalent; then to prove that $A = \left\{\sqrt{2}; \frac{2}{\sqrt{2}}\right\}$ is connected, it is sufficient to show that $B = \left\{\sqrt{2} \cdot (1-r) + \frac{2}{\sqrt{2}} \cdot r \mid r \in [0;1]\right\} \subseteq A$.[10] Let $x \in B$:

[9] *In French in the original*: "L'analyse des livres scolaires et des réponses des enseignants et des élèves […] montre que, presque toujours, en présence d'un problème de ce type [Démontrer que $A = B$], «il convient» de transformer $A$ en une suite d'expressions égales $A_1, \ldots, A_n$ telles que $A = A_1 = \ldots = A_n = = B$ de façon à «partir de $A$ pour arriver à $B$». Il est clair que l'on pourrait […] transformer à la fois $A$ et $B$ en utilisant le raisonnement «$A = C$ et $B = C$ donc $A = B$» […] Ces procédés sont rarement utilisés par les élèves et il se trouve des enseignants pour ne pas les accepter." (Antibi et Brousseau, 2000, 30).

[10] It is not necessary to use the equivalence of the notions of connectivity and convexity of the set of real numbers. It is enough to observe that there does not exist $y \in \mathbf{R}$ between $\sqrt{2}$ y $\frac{2}{\sqrt{2}}$ such that $y \notin A$.

$$\begin{aligned} x &= \sqrt{2}\cdot(1-r)+\frac{2}{\sqrt{2}}\cdot r=\sqrt{2}-\left(\sqrt{2}-\frac{2}{\sqrt{2}}\right)\cdot r= \\ &= \sqrt{2}-\left(\frac{\sqrt{2}\cdot\sqrt{2}-2}{\sqrt{2}}\right)\cdot r=\sqrt{2}-\left(\frac{\left(\sqrt{2}\right)^2-2}{\sqrt{2}}\right)\cdot r= \\ &= \sqrt{2}-\left(\frac{2-2}{\sqrt{2}}\right)\cdot r=\sqrt{2}-0\cdot r=\sqrt{2} \end{aligned}$$

**Proof according to the function definition**

Let $f(x) = x^2$ in $[0; \infty)$. We know:

$$f\left(\frac{2}{\sqrt{2}}\right)=\left(\frac{2}{\sqrt{2}}\right)^2=\frac{4}{2}=2$$

$$f\left(\sqrt{2}\right)=\left(\sqrt{2}\right)^2=2$$

This way, given that $f$ is injective, we conclude that $\frac{2}{\sqrt{2}}=\sqrt{2}$.

**Proof according to the definition as a limit process**

Let $f(x) = x$ and $g(x) = \frac{2}{x}$ in $[1; \infty)$. To show that $\sqrt{2}$ is a point that evaluates to the same values in $f$ and $g$ ($f\left(\sqrt{2}\right)=g\left(\sqrt{2}\right)$) is equivalent to showing that $x=\sqrt{2}$ is a zero of the function $h(x)=x-\frac{2}{x}$ in $[1; \infty)$ :

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0, \text{ such that } |x-\sqrt{2}| < \delta \Rightarrow |h(x)-0| < \varepsilon$$

In effect, given $\varepsilon > 0$, $\varepsilon \le 1$, one takes $\delta=\frac{\varepsilon}{1+\sqrt{2}}$, then ($x \ge 1$):

$$\begin{aligned} \left|x-\frac{2}{x}-0\right| &= \left|\left(x-\frac{2}{x}\right)-\left(\sqrt{2}-\sqrt{2}\right)\right|\le \\ &\le \left|x-\sqrt{2}\right|+\left|\sqrt{2}-\frac{2}{x}\right|= \\ &\le \left|x-\sqrt{2}\right|+\frac{\sqrt{2}}{|x|}\cdot\left|x-\sqrt{2}\right|\le \\ &\le \left|x-\sqrt{2}\right|+\frac{\sqrt{2}}{1}\cdot\left|x-\sqrt{2}\right|\le \\ &\le \left(1+\sqrt{2}\right)\cdot\left|x-\sqrt{2}\right|<\left(1+\sqrt{2}\right)\cdot\delta=\left(1+\sqrt{2}\right)\cdot\frac{\varepsilon}{1+\sqrt{2}}=\varepsilon \end{aligned}$$

In conclusion, $\text{lím}_{x\to\sqrt{2}}\, h(x)=0$ or, equivalently, $\text{lím}_{x\to\sqrt{2}}\, f(x)=\text{lím}_{x\to\sqrt{2}}\, g(x)$ and, given that $f$ and $g$ are continuous in $[1; \infty)$, it is shown that $\sqrt{2}=\frac{2}{\sqrt{2}}$.

**Proof according to the numerical definition**

The same as with the analytic proof as a limit process, it can be proved that the approximations to $x=\sqrt{2}$ are "zeros" of the function $h(x)=x-\frac{2}{x}$ (with an arbitrary margin of error, but fixed; within the limits of the calculator or computer program used). Then, the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$ is proved "margin of error to margin of error".

A program edited with a graphing calculator, programmable on a *TI-81*, is shown in table 1. The program is based on Newton's method (the tangent method). Other programs could have been edited; however, the discussion about characteristics such as efficiency, correctness, robustness and friendliness of a program is not within the bounds of this text[11].

```
PrgmD:NEWTON
:DisP "TOLERANCIA"
:InPut T
:1->X
:Lbl 1
:4X/(X^2+2)->R
:If abs(X–R)<T
:Goto 2
:R->X
:Goto 1
:Lbl 2
:DisP "SOLUCION"
:DisP R
```

**Table 1**. Program edited with a *TI-81* for obtaining zeros of the function $h(x)=x-\frac{2}{x}$, $x\in[1;\infty)$, given a margin of error *T*.

This way, for every margin of error *T*, it is shown that ($E\equiv h(x)=0$):

$$\delta\,(E(\sqrt{2}\,,T))=1\Leftrightarrow(\sqrt{2}\,,T)=\frac{2}{(\sqrt{2},T)}$$

*Concise confrontation of the proofs of the proposition* $\sqrt{2}=\frac{2}{\sqrt{2}}$

The proof, according to the definition of equivalence, can be generalized for any integer $a>0$: $\sqrt{a}=\frac{a}{\sqrt{a}}$; or, reciprocally, the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$ can be accepted as a particular case of the formula $\sqrt{a}=\frac{a}{\sqrt{a}}$. This way, the proposition $\sqrt{a}=\frac{a}{\sqrt{a}}$ can be seen as an "arithmetic in algebraic language" proof, that is, a relation be-

[11] It is said, in informatics, that a program is robust if it fulfills the following two conditions: first, it is *correct*, that is, every value introduced that satisfies the fixed entrance conditions produces an *admissible* result (that satisfies the exit conditions); the other is that the program allows the *detection of errors*, that is, for every entrance that does not satisfy the fixed conditions an error message is obtained, which indicates that the choice is faulty. If, apart from these two conditions, the program gives the user the possibility to correct an error in the entrance so that it does satisfy the fixed conditions, it is said that the program is *friendly*.

tween the practices in the arithmetic context, and those referred to in the algebraic context, is made explicit. In fact, the formula $\sqrt{a} = \frac{a}{\sqrt{a}}$ highlights permanence more than action; the same occurs with the *identity* $(a + b)^2 = a^2 + 2ab + b^2$. This change from "action to permanence" delimits the change from an arithmetic language to an algebraic one. Gascón (1994) has concluded that the equal sign in arithmetic contexts represents an action: "2 + 3 = 5" is equivalent to "2 plus 3 gives 5". However, in algebraic language, there exists a duality between the use as an action ($3x + 2 = 1$) and the static use as permanence ($a\,(b + c) = a\,b + a\,c$).

The duality of the equal sign is not exclusive to the algebraic context. The analytic equality as a process of taking the limit also has static and dynamic states, associated to the notion of limit. If we consider the constant $a$ as a sequence ($a_n = a$, $\forall\ n \in \mathbf{N}$), then $a_n$ tends to $b$: $a = b \Leftrightarrow \forall\ \varepsilon > 0,\ \exists\ N$ such that $|a_n - b| < \varepsilon$, $\forall n > N$. The duality itself has more "impact" than in the algebraic context, given that the notion of equality manifests itself as a process and as an object at the same time; deep down, the radical difficulty that is provoked is the double nature of mathematical infinity (potential-actual). For this reason, given that the infinite processes, which in many circumstances imply some limit operation, are dense in Mathematical Analysis, it appears that the duality process-object should play a central role in didactical analysis. Along these lines, Tall (1991) calls certain mathematical objects with a dual nature *procepts* —pro(cess)(con)cepts—; Cornu (1991) identifies two essentially different conceptions in students with relation to the notion of the limit of a sequence (static and dynamic); Schneider (2001) establishes the need to structure the introduction to the notion of derivative in two stages: first, an affine approximation (not dynamic); then the limit of secants (dynamic); etc.

