Using Short Open-ended Mathematics Questions to Promote Thinking and Understanding.

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The mathematics curriculum in Singapore aligns with 21st century reform-based visions of schooling around the world. There is a move to de-emphasize the use of routine mathematical questions that promotes basic algorithmic skills. Teachers need to be equipped with a variety of mathematical tasks for their pupils that can enhance their teaching. This paper shares the experience of some primary teachers who have used short open-ended problems that enabled their pupils to demonstrate their thinking and understanding of mathematical concepts in a variety of ways.

Introduction

Figure 1 depicts the framework of the Singapore Curriculum (MOE, 2000) that was conceptualized since the early 90's to encompass *mathematical problem solving* as its core. Mathematical problem solving as stated in this framework includes using and applying mathematics in practical tasks, in real life problems and within mathematics. It advocates that problems should cover a wide range of situations from routine mathematical problems to problems in unfamiliar context and open-ended investigations that make use of the relevant mathematics and thinking processes.

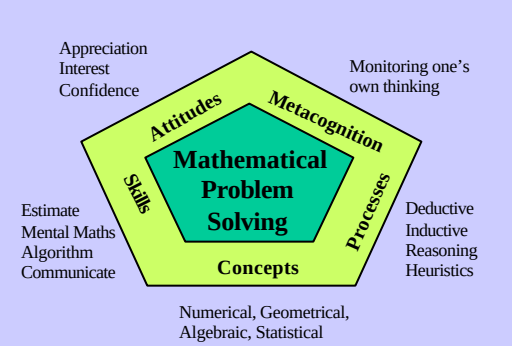

Figure 1: Singapore Mathematics Curriculum

The curriculum framework recognises that problems are both a means and an end in school mathematics instructions. Teaching via problems serves as a mean for students to construct mathematical concepts and to develop skills. Problems lead students to use heuristics such as to investigate and explore patterns and as well as to think critically. To solve problems the students must observe, relate, question, reason and infer. Success in problem solving relates to students' disposition and monitoring of their own thinking processes. Mathematics educators including schoolteachers are beginning to pay attention to the kind of tasks that they give to students. Currently, most mathematics classrooms in Singapore are engaged in the traditional approach of whole class expository teaching followed by pupil practice of routine exercises and regular written tests consisting of multiple-choice questions, short-answer and long-answer open response questions (Chang, Kaur, Koay and Lee, 2001). There is a need to equip teachers with a bank of greater variety of mathematical questions for problem solving that can enhance their teaching methods. Pupils must encounter intriguing mathematical problems where they can reason and offer evidence for their thinking, communicate and present their ideas in mathematics, and find connections across mathematics as well as in real life.

Classification Scheme for Mathematical Problems

In order for teachers to realise the encompassing role of problem solving in the curriculum, they should to be able to distinguish between and among the various types of problems and their roles. Equip with the knowledge and understanding they can judiciously select or even construct tasks for their pupils that will promote different forms of thinking activities in a mathematics lesson. Based on a systematic search of literature on problem solving and use of problems in research by Foong (1990), this paper proposes a classification scheme for different types of problems that are being encouraged for the 21st century mathematics classrooms, as shown in figure 2.

We shall adopt the commonly accepted definition of a "problem" as one where thinking takes place when a person is confronted with a problem that has no immediate solution and that the problem solver accepts the challenge to tackle it. This will exclude those school textbook exercises that are