On the other hand, it is possible to identify the proposition $\sqrt{2} = \frac{2}{\sqrt{2}}$ as the result of the search for the points that are cuts of the functions $f(x) = x$ and $g(x) = \frac{2}{x}$ in $[1; \infty)$:

$$f(x) = g(x) \Rightarrow x = \frac{2}{x} \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2} \Rightarrow \sqrt{2} = \frac{2}{\sqrt{2}}$$

The Theory of Functions allows the justification for obtaining the quadratic equation (and, implicitly, to give it a graphical interpretation. This equation should be solved with the type of arguments and procedures of algebra (and do not require interpretation in terms of the specific problem). This fact can be formulated in the following terms: the proof has a discursive component common to the theory of functions, as well as an operational component that belongs to algebra. In fact, if $\sqrt{2}$ is defined as the only positive real number whose square is equal to 2, then:

$$x = \sqrt{2} \Leftrightarrow x > 0 \wedge x^2 = 2$$

This way, $\sqrt{2}$ is the only positive solution to the equation $x^2 = 2$ and, for this reason, the proposition $\frac{2}{\sqrt{2}} = \sqrt{2}$ is justified only by verifying that $\left(\frac{2}{\sqrt{2}}\right)^2 = 2$, showing that, in this case, an identification of the proofs can be established according to the functional and algebraic definitions.

Finally, the numerical proof can be understood in analytic terms and solved in the algebraic framework. In effect, the determination of the existence and uniqueness of the limit of the sequence $(x_n)$, is given by the recursion formula (Newton's method):

$$\begin{cases} x_{n+1} = \dfrac{4x_n}{x_n^2 + 2} \\ x_1 = 1 \end{cases}$$

This expression involves discursive and operative practices that are analytic (bounding, controlled loss of information, etc.) and algebraic (manipulation of the algebraic expressions involved); it is difficult to label the different moments (aspects) of mathematical activity (as a complex of mathematical notions, processes and meaning put into play). Once the convergence of the sequence ($x_n$) is justified (a positive number), the limits are introduced in both members of the recursion rule, and the equation that results is solved:

$$
\begin{aligned}
x_{n+1} = \frac{4x_n}{x_n^2+2} \quad &\Rightarrow \quad \lim_{n\to\infty}\left(x_{n+1}\right) = \lim_{n\to\infty}\left(\frac{4x_n}{x_n^2+2}\right) \Rightarrow \\
&\Rightarrow \quad x = \frac{4x}{x^2+2} \Rightarrow \\
&\Rightarrow \quad x^3 + 2x = 4x \Rightarrow x = \sqrt{2}
\end{aligned}
$$

The proof of the proposition "$\sqrt{2} = \frac{2}{\sqrt{2}}$" has exemplified how the different definitions of the notion of equality condition mathematical work. The evolution of the notion of equality has followed an inverse process: the mathematical process has conditioned the meanings attributed to the notion of equality and, only afterwards, when this notion is taken as an object of study, is the meaning formalized in definitions (that emerge from certain systems of mathematical practices relative to problems in specific areas). Hence, the fundamental task consists of reconstructing this process, that is, of determining the discursive and operative practices that have caused the definitions of the notion of equality to emerge in different contexts of use. In section 3 we sketch these practices.

## SUBSYSTEM OF PRACTICES ASSOCIATED TO DIFFERENT CONTEXTS OF USE

The Anthropological Theory of Didactics (TAD) and the Onto-Semiotic Approach (OSA) share the same anthropological assumptions about institutional knowledge. In fact, the operative and discursive systems of practices are to OSA what mathematical praxeologies are to TAD. However, the description of the systems of practices is not equivalent to the description of the praxeologies. TAD characterizes mathematical activity starting with the tasks and the techniques, to finally arrive at the technological-theoretical discourse (without challenging the nature of the objects that intervene in the discourse). The notion of praxeology models mathematical knowledge as a human activity, understanding by knowledge the product (refined) of a systematic, intentional, historical and social study. The most immediate product of the study is the techniques ("know-how"), whose validity is subjected to a technological-theoretical discourse which justifies ("knowledge", by-product). This way, "know-how" and "knowledge" make up the two faces of a praxeology ("praxis-logos").

> "A mathematical organization always arises as an answer to a question, or set of questions. It is not specified what a mathematical organization *is*, but a sketch is given of its structure, postulating that it is made up of four principal components: *types of problems*, *techniques*, *technologies* and *theories*. If we put the emphasis on the dynamic relations that are established between the components, with the object of carrying out the necessary mathematical activity to be able to respond to the challenging initial question, then two inseparable aspects appear: mathematical practice, or 'praxis' (formed by *tasks* and *techniques*) and the rational discourse, or 'logos' about the actual practice (formed by *technologies* and *theories*)." (Bolea, Bosch and Gascón, 2001, 251).

This way, the work that is shown allows the determining of a praxeology associated to the *generating question* (Chevallard, 1999, 232): Given $a, b \in \mathbf{R}$, do $a$ and $b$ represent the same number? The task is to show that two numbers represented by different ostensive presentations are equal. The techniques that are associated to each one of the definitions: transform by successive equivalences one of the numbers, until the other is obtained; justify that, given $a, b \in \mathbf{R}$, that simultaneously $a \le b$ and $b \le a$ are true; reason out that, given $a,b \in \mathbf{R}$, none of the following two inequalities can be true: $a < b$ y $b < a$, and that by the law of trichotomy $a = b$; determine that the Euclidean distance between two numbers is zero; etc. The technology allows the justification of the steps that are carried out by means of each one of the techniques; indeed, in the institution where the study process is carried out, it is common to accept that the technical movements made (multiply by one, raise to the second power, etc.) and the mathematical objects employed (square root, order relation, connectivity, etc.) have been previously justified or defined. The theory refers to the fundamental structural, topological and analytical notions of real numbers that play the role of support and reference and that, in the majority of cases, represent abstractions or generalizations of the presuppositions and technologies stated. The "level of justification", that is, the development of the technological-theoretical block, is inherent to each institution.

> "The style of rationality that is put into play in an institution varies, of course, according to its sectors, and also varies as the institution evolves in time, with its own institutional history, such that a given institutional rationality can seem…highly irrational in the eyes of another institution". (Chevallard, 1999, 226)[12].

However, OSA is interested in theorizing about the notion of meaning in didactics, which is done by means of the semiotic function and its associated mathematical ontology. The starting point of OSA is to try and characterize the nature and the meaning of mathematical notions; it begins with the elements of the technological discourse (notions, properties, arguments, etc.) and it is thought that their nature is tied to the corresponding systems of practices and contexts of use. In this realm, the necessary and sufficient conditions of each one of the definitions are the explicit antecedent (expression) of a semiotic function whose consequent is the notion of equality; the properties determine the meaning of the notion of equality as a technological-theoretical object.

> "A deeper study would show that many difficulties caused by the "=" sign appear in the language-object: the equation, the identity, the calculation, and in the working language in which the object is immersed: as the descriptor of transformations, as a meta-theorem or as 'inference'." (Antibi and Brousseau, 2000, 31)[13].

Nevertheless, the definitions that are given do not determine the meaning of the notion of equality by themselves; they only represent the "visible aspect" of the systems of practices, the crystallization of certain ways of doing and justifying, of operating and elaborating the discourse. To describe the operative and discursive systems of practices in relation to a mathematical object, in the first place it is necessary to identify the principal notions, properties, language, arguments and procedures that are used in a wide range of prototypical problems in different contexts of use and, in second place, to describe the relations between the mathematical objects involved in the different subsystems of practices that are developed. A detailed description of these subsystems of practices goes beyond the objective of this article[14]. A brief sketch will be given in the following paragraphs.

The fundamental notions associated to the notion of equality are:

(i) the numerical context: approximation and margin of error, as they determine the acceptable intervals;

(ii) the arithmetic context: identity and order relation;

(iii) the algebraic context: equivalence and function (in the majority of cases, algebraic functions);

(iv) the analytic context: function (in particular, transcendental) and limit (convergence);

(v) the topological context: distance (measure) and connectivity.

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12 *In French in the original*: "Le style de rationalité mis en jeu varie bien entendu dans l'espace institutionnel, et, en une institution donnée, au fil de l'histoire de cette institution, de sorte qu'une rationalité institutionnelle donnée pourra apparaître… peu rationnelle depuis telle autre institution." (Chevallard, 1999, 226).

13 *In French in the original*: "Une étude plus approfondie montrerait que de nombreuses difficultés proviennent aussi de ce que le signe «=» est mobilisé en même temps dans la langue objet: l'équation, l'identité, le calcul, et dans la langue de travail sur l'objet: comme descripteur des transformations, comme métathéorème, ou comme «inférence»". (Antibi et Brousseau, 2000, 31).

14 Well understood, the detailed description of the system of practices associated to a mathematical notion supposes a *rational reconstruction* (Lakatos, 1976) or the elaboration of an *anthropology of knowledge* (Chevallard, 1985) of the notion.

In the arithmetic context the mathematical language is not very formal. The discourse relies on the manipulation of concrete values and the argument is based on the properties of symmetry and transitivity of equality, implicitly accepting that "the symmetric and transitive relations are formally part of the nature of equality" (Russell, 1903, 257)[15]. In fact, this lack of formality implies an abusive use of the equality sign in arithmetic contexts.

> "In exercise books of arithmetic 2 + 7 = 9 + 7 = 16 + 7 = 23 + … is a well-known feature. We reject it. It is not because it cannot be justified —not without long hesitation has this notation been forbidden. There are still rudimentary traces of the mathematical style which would allow such formulae. It could be maintained with appropriate such as (((((2 + 7 = 9) + 7) = 16) + 7 = 23) + …, or by the convention that with no further comment every formula is read by progressing from left to right. In fact this is the rule with all expressions that contain additions and subtractions only." (Freudenthal, 1986, 299–300).

In the algebraic context, the objects are represented by means of symbolic-literal language, whose objective is to generalize the concrete operations, constructing a system of signs that are easily recognized and establishing an operational and discursive structure that allows the reduction of mathematical objects to canonical expressions, so that just "by simple observation" the objects can be described. In the numerical context the discourse is organized by means of sentences that determine the order in which programmable algorithms are carried out. In this way, the language is typical of programming. Equality has two functions: *logical*, in sentences of the type "if $a = b$ then…", and *arithmetic*, assigning the value $a$ to $b$ ($a \to b$).

In the analytic context, the language of infinitesimals is used and the "classical" argument en terms of the "$\varepsilon$ -$\delta$" notation.

> "To say that a real quantity $\lambda \in \mathbf{R}$ is zero exactly when
>
> $$|\lambda| \leq \varepsilon, \forall \varepsilon > 0,$$
>
> occurs, forms part of the *language and reasoning style* of the analyst (in the same way, an analyst immediately says that two numbers $a, b \in \mathbf{R}$ are equal if $\forall n \in \mathbf{N}$, $|a - b| < \frac{1}{n}$). An algebraist would probably say (admitting that we are in a field of characteristic zero) that $\lambda = 0 \Leftrightarrow \lambda + \lambda = \lambda$ (something that forms part of the *language* and *reasoning style* of the algebraist). If we return to the analytic "slang" '*epsilon-delta*' we see that there is an underlying *structure* […] *The principal characteristic that the language of Mathematical Analysis has is the form in which its concepts are systematically structured, as well as its typical methodological philosophy*." (Induráin, 2001, 64–65).

Finally, the topological language shares part of the algebraic symbolism (transformation by equivalences), set theoretic (belonging to, contained in, etc.) and analytic-arithmetic (sum and product operations, order relation, etc.).

The basic arithmetic propositions[16] highlight some of the fundamental properties of the real numbers provided with the binary operations sum and product; In fact, "It is also important to take into consideration that

[15] Lorenzo (1974, 55) formulates this fact in the following terms : "*Calculus with equality*: Rule or reasoning process […] by means of which the same uniform operation applied to two equal numbers will give identical results […] It is standard that the translation to symbolic language is '$a = b \Rightarrow f(a) = f(b)$' constituting one of the characteristic premises of the equality relation, while the other premise would be a form of the identity principal 'every quantity is equal to itself'." This formulation remits us to the functional definition presented in this article.

[16] The propositions are mathematical properties that mathematical conventions and culture have privileged. There is no mathematical justification for considering them as primary elements. Hence, in the OSA the propositions play a similar role to the definitions, that are also considered as a particular type of property: one that differentiates an object unmistakably in a given universe.

the symbol for equality did not evolve independently from the symbols for arithmetic operations and operations with variables" (Sáenz-Ludlow & Walgamuth, 1998, 155). Often the arithmetic propositions are modelled by means of algebra (Bolea, Bosch y Gascón, 2001, 257–265); in this sense the algebraic propositions generalize the arithmetic ones and make up an explicit justification of these propositions. Other types of algebraic properties deal with the structure of **Q** (and, by extension, of **R**): from the proof of basic laws such as the *cancelation for the sum* ($a + c = b + c$ implies $a = b$) or the *cancelation for the product* ($a \cdot c = b \cdot c$ and $c \neq 0$ implies $a = b$) to the justification that the "subtraction" and "division" operations or the proof of the *rule of signs*, all are deduced from the axioms of the definition of the rational (real) numbers and imply the notion of equality[17].

Furthermore, the analytic propositions suppose, in many cases, a radical rupture with the arithmetic-algebraic ones. This rupture is identified, in many cases, with the need to carry out infinite processes that often require taking a limit (or, in a more abstract setting, the notion of convergence), or by the presence of a transcendental function. Thus, the proposition:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = \operatorname*{lím}_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

that determines the "transcendental (not algebraic) Euler number ($e$) that can be expressed in at least two equivalent ways" is an analytic proposition. In the same way, the statements about and proofs of the basic properties of limits imply the notion of equality; for example the uniqueness of the limit of a sequence (convergent[18]).

The propositions in the numerical context always suppose the acceptance of a margin of error. It is not about determining that $0.\overline{9} = 1.\overline{0}$ (with infinite nines or zeros) but that it is possible, for a specific margin of error, to have a finite number of nines and zeros such that the difference between these two numbers is less than that margin of error. This manner of stating and proving the propositions establishes the fundamental difference between the numerical and analytic contexts. Thus, in an analytic context, the previous proposition would be stated in the following way:

> "*Theorem*. A real number $x$ has exactly one decimal expansion or else $x$ has two decimal expansions, one ending in a sequence of all 0's and the other ending in a sequence of all 9's." (Ross, 1980, 108).

A fundamental problem in the arithmetic and algebraic contexts is obtaining the canonical representatives; to solve this problem, the characteristic action is the manipulation of objects by means of equivalences (that are justified by the axioms of the real numbers or by previously established properties). This problem explains the importance that the arithmetic and algebraic models of the notion of equality have in contemporary institutions, given that they provide an alternative to a fundamental task of all mathematical activity.

> "The way in which the purpose is identified is generic: given a system of mathematical objects, it is very useful to provide, when it is possible, a canonical writing system of these objects, so that two objects of the system can be compared without ambiguity." (Chevallard, 1999, 244)[19].

In the numerical and analytic contexts a fundamental problem is comparison; however, the two contexts are differentiated by the type of entities compared: in the numerical context they are intervals, whereas they are numbers (unique, points) in the analytic context. The determination of a number in the numerical context

[17] Aliprantis & Burkinshaw (1999, 12–20), for example, state and prove a collection of these problems.

[18] If the existence of a limit has not been proved, it is justified that "every sequence has *at the most* one limit"; if the existence has been shown for a particular sequence or, if the existence of the limit is taken as a hypothesis, it is shown "that the limit is *unique*".

[19] *In French in the original*: "La raison d'être ainsi identifiée est générique: étant donné un système d'objets mathématiques, il est très utile de se doter, chaque fois que la chose est possible, d'un système d'écriture canonique de ces objets, et cela afin de pouvoir comparer sans ambiguïté deux tels objets. " (Chevallard, 1999, 244).

supposes obtaining an approximation "sufficiently good", that is, with a pre-established margin of error that fixes an admissible interval for the number. The type of compared entities also sets the characteristic procedures in each of the contexts: in the numerical context the principal action consists of the construction of a logical sequence of sentences that obtains, for a range of pre-established entrance values, a dichotomized answer (true or false, yes or no, 1 or 0, etc.) to the question "given $a, b \in \mathbf{R}$, $a = b$ for an *admissible margin of error*?"; in the analytic context, the comparison is carried out by the controlled loss of information, that allows sufficient conditions to be set, so that the question "given $a, b \in \mathbf{R}$, $a = b$ for every *margin of error* ?" can be answered.

In the topological context, the fundamental problem is establishing the equality or difference between numbers, that is, it is not about "measuring the difference", but about setting a characteristic function Δ such that, given $a, b \in \mathbf{R}$: $\Delta(a, b)=1$, if $a \neq b$; $\Delta(a, b)=0$, if $a = b$. This determines one of the differences between the topological and analytic contexts. According to Induráin (2001, 96) one of the basic processes of analysis is *separating* or *calculating distances* which rests on the notion of *distance* or *metric*[20]. Hence, in an analytic context, the interest lies in determining the equality of or difference between two numbers and, in case they are different, *how much.*

Then, for example, let $(a_n / a_{n-1})$ and $(b_n)$ be sequences, defined in a recursive way by::

$$\begin{cases} a_1 = 1 = a_2 \\ \forall n \geq 3 : a_n = a_{n-1} + a_{n-2} \end{cases}; \quad \begin{cases} b_1 = 1/2 \\ \forall n \geq 2 : b_n = \dfrac{1}{1 + b_{n-1}} \end{cases}$$

And let $a$ y $b$ be the limits of these sequences. Then, the topological assertion $\Delta(a, b) = 1$ supposes only the assertion that $a$ and $b$ are different; whereas the metric declaration $d(a; b) = 1$ establishes the magnitude of the difference between both limits ( $a = (1+\sqrt{5})/2$ y $b = (-1+\sqrt{5})/2$ ). Indeed, to justify that $\Delta(a, b) = 1$ it is sufficient to justify that $a > 1 > b$.