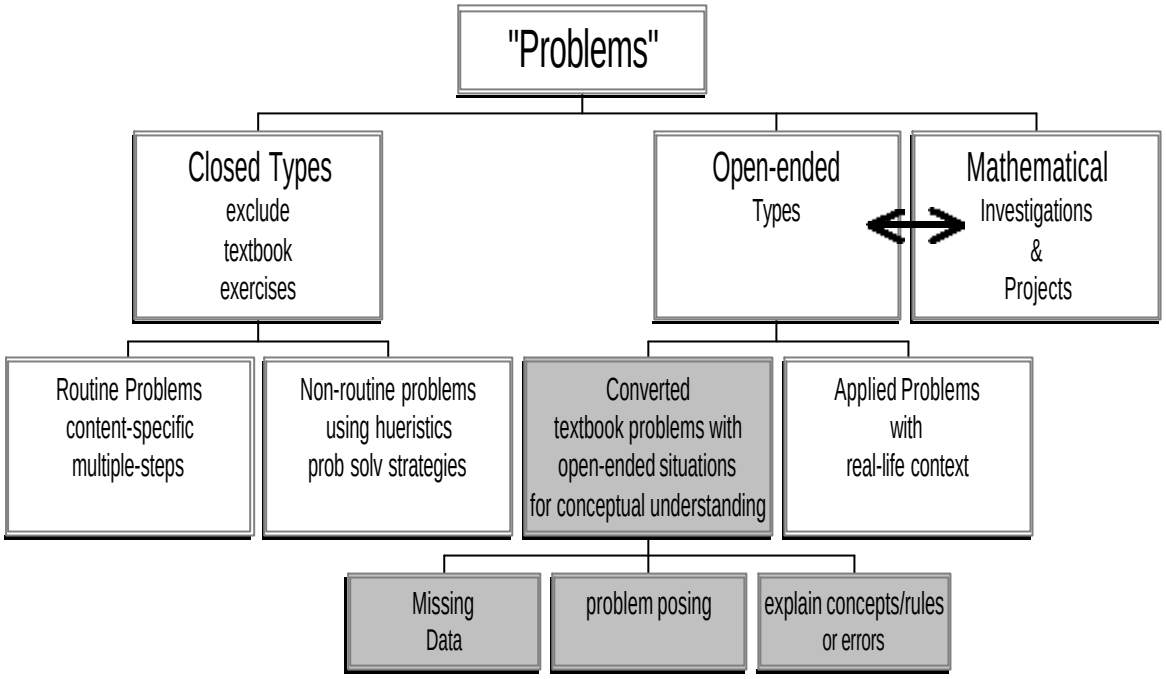

Figure 2: Classification Scheme for Mathematical "Problems"

used for practice of an algorithm or skill such as in computational sums or in simple one to two-step translation word problems. In this scheme, basically, most *problems* can be broadly classified as "closed" or "open-ended" in structure. Problems in this classification scheme have their different roles in mathematics instruction as in teaching ***for*** problem solving, teaching ***about*** problem solving, or teaching ***via*** problem solving. These roles will be illustrated in the following discussion on the different categories of problems.

Closed Problems

Closed problems are "well-structured" in terms of clearly formulated tasks where the one correct answer can always be determined in some fixed ways from the necessary data given in the problem situation. These closed problems would include
content-specific routine multiple-step problems as well as non-routine heuristic-based problems. To tackle these problems, the solver through productive thinking rather than simple recall must generate some process skills or some crucial steps in the solution method. Figure 3a shows examples of the routine content-specific multiple-step problems, also known as "challenge problems" to local teachers for the topic on Fraction. Teachers use such challenge problems for its role in *teaching **for** problem solving* where the emphasis is on learning mathematics for the main purpose of applying it to solve problems on a particular topic. Such type of challenge problems in the Singapore syllabus are used to assess so called higher–order analytical thinking skills of the pupils. Similar structured word problems of this such kind are posed as challenge sums across related arithmetic topics like whole numbers, fractions, ratio and percent.