In the next section we will show how to structure the subsystems of practices associated to the different contexts of use the emergent objects of these subsystems (including the definitions). We will propose a diagram of the different objects associated with the notion of equality and the correspondences that exist between them. By means of "levels", we will identify the contexts of use of the notion of equality, the systems of practices associated with the notion, the emergent objects of such systems, the language (voice "equality", sign "=") and, finally, the formal structure to which all mathematical work (operative and discursive) explicitly or implicitly refers in relation to the notion of equality.

## OBJEcTS, Meanings and models ASsOCIAted with the notion of equality

The interpretation of the meaning of the mathematical objects in terms of "operative and discursive systems of practices", relative to a determined institution, leads to the postulation of a socio-epistemic relativism with relation to mathematical objects, a consequence of adopting the anthropological point of view of mathematics (Godino, 2003). This socio-epistemic relativism contradicts the apparent absolute and universal character that the professional mathematician attributes to mathematical objects. However, as we understand, this dilemma can be resolved by accepting that the mathematician identifies the same formal structure in the variety of objects and practices (operative and discursive); a structure that he considers as "the mathematical object" that represents the reference implicit in the variety of systems of practices and emergent objects in the different contexts of use. In the case of equality, the formal structure can be described briefly as:

$$a = b \Leftrightarrow a \text{ y } b \text{ represent the same number}$$

[20] According to this author, the other four fundamental analytic processes and the notions on which they rest are: *counting* with respect to the notion of *number*; *comparing* and *ordering* with respect to the notion of *order*; *approximation* or calculating the *tendency of a magnitude* with respect to the notion of *limit*, of *convergence* or of the more advanced idea of *continuity*; *measuring* with respect to the notion of *measure* or *integral*.

Thus, the formal structure represents a description of the notion of equality without an explicit reference to concrete practices or contexts.

Figure 1 shows, schematically, the diversity of objects associated with the notion of equality. Each definition represents an emergent object of the system of practices in a determined context of use. Each pair "definition - system of practices" (and, in general, "emergent object- system of practices") determine a *model of the notion of equality*; that is, an effective or potential relation with the notion of equality (understood as a system) that a subject (or an institution) establishes, starting with knowledge *a priori* of the notion. The model is a coherent form of structuring the different contexts of use, the mathematical practices relative to those contexts of use, and to the emergent objects of such practices.

According to the different contexts of use, mathematical practice is structured around certain privileged mathematical notions and techniques. Furthermore, mathematical practice establishes basic criteria of proof (about how the deductive processes evolve according to the nature of the conditions) and determines guidelines on how to finish a proof (obtaining a semantic tautology or accepting a logical tautology). In fact, the stability of the models in the educational institutions is based on a process of "objectifying the models" that consists in the establishment of a set of *discursive entities* (notions, arguments, properties), another of *praxemic entities* (problems, procedures) and a language (graphical, symbolic, oral, etc.) specific to the mathematical notion that is to be introduced or developed. The structuring of the praxemic and discursive entities and the integrations of the language form a *local* epistemic network or configuration (associated with a specific context of use). Each local epistemic configuration "synthesizes" a partial aspect of the meaning of the corresponding notion; that which is associated with the modelling system.

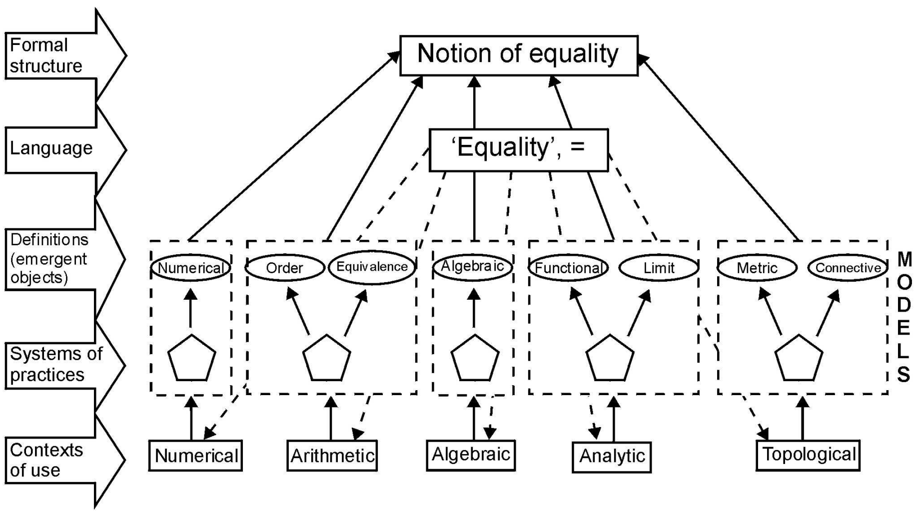


**Figure 1**. Structuring of the models and meanings associated to equality.

However, figure 1 shows a static structuring of the models and meanings associated with the notion of equality, where only the relations of level are indicated (by arrows) between the different objects involved: with relation to the notion of equality, in each concrete context of use, a model is associated (system of practices – emergent objects), that determine a meaning (partial) of the notion. The mathematical activity, however, flows between the different levels. From the systems of practices the praxemic, discursive and linguistic entities emerge and, gradually, they are integrated in the practices as operational rules (know-how), as instruments of argumentation or regulation (knowledge) or as means of expression and communication. The *ecological metaphor* (Godino, 1993) represents a relevant focus for the description of the dynamics of the proposed structure.

The notion of *holistic-meaning* that we will introduce in section 5 will allow us to interpret “equality” as a network of models associated with it (*global configuration*). Furthermore, the notion of *holistic-meaning* allows us to establish what we express when we assert that a person understands the notion of equality.

## HOLISTIC-MEANING OF THE NOTION OF EQUALITY

Mathematical practice has assumed the phenomenon of homonymy in relation to the notion of equality; to avoid designating “each” equality with a different term, the name is maintained (equality) and the sign (=) is common in all the contexts of use, accepting the specific meaning that is attributed to the notion of equality in each one of them[21]. In fact, from the strictly formal point of view, it is accepted that the definition of a mathematical object makes up its meaning. Hence, the problem of homonymy is solved by selecting one of the definitions and proving, afterwards, the equivalence of the rest of them in a theorem. For example, if one is working in an eminently analytic context, it will be accepted that two numbers $a$ and $b$ are equal if, for every $\varepsilon > 0$, $|a - b| < \varepsilon$ (def.7) and the objective is to prove the following[22]:

*Theorem 1 (Equality)* Given two real numbers $a$ and $b$, the following propositions are equivalent:

1. $a = b$.
2. $d(a; b) = 0$.
3. $\{a\} \equiv \{b\}$.
4. $\{a; b\}$ is connected.
5. $a \leq b \wedge b \leq a$.
6. For every equation $E$, $[\delta (E(a)) = 1 \Leftrightarrow \delta (E(b)) = 1]$.
7. Let $F_i(D)$ be the set of injective functions over a domain $D$, then $\exists\, f \in F_i(D)$, $\{a, b\} \subseteq D$, such that $f(a) = f(b)$.

Proving theorem 1 supposes declaring that the given definitions designate the same object (equality); even more, the definition of equality as a limit process is considered the “definition”, while the rest of the definitions are unmistakable “characterizations” of the original definition. The difference between a definition and a characterization obeys mathematical conventions, of more or less explicit cultural use. Hence, a characterization of a mathematical object is a definition of it that competes with a previous definition that has been deemed natural within certain institutional practices. Because of this, the consciousness of the agreement upon which it is based has been lost. In the academic institution the arithmetic definition of equality as equivalence has been privileged and, for that reason, the rest of the definitions are considered characterizations. This supposes, when the first notions of analysis are introduced, the need to reconstruct the forms of reasoning in relation to the notion of equality that, in particular, help the evolution from the technique of proof by successive equivalences to the technique of proof by the controlled loss of information and sufficient conditions.

> “Work in Mathematical Analysis is clearly based on algebraic competencies that require, as soon as the work does not limit itself to algebraic analysis, a reconstruction of the relationship with equality. This reconstruction is accompanied by an oscillation in forms of reasoning: passing from successive

[21] This fact shows the impossibility of avoiding obstacles in the learning process : the discourse constantly must decide among the phenomena of synonymy and (antagonically) homonymy. The case of equality is not pathological; for example, Wilhelmi (2003) has observed the same phenomenon with respect to the notions of *continuous function* and *absolute value*.