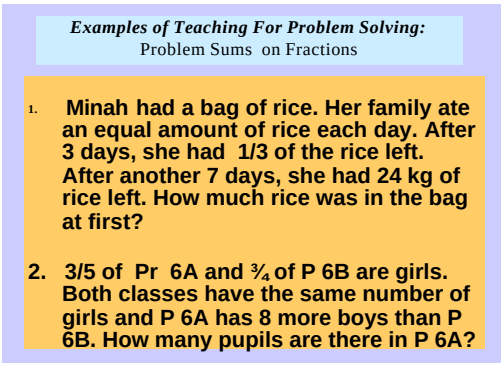

Figure 3a: Routine challenge sums

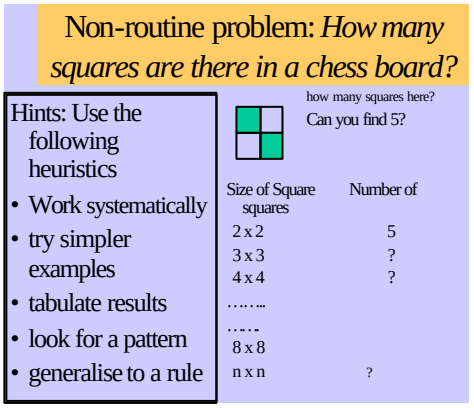

Figure 3b: Non-routine heuristic problem

In tandem with the current call for more emphasis on process skills, there are recommendations in the syllabus for teachers to use non-routine problems to teach thinking skills and problem solving heuristics. Non-routine closed problems of the type shown in Figure 3b are beginning to appear in the mathematics classroom where teachers use them in the role of *teaching **about** problem solving.* The emphasis is on using heuristic strategies like *use guess and check, working systematically, try simpler cases, tabulate data, look for a pattern , generalize, etc.etc.,* to solve an unfamiliar problem that is usually not domain-specific to any topics in the syllabus.

Open-ended Problems

This category of problems are often considered as "ill-structured problems" for they lack clear formulation as there are missing data or assumptions and there is no fixed procedure that guarantees a correct solution. Much of the real-world problems, e.g. *Design a better time-table or How much water and money can a school save during the "Save Water" campaign?* would fall into this category. Mathematical investigation in the form of major open-ended projects that explore and extend a piece of pure mathematics for its own sake or those real-world problems that require mathematical modeling meant for higher level mathematics will not be discussed in this paper. Such open-ended projects usually require students to demonstrate their ability in the form of a detailed report on how they carry out an extended piece of independent work in mathematics showing their creative application of mathematical knowledge and skills.

Converting Textbook Exercises into Short Open-ended Problems

For the purpose of this paper, we will focus on the types of short open-ended problems that teacher can convert from closed questions found in textbook exercises. Such tasks can be incorporated into normal lessons and would not require an extensive length of several periods or weeks for pupils to do. Teachers can use short open-ended problems for its role in *teaching **via** problem solving* that emphasizes learning mathematical concepts and skills through a problem situation. Caroll (1999) found that short open-ended questions provided teachers with quick checks into students' thinking and conceptual understanding. They are no more time-consuming to correct than the worksheet questions that teachers normally give. When used regularly, the pupils can develop the skills of reasoning and communication in words, diagrams, or picture.

Consider a typical question that one may find in a primary textbook, Figure 4. Formulated in such a closed structure, the teacher and pupils would have in mind the expected standard response of seeing it as a multiplication sum. However, in the same context of a Polar bear, teachers in the Netherlands (Van den Heuvel-Panhuizen, 1996) posed an open-ended situation for the pupils to solve, see Figure 5.

Closed Question

- **A Polar bear weighs about 20 times as heavy as Ali. If Ali weighs 25 kg. What is the mass of the Polar bear?**
- **Expected Pupils' responses:**
- **Cue word: *"20 times as heavy"***
- **Concept: *"multiplication" situation***
- **Procedure: *25 x 20 =….***

$$\begin{array}{r} 25 \\ \times\ 20 \\ \hline \end{array}$$

Figure 4: Closed question with an expected standard response.