[22] In theorem 1 the definition of numerical equality is excluded, which is of an essentially different nature. In section 2.2 we have commented the relation between the numerical definition and the analytic definition as the process of taking the limit.

equivalences based on the preservation of equalities, to reasoning by sufficient conditions, based on the controlled loss of information in the treatment of inequalities, as the equality is converted to an inequality that is satisfied for any strictly positive $\varepsilon$." (Artigue, 1998, 239)[23].

The equivalence of definitions is confirmed in the mathematical realm not in the *cognitive* sense (given that, in particular, the definitions do not generate the same procedures and strategies) nor in the *instructional* sense (given that they do not come motivated by an equivalent introduction to the topic) and also not in the *didactical* sense (given that the social meaning differs and provokes different affiliations between the subject and the equality object, generating clauses that cannot be compared within the didactical contract). Hence, theorem 1 does not represent a suitable instrument to structure the models associated with the notion of equality. The study of models of equality (together with the associated definitions), and of their application for the proof of the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$ , shows, in relation to the meaning, the need of a *flexible transition* between the different models. Wilhelmi (2003) defines *flexible mathematical thinking* as the action carried by a subject that allows the routine transition among the different models associated with a mathematical object, recognizing the specific limitations of each one of them; furthermore, flexible mathematical thinking lets the subject establish solid links between the models and one or more mathematical contexts, allowing him to establish an efficient control of the activity and capacitating the subject to assume mathematical responsibility of the results he produces. The *holistic-meaning* incorporates the relations between the models and the tension, relationships and contradictions that are also established (and that flexible mathematical thinking allows to identify, describe and control).

Now, how is the *holistic-meaning* of the notion of equality described? This notion is determined (theoretically) by the relations that are established between the models associated to it. Figure 2 represents a scheme of this notion.

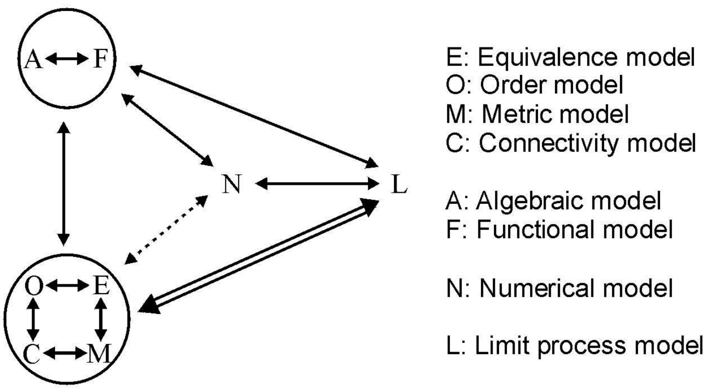


**Figure 2**. Representation of the holistic-meaning of the notion of equality.

In figure 2, the lengths of the arrows that describe the relations are not anecdotic; neither is the fact that some are single, others double and one is discontinuous. The length determines the distance of the models; distance measured as a *time interval* (two models are close if their introduction can be carried out in a reasonable way within the same unit of time in the study process) or as p*lace interval* (two models are close in the measure in which they are used simultaneously in a broad class of situations). The double arrows refer to a dialectic interaction between the models: one model is understood essentially by opposition to another

[23] *In French in the original*: "Le travail en analyse s'appuie à l'évidence sur des compétences algébriques mais il impose, dès que l'on ne se limite pas bien sûr à une analyse algébrisée, une reconstruction du rapport à l'égalité. Cette reconstruction s'accompagne d'un basculement des modes de raisonnement : on passe de raisonnements par équivalences successives basés sur la conservation d'égalités à des raisonnements par conditions suffisantes basés sur la perte contrôlée d'informations dans le traitement d'inégalités, l'égalité devenant une inégalité satisfaite à $\varepsilon$ près pour tout $\varepsilon$ strictement positif. " (Artigue, 1998, 239).

model. The interaction, then, is not circumstantial to some naturalized practices or a culturally structured form of knowledge, but it has an undividable relationship with the actual models. The discontinuous arrow determines not an essentially epistemological relation but a material one (the instruments of calculation used) or practical (specific uses in contemporary educational institutions). Finally, the circle that surrounds the models establishes a system of models that can only be differentiated by carrying them out, that is, by the use of the models, by the contexts in which they appear and by the meaning attributed to them.

Thus, the distance between the arithmetic model as equivalence (E) and the analytic model of taking the limit (L) can be comprehended by the weakness of their mutual attraction: the lack of ostensive instruments and the almost total absence of an explicit discourse that would contribute to an approximation of the two models. In other words, the weak attraction is a manifestation of the evident imbalance that exists between the arithmetic and analytic models.[24].

When the L model "erupts", a constant interaction with model E is produced. Indeed, for example, just the writing of $\lim(a_n) = a$ implies the acceptance *ipso facto* of both models. In this context, the models E and L remain close, given that they are utilized simultaneously in a broad class of situation (adjacency); however, they are very far apart in the curriculum of contemporary academic institutions (and in the texts) as it is not feasible to promote their emergence in the same time interval in the study process (distance in time). What are the implications derived from looking at the place where the notions are introduced (related to the practices) and the distance in terms of time of their introduction (related to the curriculum)? This difference seems to create a necessary condition for the appearance of an *obstacle*, that is, of knowledge that is useful and relevant (equality as an equivalence) within a block of situations (proof of the equality of algebraic numbers), but that is not useful any more when confronted with a different context (analytic) or another class of situations (proof of the equality of transcendental numbers) and the sole presentation of new knowledge that generalizes, restructures or substitutes the original knowledge (analytic definition $\varepsilon$-$\delta$), is not sufficient for the stability of the future operational and discursive practices (analytic). The nature of this obstacle is essentially didactical[25], that is, it is the result of a didactical transposition that the teacher cannot re-negotiate, at least in the classroom environment.

From what has been said, it can be deduced that the academic practices privilege model E and tend, in many cases, to reinforce and perpetuate it, including in applications where it should be prohibited. In fact, model L is understood, in many cases, only as a sub-product of the notion of limit; in the practice, the designation of equality is restricted to the notion of arithmetic equality as equivalence, "as the possibility of obtaining a literal or semantic tautology". This presupposition conditions the types of practices that are naturalized within contemporary academic institutions in relation to the notion of equality; concretely, the notion of equality is considered as a notion *paradidactical* (Chevallard, 1985). This phenomenon has a *cultural* origin, related to the didactical and epistemological knowledge existing in the noosphere (and is made concrete in the didactic transpositions that are elaborated).

Model L can only be explained by dialectic opposition to Model E, as it is not possible to reduce the interaction between these models to naturalized practices or a cultural structuring of knowledge: the relation is inherent to the models themselves. This does not happen when comparing the order, metric, topological, and equivalence (algebraic and functional) models, which can be understood by themselves and do not need an explicit reference to another model. Furthermore, the distance between these models is minimal, being possible to differentiate one model from another only by the practical context in which it appears and by the effects that it has on the system.

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[24] In reality, from the cognitive point of view, it is not evident that both are models of the same notion. In fact, it is necessary to do a detailed analysis of how the idea of de "adjacency" influences the notion of equality as equivalence (in particular, at the moment of the emergence of decimal numbers in the school setting).

[25] In asserting that "the nature of the obstacle is *essentially didactical*" we are implicitly accepting another interpretation: in particular, epistemological. Indeed, we think that it would be necessary to study "the composition" of the obstacle, but this study is out of the realm of this article.

The numerical model, as we have shown, plays the role of a linking element between the algebraic and analytic models, and is situated in the middle of those two; this does not imply that the relationship between the algebraic and analytic models is always carried out through the intermediation of the numerical model. However, the relationship between the numerical and arithmetic models is carried out in function of the material means used (the equal sign in the calculators represent a numerical approximation with an order of approximation fixed by its technical characteristics) and the naturalized practices in contemporary institutions ($\pi = 3.1416$ is said to be "approximately equal" or a "sufficient approximation", but doesn't this imply, in itself, the acceptance of an admissible error?).

Finally, what does it mean to say that a person understands the notion of equality? In a few words, it means that he/she can interpret figure **2** in a suitable way, that is, that he/she is capable of differentiating the different models of equality, of structuring these models in a complex and coherent whole, and of confronting the operative and discursive needs with relation to the notion of equality in the different contexts of use (see section 3).

## CURRICULAR IMPLICATIONES OF HOLISTIC-MEANING

The analysis of the notion of equality that was carried out is neither circumstantial nor isolated. Wilhelmi (2003) makes an implicit use of the holistic-meaning as an interaction of mathematical models for the systemic description of the notions of "continuous function" and "absolute value", interpreting these notions as epistemic configurations where the different models, in levels of abstraction and generality (continuous function), or in levels of formalization and syntactic expression (absolute value), are given a hierarchy. Furthermore, the analysis fixes a framework for these notions within the didactic system, that is, a global perspective of which techniques are to be taught in a global teaching project. The description of the reference meaning of a statistical object, "median" that is presented in Godino (2002) as a list of objects classified in six categories (problems, procedures, language, notions, properties and arguments), can be understood as the "basic foundation" of the holistic-meaning of the notion of median (although it is necessary to carry out an analysis of relationships that are only pointed to in that work)[26].