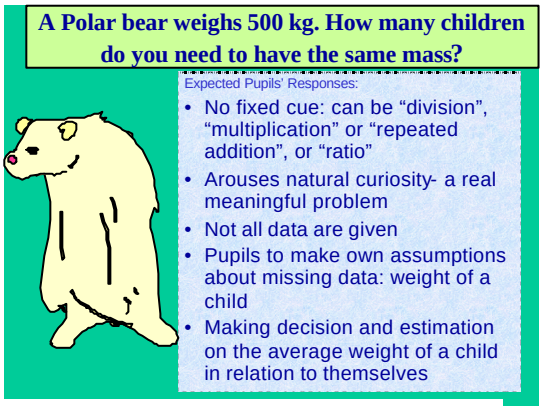

Figure 5: Open-ended situation for a variety of responses.

According to Van den Heuvel-Panhuizen, the Polar Bear problem (Figure 5) represents an important goal of mathematics education. In an open-ended situation, pupils in addition to applying calculation procedures are also required to solve realistic problems where there is no known solution beforehand and not all data is given. It would require pupils' own contributions, such as making assumptions on the missing data. Without giving the children's weight, it becomes a real problem and the pupils have to think about and estimate the weight of an average child. There is no cue word for students to figure out which operation to use as in the closed question, Figure 4. For the Grade 3 children in the Netherlands, the open-ended situation enabled the pupils to show what they knew and

it also enabled the teachers to acquire richer information on how their pupils tackled problems. The children used their acquired knowledge of measurements, they used varied 'division' strategies and different models and notation schemes to support their thought processes.

Singapore Primary Teachers' Workshop and Pupils' Work

Since 1999, several local teachers have attended workshops conducted by the author on the use and construction of open-ended problems in the teaching of primary mathematics. These workshops constituted as one of the many units in their in-service training programme. In the workshop, teachers were exposed to the different kinds of short open-ended questions, they were given opportunity to convert textbook questions into open-ended problems and then they had to trial their questions on their own pupils in school. The features and types of short open-ended questions that they can construct are shown in Figures 6 and 7.

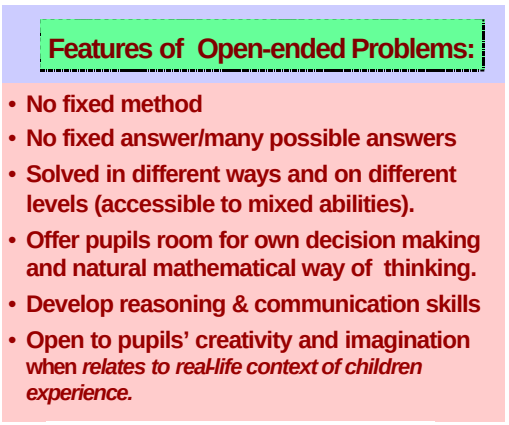

Figure 6: Problem Features

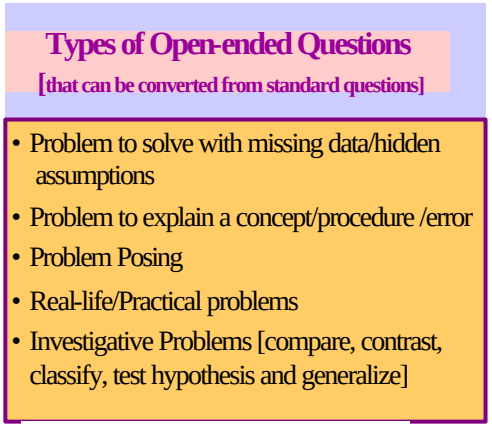

Figure7: Problem Types

Samples of Teachers' Open-ended Questions and Pupils' Work

The teachers found little difficulty in constructing the questions but initially many of them were apprehensive about giving their pupils such tasks. The reason being that it has never been part of their teaching where they require pupils to give explanations and reasons for their solutions. The practice has always been pupils are given problems that have only one answer and one taught method of finding it. When the teachers had to trial the open-ended questions with their pupils as part of their assignments in the workshop, many of them were surprised by the various and rich responses that most of their pupils could give. Of course, there were also reports by some teachers that many pupils seem to lack the reasoning and communication skills that they had hoped to see in the pupils' work. Some of the teachers gave the problems to their pupils to work in small groups to generate discussion among them. The following are some samples of two teachers' questions and their pupils' responses:

Problem with missing data

Miss A, a primary two teacher formulated this open-ended question as an antithesis to the routine exercises that her pupils were doing on simple word problems in addition and subtraction:

> There are some apples on the table and some apples in a small basket. If there are 50 apples altogether, how many apples are on the table. Explain your answer.

She decided to have the pupils worked individually as she felt that they were too young and were not used to working in groups for mathematics. To her surprise, most of her pupils were able to respond appropriately and the variety of the children's thinking was enlightening. Some children could reflect upon analysing the question, that it cannot be solved as there was not enough given information. There was a child who was able to generalize that there were many possible answers. Different strategies were used: subtraction, division, and even model drawing. The uses of verbal, symbolic and pictorial mathematical representations were manifested in these children's work when they were given the opportunity to be creative in an open-ended situation such as this. Some pieces of the primary two children's work are shown in Appendix 1.

Problem Posing

Miss B, formulated this open-ended problem posing, Figure 8, for her primary five girls to assess their understanding on the concept, *average*. She analysed their work and classified them. From the pupils' work she realized that although many of her pupils were able to apply the procedure for finding average in their standard workbook, there were some who had no understanding of the concept based the questions that these pupils had constructed. Some have misconception of average as *the total* or as *in sharing or division.* Interestingly, she also found some pupils have very poor measurement sense in the context they used for their constructed questions. They would use unrealistic measurement units like *93 cm for a toddler, 81 kg for an obese hamster, or a family using only 93 litres of water in a week!* This might also indicate that pupils could be modeling after some textbook or teacher questions that often do not necessarily relate to the real world. Another insight to some of her female primary 5 girls was their imaginative or otherwise, creations that used context of *boyfriends, bald man and underwear's.*