Vinner (1991) suggests that one of the goals of the teaching of mathematics should be that of early channelling of daily thinking habits towards the technical-scientific way of thinking, and concludes that, when acquiring knowledge, the definition is the best representation of the conflict between the structure of mathematics and the cognitive process. However, in a teaching program based on the *pedagogical theory of the curriculum*[27] the definition is sub-valued. From the theoretical perspective of this author, the important question is the timing and sequence of content: a "sufficient" set of notions, techniques and propositions are introduced so that, progressively, new notions can be defined, new procedures discovered and new theorems stated. Frequently, the introduction of the notions is done in an ostensive manner, resulting almost irreme-

[26] Up until now, the notion of holistic-meaning has been used to structure and describe mathematical notions; however, it can be asked if the notion of holistic-meaning can be relevant to the description of other primary entities that are not notions (arguments, procedures, problems or propositions). *A priori*, it is admissible to accept that the notion of holistic-meaning could be used, in particular, to observe in which contexts and in what forms a specific type of argument (in particular, a technique of mathematical proof) is used, so that the problems and propositions to introduce and develop the argument could be selected. For example, the proof by mathematical induction is used in different contexts (analytic, combinatoric, etc.) and it would be advisable to describe and structure the operative and discursive practices in relation to this method of definition and proof, and classify in relation to mathematical induction the following: (i) the problems, according to their specificities (could they be solved with alternative methods ?), (ii) the procedures, according to their effectiveness (with what cost do they allow a solution, or justification to a specific class of problems ?), (iii) the notions, with respect to the frequency of their use (the notion of sequence is inherent to the method of mathematical induction; then what other notions are also inherent to this method ?).

[27] Gimeno and Pérez (1983, 189–250) describe the *Curriculum Theory* from their original pedagogical perspective which determined, in particular, the elaboration of the "basic curricular designs" (MEC, 1989). Chevallard, Bosch and Gascón (1997, 141–147) give their critical vision of that theory from the perspective of mathematical didactics.

diably in knowledge that is inefficient when confronting complex situations. Then, the educational system proceeds to define the notions, once again ostensively and formally. At this moment it is generally accepted, albeit implicitly, that the notions are acquired by means of their definitions, and that the students are capable of using them to solve problems and prove theorems. It is undeniable that a transparency between the mathematical object and its definition is assumed.

In the *Epistemological Program* (Gascón, 1998) mathematical knowledge is explicitly problematized and it is not assumed that the definition of a mathematical object is its "measure". Indeed, looking at the teaching and learning of a mathematical topic, it is necessary to explicitly model the object. These models condition the structuring of the curriculum in an institution, given that they represent transpositions of mathematical work. In these transpositional processes it is important to determine the techniques that one wishes to teach (and the justification of these techniques), that condition the systems of practices when confronted with a certain class of problems. This way, the holistic-meaning of a mathematical object thus described, as the interaction of mathematical models associated to such an object, makes up the macro and micro didactical tools.

The determination of the techniques to be taught allows the establishment of orientations in terms of the ecology of knowledge, and the elaboration of a relevant didactical transposition, that is made concrete in the construction of curriculum or in the determination of general guidelines for the creation of textbooks in an institution (macro-didactic level). On the other hand, it is possible to establish criteria of the description and comprehension of students' conceptions in the construction and communication of specific mathematical knowledge (that constitute the nucleus of the operative and discursive systems of practices in relation to the notion of equality, see section 3). Potential forms of negotiation of constructivist learning can be developed as well, where the teacher must anticipate, in "real time" the students' procedures and create mechanisms to collect and interpret information, as well as possessing action and decision strategies, adapted to concrete situations (micro-didactic level). In the next section we highlight some macro and micro-didactical consideration with respect to the notion of equality.

## 7. SYNTHESIS AND CONCLUSIONS

The evolution of mathematical didactics has brought about a progressive extension of its principal object of study: from the search for mechanisms of direct action in the processes of teaching and learning (*normative* or *technical* didactics), to the analysis of events and phenomena of teaching and learning (didactics as a *scientific discipline*). Strong epistemological studies have contributed in a definite way to this extension, studies whose goal is to fix an objective reference that works for the analysis of actual or potential projects.

The epistemological study that we have carried out fixes a framework which can be used to evaluate the "effectiveness" of possible *didactical interventions*[28], which result from the representative adaptations of the institutional reference meaning in relation to the notion of equality. Hence, the conclusions that we will make are theoretical, and do not look for immediate and "effective" academic changes, but to *regulate* possible didactical procedures in the introduction or development of the notion of equality.

### 7.1. Macrodidactics

The analysis of the notion of equality presented in this paper highlights, in particular, the need to elaborate instruments that are better adapted to an integrated teaching approach in terms of arithmetic, algebra and analysis; concretely means are needed to avoid the phenomena of *linearity and reductionism*. Linearity can be described by asserting "arithmetic precedes algebra, which precedes analysis". It is understood with this that they are "chains" and that the learning of each one establishes previous necessary conditions for the learning of the "next". The teaching scheme is:

[28] "Didactic interventions are *regulations* whose purpose is to maintain a balance, more than to produce direct effects, and these regulations are specific to the mathematical notion. " (Brousseau, 2000, 25).

$$\text{Arithmetic} \rightarrow \text{Algebra} \rightarrow \text{Analysis}$$

Reductionism can be described in the following way: algebra is understood as generalized arithmetic (with letters) and analysis as an algebra of functions. Hence, the teaching of algebra is centered on symbolic manipulation and on generalizing concrete arithmetic methods (implemented on concrete numbers); according to Gascón (1994), this has led to a real disarticulation in the class of problems of generalized arithmetic. Additionally, the teaching of analysis has tried to show the power of the formal manipulations outlined in the teaching of algebra. In the practice, these two reductionisms invert the previous scheme:

$$\text{Arithmetic} \leftarrow \text{Algebra} \leftarrow \text{Analysis}$$

The notion of equality cannot be restricted to a unique mathematical context. In effect, the sign "=" is determined by the set of relations that are established between the models associated to it, and that emerge from the usual practices in contemporary educational institutions. These practices have privileged the arithmetic model, emphasizing the importance of transformations "by equivalences" and the simplification of arithmetic and algebraic expressions to obtain "canonical" representations of mathematical objects.

This focalization on the type of tasks has caused the fundamental relations between the systems made up of the arithmetic, order metric and topological models, and the analytic and numerical models to be dialectic; hence it is not possible, for example, to comprehend the analytic notion of equality as a limit process if not by opposition to the arithmetical notion of equivalence relation between two objects[29].

The analysis of the notion of equality is a sample of how the atomized and linear teaching of arithmetic, algebra and analysis (in this order) does not contribute to learning; a triangular conception, which allows, in each concrete problem, the interaction of numerical, algebraic and analytic approximations to obtain a solution is necessary.

For example, given the Fibonacci sequence $(a_n)$, for the determination of the limit of the sequence $(a_{n+1}/a_n)$, numerical analysis allows the calculation of a "tentative" approximate value (interval of plausible solutions); the algebraic approach allows the formalization of the calculation of the limit (exact value); the analytic method allows the argument of the existence and uniqueness of the limit. The assessment of the procedures and arguments used in each case is fixed by its efficiency in the solution or justification of each one of the tasks. There is no *moment* (Chevallard, 1997) "more important" than another: the determination of an approximate value (*exploratory moment*) allows the establishment of an interval of acceptance or rejection of the exact value obtained *(technological moment*), and the justification of the existence and uniqueness of that value (*theoretical moment*), all confirm the relevance of the calculations that were carried out.

This study suggests the displacement of the focus of interest in the teaching of mathematical analysis: from the formal analysis (for example, literal manipulations of algebraic functions) to the communication and construction of knowledge in a more intuitive way, for example, graphical and numerical (where comparison and approximation represent fundamental processes). In this context, the new technologies (graphing and programmable calculators, and specialized software) should play a central role in the introduction of the notions, processes and meanings of analytic objects. In fact, "research in mathematics education in the calculus context cannot be deployed in isolation from the technological dimension". (Artigue, 1998, 258)[30].

Thus, a supposed functional necessity in the teaching of numerical calculus is what postpones the introduction and development of the fundamental notions of mathematical analysis. The numerical study of the relations between objects constitutes a way of accessing the notion of equality, making possible a flexible transition between the different models that are proposed. There is no opposition between the analytic and numerical justifications, nor is there a pragmatic gradation of them (this can only establish itself in relation to a theory and some pre-established criteria about the mathematical discourse)

[29] Indeed, our epistemological study confirms the empirical and theoretical analysis carried out by other authors. (Bloch, 1995; Artigue,1998; Wilhelmi, 2003).