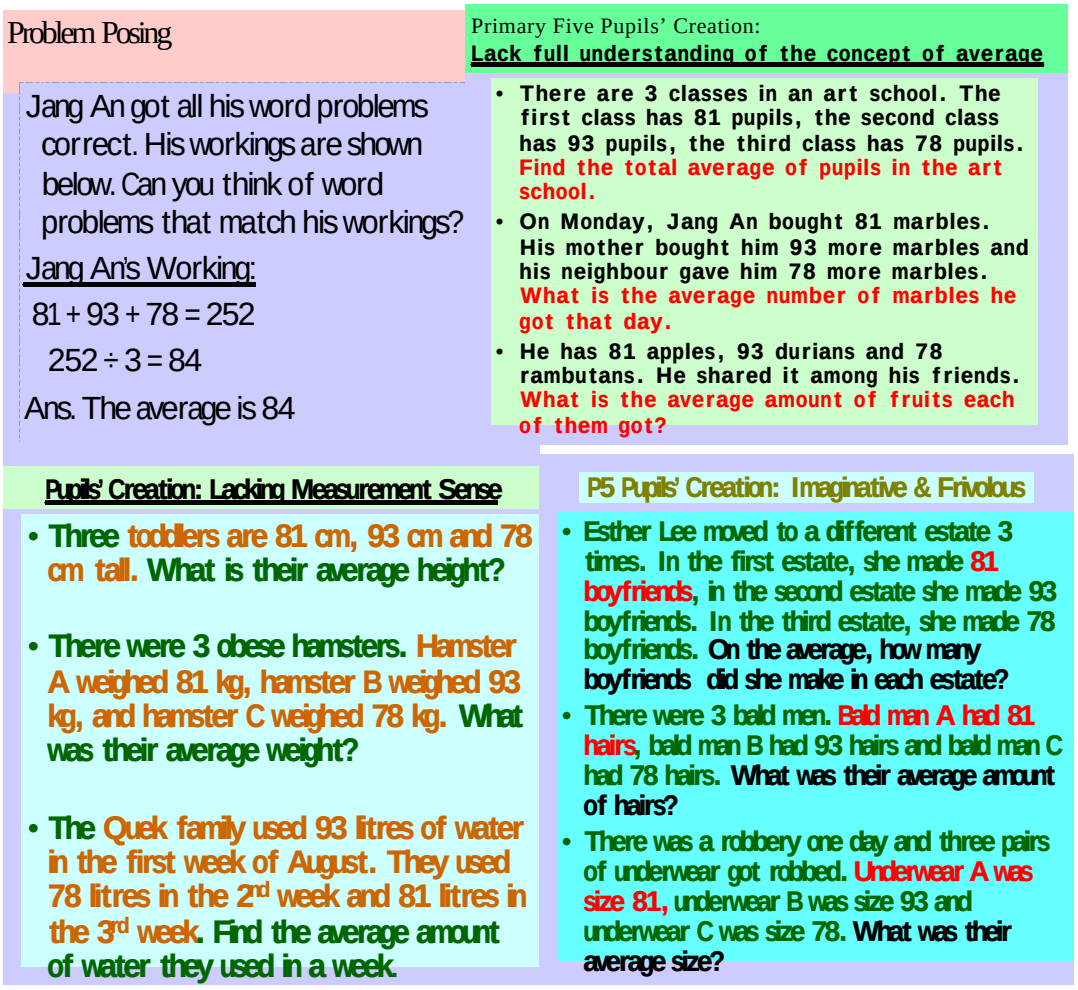

Figure 8: Problem Posing and Pupils' Creations

Due to the limitation of space, there are many other teachers' constructed questions and interesting pupils' responses that could not be accommodated in this paper. From the samples that we have presented here, the teachers who have used such short open-ended questions in their classrooms found many advantages that enhance their professionalism. This approach has enabled teachers to see pupil's thinking rather than the teacher's own thinking through closed questions that have predetermined method and answer. In the traditional approach, there has been a tendency for pupils to see mathematics as merely practicing one-step, two-step or many-step procedures to find answers to routine problems. On the other hand, the open-ended questions if given on a regular basis would instill in them that understanding and explanation are crucial aspects of mathematics. From the

evidence presented in the pupils' work, the pupils were capable of communication in mathematics using words, diagrams, pictures or manipulatives. For concluding remarks to encourage others to implement open-ended questions in their teaching, excerpts of three of the teachers' observation and reflection after they had trialed their problems with their own pupils, are expressed in the following:

Teacher X: *"A lesson such as an open-ended problem solving was indeed a fruitful one for my pupils and me. There was another avenue to motivate pupils to learn mathematics. ….When the pupils got started to work in pairs, I could see that they were thinking very hard as to how the work could be done. I heard comments like' How can a person be so heavy?' and 'It is not possible to have so many passengers in a bus?'. I was glad that they tried to explain and justify to their peers…"* (primary four)

Teacher Y*: "From my pupils' work, I feel that critical thinking can be taught. These pupils were frightened at first to take the first step, for fear of getting the answers wrong. But after seeing their friends' answers and reasons, these fears were lifted…..Lastly, the pupils do include their own personal experiences and knowledge. I was quite impressed."* (primary four*)*

Teacher Z: *" ….Another advantage of open-ended investigation is that learning is more active. Pupils had a great time making wild guesses, trying to defend their procedure. However, our pupils are still not accustomed to accepting other pupils' alternative answers, it will take them some time to recognize the fact that there are sometimes more than one solution to a problem.…… Open-ended investigations here no doubt require more preparation and time on the part of the teachers, but judging from the positive learning outcome of the activity carried out from most of the pupils, the pupils will begin to see mathematics can be meaningful and interesting.* (primary six)

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