[30] In French in the original: "la recherche en didactique de l'analyse ne peut se déployer en faisant abstraction de la dimension technologique" (Artigue, 1998, 258).

### 7.2 Microdidactics

It is necessary formulate strategies of didactical engineering for the development of the object "equality", as has been shown in this article. Within the Theory of Didactical Situations (TSD) (Brousseau, 1998) this would suppose the search for a fundamental situation, capable of generating (in the majority of the students) stable tensions with the majority of models linked to the notion of equality, as well as useful associations to the contexts of use in which these notions are registered.

The determination of a fundamental situation of the notion of equality is complex. We have reasoned how the analytic model of equality is understood, in many circumstances, as a sub-product of the notion of limit; for instance:

$$a = b \Leftrightarrow a_n = a,\ \forall\ n \in \mathrm{N},\ \lim a_n = b$$

This fact, together with the dialectical relation between the arithmetical models, as equivalence, and the analytic models, as a limit process, represents an epistemological support for the conjecture that the search for a fundamental situation of the notion of equality is equivalent to the search for the fundamental situation of sequential limit. The idea of relating the search for a fundamental situation for the notion of equality with another has been widely debated: (limitations of the oil tanker, Di Martino, 1992); the possibility of obtaining situations with an essentially adidactic component (Bloch, 1999); the use of graphing and programmable calculators *TI-81* for the introduction of the sequential limit (Wilhelmi 2003); etc.). Similarly, the conjecture of the equivalence of the two research problems is sustained in other didactical research. For example, Cornu (1991) distinguishes different models of the notion of limit, in which the notion of equality is implicitly problematized when there is an infinite process by means of characterizations such as "approximation", "tends to" or "distance". Tall & Vinner (1981) introduce the notion of *cognitive conflict* to highlight situations in which the "intuition" and the "formal calculations" are not compatible; for example, the justification that $0.\overline{9} = 1$ by means of the general formula of the infinite sum of a geometric progression of ratio less than one[31] is "opposed" to the intuition (incorrect) that "$0.\overline{9}$ has more and more nines and, then, gets closers and closer to 1, but never reaches it". Finally, Artigue (1998, 239) is much more explicit in establishing a relation between the notion of limit (sequence) and equality:

> "First one tries to find meaning in pre-constructed objects, using a system of practices; at a second stage, the objects are not seen as objects constructed according to definitions. Contemporary teaching of the limit concept, essential in Mathematical Analysis, is an evident examples of this, and of the necessity of invoking previous mathematical reconstructions [see previous quote, Artigue, section 5] to help understand, it seems to us, the separation between the capacity to give a concept an intuitive meaning, illustrating with examples and counterexamples, and the capacity to operationally manipulate those concepts, giving it the status of a constructed object, subject to formal proofs." (Artigue, 1998, 239)[32].

The complexity referred to does not imply the rejection of the search for a fundamental situation for the notion of sequential limit or for the notion of equality. Legrand (1996) has defended the paradoxical thesis according to which the search for fundamental situations is consistent both for research in mathematical didactics as well as for teaching, whether it is found or not. Grosso modo, Legrand justifies the search for fundamental situations because they constitute a fruitful instrument in the analysis of knowledge, of formu-

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[31] The expression $0.\overline{9}$ can be interpreted as a geometric progression of ration 1/10 and first term 9/10.

[32] *In French in the original*: "On travaille d'abord sur des objets préconstruits auxquels on essaie de donner sens par un ensemble de pratiques; ce n'est que dans un second temps que ces objets sont censés prendre le statut d'objets construits assujettis à des définitions. L'enseignement actuel du concept de limite, central en analyse, en est un exemple évident et les besoins mathématiques des reconstructions ci-dessus évoquées [see previous quote, Artigue, section 5] aident bien à comprendre, nous semble-t-il, ce qui peut séparer la capacité à donner un sens intuitif au concept, à l'illustrer par des exemples et contre-exemples, de la capacité à manipuler opérationnellement le concept avec son statut d'objet construit, assujetti à des preuves formelles." (Artigue, 1998, 239).

lating teaching projects, and of the selection of didactical interventions. Furthermore, "researching fundamental situations is a prerequisite for the teacher who wants to manage and embark on a real 'scientific debate' with precise knowledge" (Legrand, 1996, 223)[33].

The Onto-Semiotic Approach (OSA) (Godino, 2003) determines a solution for the elaboration of "quality teaching", that is, teaching that combines *know how* (technical) and *meaning* (the realm of applications of the techniques), and that articulates the epistemological analysis typical of the search for a fundamental situation (whether or not it is obtained) with the methodological and time restrictions within a concrete institution. In particular, in relation to the notion of equality, the objective consists in establishing a system of institutional practices that make possible the explicit interaction of the arithmetic model of equality with the rest of the models and, especially, with the analytic model, in such a way that the notion of equality, understood as a system, brings a balance in relation to the personal meaning that students attribute to them.

### 7.3. Theoretical

The Theory of Didactical Situations (TSD) postulates that all "knowledge" can be modelled by one or various adidactical situations that preserve the meaning attributed to such knowledge. The notion of knowledge, within TSD refers explicitly to knowledge which is an object of study (that can be made explicit, that can be communicated, and that can be validated or invalidated) in a determined culture and society. The notion of *holistic-meaning* (network of models) represents the structuring of objectivated knowledge, and creates a reference for the modelling process. Furthermore, the notion of holistic-meaning can be used in the *a priori* analysis of the search for a fundamental situation for the introduction of development of a specific mathematical notion; concretely, to determine the degree of representativity of the situation in relation to the intended institutional meaning.

In the same way, the notions of model and holistic-meaning of a mathematical object are theoretical tools for the epistemological analysis of the discursive mathematical objects, that is, the cultural products that result from mathematical activity. The notions of model and holistic-meaning propose an answer to the questions: what is a mathematical notion?; what does it mean to know that notion?; in particular, what is the notion of equality?; what does it mean to know the notion of equality?

On the other hand, as we have mentioned, the operative and discursive systems of practices are to OSA what the mathematical praxeologies are to the Anthropological Theory of Didactics (TAD). Furthermore, the notion of praxeology represents a component of the general scheme of mathematical activity (and of the products that are obtained from this activity) proposed within OSA. OSA considers that each mathematical notion is the antecedent (expression) of a semiotic function (Godino, 2002) whose consequent (meaning) is the configuration formed by the system of practices, contexts of use of the expression, and the network of emergent objects of such a system of practices (figure 3).

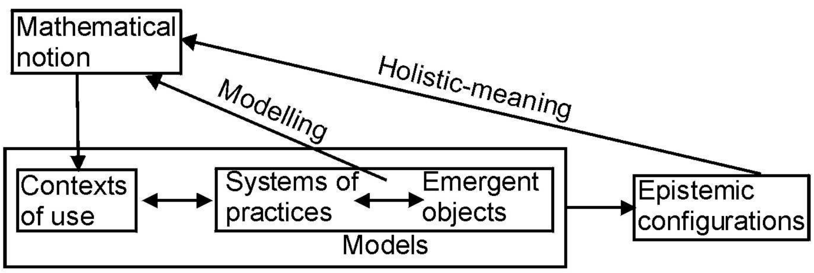


**Figura 3**. Tools for epistemological analysis in OSA

[33] *In French in the original*: "la recherche des situations fondamentales est un passage obligé pour le professeur qui veut engager et gérer un réel 'débat scientifique' sur un savoir précis." (Legrand, 1996, 223)

Hence, the essential difference between TAD and OSA is determined by the form in which both describe and analyze mathematical activity. OSA is focused on the determination of a specific mathematical ontology in the description of the possible semiotic functions involved in mathematical activity and in the characterization of the nature of this activity. TAD is centered on the determination of praxeologies (the mathematical objects can be described only as constituents of the praxeologies) and in the analysis of the ecological issues within the institutions (the mathematical objects represent the product of an institutional activity).

> “TAD situates *mathematical activity* and, accordingly, the *study of* mathematics as an activity, in the *collection of human activities and that of social institutions* […] It is admitted that, in effect, *every* regularly carried out human activity can be subsumed under a *unique*, model that is summarized here with the word praxeology.” (Chevallard, 1999, 223)[34].

Finally, the notions of praxeology and epistemic configuration are powerful tools for didactical and curricular analysis. The theories of curriculum structure the subject that will be the object of study, in terms of *conceptual*, *procedural* and *attitudinal* contents, ignoring the specificities of each discipline. The application of the model of description and analysis of mathematical activity proposed by TAD, when structuring the curriculum, would suppose the identification of techniques that are necessary to carry out the types of tasks that students would need to confront, highlighting the tensions that force the use of a specific technique and the justification of the necessity of its “life” in the academic context (underlying technologies and theories). However, in the study process the quadruple (task, technique, technology, theory) is not always made explicit.

> “We can imagine an institutional world in which all human activities would be governed by well adapted praxeologies, in order to deal with every task in an efficient, reliable and intelligible manner. However, a world with these characteristics does not exist […] Generally, this praxeological poverty is translated to the absence of techniques.” (Chevallard, 1999, 230–232)[35].

OSA, on the other hand, distinguishes six categories of primary objects that form a system of practices: problems, procedures, language, notions, properties and arguments. An epistemic configuration is a system of objects (and of semiotic functions that are established between these objects) in relation to the communication, assessment, formulation and resolution of a mathematical situation. Furthermore, OSA describes the analysis of the proofs of the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$ in terms of the notion of operative and discursive system, without needing to explain the general techniques of the proofs or the procedures to justify these techniques. In effect, the nucleus of the discourse is made up of *definitions*, that do not represent elements of a praxeology (understood as the quadruple) but in ostensive objects of an epistemic configuration in relation to the proof of the proposition $\sqrt{2}=\frac{2}{\sqrt{2}}$ (*problem*); in fact, the *argumentation* of the proofs is carried out by means of a formalized *language*, based on certain *notions* (real line, order relation, distance, connection, equation, injective function, neighborhood-limit, error-approximation), supported by the *properties* of the notions involved and in logical laws (the excluded third, for example), and it is carried out by means of specific *procedures*

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34 *In French in the original*: “La TAD situe l’activité *mathématique*, et donc l’activité d’*étude* en mathématiques, dans *l’ensemble des activités humaines et des institutions sociales* […] On y admet en effet que *toute* activité humaine régulièrement accomplie peut être subsumée sous un modèle *unique*, que résume ici le mot de *praxéologie*.” (Chevallard, 1999, 223).

35 *In French in the original*: “On peut imaginer un monde institutionnel dans lequel les activités humaines seraient régies par des praxéologies bien adaptées permettant d’accomplir toutes les tâches voulues d’une manière à la fois efficace, sûre et intelligible. Mais un tel monde n’existe pas […] Ordinairement, la pénurie praxéologique se traduit d’abord par un manque de *techniques*.” (Chevallard, 1999, 230–232).

(the selection of the representatives of a mathematical object, the definition of intervals or function, the editing of a program, etc.)

**Recognition**

Reference Projects: Resolución UPNA nº 1.109/2003, de 13 octubre, y MCYT-FEDER BSO2002-02452.

## REFERENCES

ALIPRANTIS C. D., BURKINSHAW O. (1999), *Problems in real analysis. A Workbook with solutions*, 2nd Edition. San Diego, CA: Academic Press.

ARTIBI A., BROUSSEAU G. (2000), La dé-transposition de connaissances scolaires. *Recherche en Didactique des Mathématiques*, 20(1), 7–40.

ARTIGUE M. (1998), L’évolution des problématiques en didactique de l’analyse. *Recherche en Didactique des Mathématiques*, 18(2), 231–262.

AYALA R., DOMINGUEZ E., QUINTERO A. (1997). *Elementos de la topología general.* Madrid: Addison – Wesley Iberoamericana.

BLOCH I. (1999), L’articulation du travail mathématique du professeur et de l’élève dans l’enseignement de l’analyse en première scientifique. Détermination d’un milieu. Connaissances et savoirs. *Recherche en Didactique des Mathématiques*, 19(2), 135–194.

BLOCH I. (1995), *Approche didactique de l’enseignement des premiers concepts de l’analyse*, Mémoire DEA. Bordeaux: Univesité Bordeaux I, LADIST.

BOLEA P., BOSCH M., GASCÓN J. (2001), La transposición didáctica de organizaciones matemáticas en proceso de algebrización: el caso de la proporcionalidad. *Recherche en Didactique des Mathématiques*, 21(3), 247–304.

BOYER C. (1969), *Historia de la matemática*. Madrid: Alianza Universidad, 1996.

BROUSSEAU G. (2000), Educación y didáctica de las matemáticas. *Educación Matemática*, 12(1), 5–38.

BROUSSEAU G. (1998), *Théorie des Situations Didactiques*. Grenoble: La Pensée Sauvage.

BROWN J. R. (1998), What is a definition? *Foundations of Science*, 1, 111–132.

CANTORAL R., FARFÁN R. M. (2003), Mathematics Education: a vision of its evolution. *Educational Studies in Mathematics*, 53, 255–270. Dordrech, HOL: Kluwer.

CHEVALLARD Y. (1999), L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherche en Didactique des Mathématiques*, 19(2), 221–266.

CHEVALLARD Y. (1997), Familière et problématique, la figure du professeur. *Recherche en Didactique des Mathématiques*, 17(3), 17–54.

CHEVALLARD Y. (1985), *La transposition didactique*. Grenoble: La Pensée Sauvage.

CHEVALLARD Y., BOSCH M., GASCÓN J. (1997), *Estudiar matemáticas. El eslabón perdido entre enseñanza y aprendizaje*. Barcelona: ICE – Horsori, Universitat de Barcelona.

CORNU B. (1991), Limits. In D. Tall (ed.), *Avanced mathematical thinking* (pp. 133–166). Dordrecht, HOL: Kluwer.

DI MARTINO H. (1992), *Analyse du contrôle épistemologique d’une situation didactique: La situation du pétrolier*, Mémoire DEA. Grenoble: IREM Grenoble.[URL: Http://www.ac-grenoble.fr/irem/publications.html]

DOUADY R. (1986), Jeux de cadres et dialectique outil-objet. *Recherche en Didactique des Mathématiques*, 7(2), 5–31.

FREUDENTHAL H. (1986), *Mathematics as an educational task*. Dordrecht, HOL: Reidel.

GASCÓN J. (1998), Evolución de la didáctica de las matemáticas como disciplina científica. *Recherches en Didactique des Mathématiques*, 18 (1), 7–33.

GASCON J. (1994), Un nouveau modèle de l’algèbre élémentaire comme alternative à ‘l’arithmetique généralisée’. *Petit x* 37, 43–63.

GIMENO J., PÉREZ A. (1983), *La enseñanza: su teoría y su práctica*. Madrid: Akal.

GODINO J.D. (2003), *Teoría de las funciones semióticas. Un enfoque ontológico-semiótico de la cognición e instrucción matemática*. Granada: Dept. de Didáctica de la Matemática, Universidad de Granada. [URL: Http://www.ugr.es/local/jgodino]

GODINO J.D (2002), Un enfoque ontológico y semiótico de la cognición matemática. *Recherche en Didactique des Mathématiques*, 22(2/3), 237–284.

GODINO J.D., BATANERO M.C. (1994), Significado institucional y personal de los objetos matemáticos. *Recherche en Didactique des Mathématiques*, 14(3), 325–355.

GODINO J. D. (1993), La metáfora ecológica en el estudio de la noosfera matemática. *Quadrante*, 2(2), 69–79.

INDURÁIN E. (2001), *Memoria sobre concepto, método, organización, funciones y programas en la docencia e investigación en el área de Análisis Matemático*. Pamplona, ESP: Universidad Pública de Navarra.

LAKATOS I. (1976), *Proofs and Refutations*. Cambridge, UK: Cambridge University Press.

LEGRAND M. (1996), La problématique des situations fondamentales. Confrontation du paradigme des situations à d’autres approches didactiques. *Recherche en Didactique des Mathématiques,* 16(2), 221–280.

LEIKIN R., WINICKI-LANDMAN G. (2000), On equivalent and non-equivalent definitions: part 1”. *For the Learning of Mathematics*, 20(1), 24–29.

LORENZO J. de (1974), *La filosofía de la matemática de Poincaré*. Madrid: Tecnos.

MEC (1989). *Diseño curricular base*. Madrid: Ministerio de Educación y Ciencia.

ROSS K. A. (1980), *Elementary analysis: the theory of calculus*. New York: Springer-Verlag, 2000.

RUSSELL B. (1903), *Los principios de la matemática*. Madrid: Espasa-Calpe, 1967.

SÁENZ-LUDLOW A., WALGAMUTH C. (1998), Third graders’ interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, 35, 153–187. Dordrech, HOL: Kluwer.

SCHNEIDER M. (2001), Praxéologies didactiques et praxéologies mathématiques à propos d’un enseignement des limites au secondaire. *Recherche en Didactique des Mathématiques*, 21(1/2), 7–56.

TALL D. (ed.) (1991), *Avanced mathematical thinking*. Dordrecht, HOL: Kluwer.

TALL D., VINNER S. (1981), Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151–169.

VINNER S. (1991), The role of definitions in the teaching and learning of mathematics. In D. Tall (ed.), *Avanced mathematical thinking* (pp. 65–81). Dordrecht, HOL: Kluwer.

WILHELMI M. R. (2003), *Análisis epistemológico y didáctico de nociones, procesos y significados de objetos analíticos*. Sección 2: Tesis doctorales, nº23. Pamplona: Universidad Pública de Navarra.