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A *MATHEMATICA* NOTEBOOK ABOUT ANCIENT GREEK MUSIC AND MATHEMATICS

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Abstract It is known that Information and Communication Technology are becoming increasingly useful in teaching mathematics, but so far it has not been considered as well that this technology, and symbolic calculus languages as Mathematica or Maple in particular, can pave the road to new connections between mathematics, computer and humanities.

Music and Mathematics have always been deeply connected. At the beginning of our mathematics there is Pythagorean mathematics that was first and foremost music-driven, and the end of ancient music was connected to the equal-temperament, linked to the idea of real number.

We give an outline of a Mathematica notebook about ancient Greek Music, that includes a short introduction to mathematical music theory, a part about Greek mathematical music theory and the difference between Pythagorean and equal-tempered tuning, a system to play ancient music, in which we can choose the system (enharmonic, chromatic, diatonic) and the mode (Doric, Lydian, Phrygian, etc.), and a short report about the different theories (geometric, arithmetic and music-theoretic) about the discovery of incommensurability. It is noteworthy that playing with this system the first stasimo of Euripides' tragedy Orestes, both in well-tempered and Pythagorean tuning, usually all participants recognize a difference.

It is well known the role played by music in the ancient Greek civilization.

Music for us is a "separated" activity, a specific business and a purely aesthetical enjoyment. We hear music quite random: walking in the road, driving the car, from TV and hi-fi at home. Rarely we 'listen' music, in concert, and also those performances are sharply autonomous.

Ancient music was instead always 'contextual', its performances were always intertwined with specific activities, working, wedding, praying, etc., and almost every activity had its specific musical background (SACHS, 1962).

These characters were shared by ancient Greek music, probably a legacy of Babylonian music. It was basic in the social and political establishment: Terpander (VII century B.C.) and Timotheus (IV century B.C.) were both condemned in Sparta for having added more strings to the lyre.

And Plato shared the opinion of the musicologist Damon that:
"For a change to a new type of music is something to beware of as a hazard of all our fortunes. For the modes of music are never disturbed without unsettling of the most fundamental political and social conventions" (*Respublica 424c*).

Ancient Greek poetry and tragedy was inseparable from music (even the term 'lyric' is from the 'lyre') and even the term "music" etymologically stems from the "Muses", daughters of Mnemosine, the "memory", and Goddesses of poetry, music and dance, and originally included all those artistic fields.

Music was the ground of the Platonic theory of education, appeared both in elementary (with gymnastic) and superior (part of the *Quadrivium*) education, and the ancient "games" (Olympic, Isthmic, Pythian, etc.) were based on music and gymnastic.

In Philolaus the same structure of kosmos' harmony reflected musical consonances, and in Plato's *Timeus* the whole cosmology is built on the music-theoretical ratios.

In this context music had also its own magic power.

Music moved the stones to build Thebes' walls and Orpheus' music could bewitch animals and trees, and even rescue Eurydice from Hades. The lyre was Apollos' emblem and Atheneus wrote that "Greek ancient wisdom seems to have been tied most of all with music. To this extent they judged the most musical and the wisest Apollos among the gods and Orpheus among the semi gods".

Nevertheless ancient Greek music is almost absent in our perception of Greek civilization (in our universities, in our books and also in the actual performances of ancient Greek tragedies and lyric poetry). The problem is that it is almost impossible for us to appreciate a reliable ancient Greek musical performance: we have few original pieces, the texts are hardly understandable and their execution is often accomplished only in modern musical terms.

An often ignored problem is that our musical performances are accomplished in the well-tempered tuning, that appears only in the XVII century, linked to the idea of *real number*, whereas all the earlier music (Middle Ages music included) was instead performed in other tunings based on rational numbers, more or less connected with the Pythagorean theory of music.

We can not give here elements of mathematical music theory. The interested reader can find many good basic books on the subject. Among those specifically concerned with ancient music we can recommend (Burkert, 1972, Sachs, 1943, 1962, van der Waerden, 1963, West, 1992).

The notes were played in the Pythagorean mathematical music theory on the so called "canon": a rod with a chord, that was divided in 12 equal frets. With bridges on the frets it was possible to play 12 different notes, and hence the consonances between two notes could be regarded as intervals on the canon, and represented as ratios between integers from 1 to 12, more precisely "superparticular ratios" n:n+1 (the octave, 1:2, the fourth, 3:4, and the fifth, 2:3). On the canon 6:12 was the octave, 6:8 or 9:12 was the fourth, 6:9 or 8:12 was the fifth. Hence the difference between the fourth and the fifth, the so called "tone", was the ratio 8:9.

The main mathematical difference between ancient and modern scales on a chord is that the former were centred on three harmonies yielded by simple superparticular ratios, whereas the well-tempered (or equal-tempered) scale is based on a logarithmic division of the chord (the n-th semitone is obtained by the n-th power of the 12-th root of 2).

The Pythagorean musical theory was explicitly connected by the Pythagoreans with the means: the arithmetic mean, a-b=b-c, the geometric a:b=b:c, and the harmonic or subcontrary a-b:a = b-c:c. The intervals were ratios, so that to cut an interval a,b in two equal parts meant to find a x such that a:x = x:b, in other words, by the geometric mean.

However the first and third can be easily recognized in the fourth 12 - 9 = 9 - 6 (such as C-F) and the fifth 12-8 : 12 = 8-6 : 6 (such as C-G) consonances, whereas the geometric mean does not correspond to any consonance in one octave of the "canon", for it would yield $12:(6*\sqrt{2})) = (6*\sqrt{2}):6$. With the word of Szabo (1978, 174): "an octave cannot be divided into two equal subintervals by a number".

In the modern well-tempered tuning the intervals are identical in all parts of the scale, as in a sort of homogeneous (constant curvature) Riemannian space, but those harmonic consonances are not perfectly tuned.

Among the ancient tunings the Pythagorean produced perfectly consonant fifths. Being the intervals not identical along the chord, as in a sort of heterogenous space, in ancient Greek music it was impossible "to change tonality" preserving the melody, and there were hence substantially different modes (Dorian, Lydian, Phrygian, etc.) whose employment was not just a question of vocal accommodation achieved with a change of tonality, but gave specifically different emotional effects.

In addition equal temperament gives twelve logarithmically equal semitones (each conventionally set equal to 100 cents) whereas Pythagorean tuning gives five tones (each of 204 cents) and two semitones (90 cents) whose sum is clearly not a tone.

Moreover, to cut the musical intervals by the geometric mean of the superparticular ratios meant substantially to find the way to connect the seven modes of Greek music (Dorian, Phrygian, Lydian, etc), which were considered by Plato (and credibly even by the Pythagoreans) basic for the harmonic behaviour of the citizen and the city as well.

Plutarchus reminds us that the crucial problem was the division of the tone (9/8, i.e. the interval between the fourth and the fifth) in two 'equal', i.e. 'proportional', parts, and that the Pythagoreans discovered it to be impossible (see in the following). An equivalent problem was whether the octave could be divided in 6 tones (according to Aristoxenus, who rejected the relevance of the mathematical impossibility of cutting the tone) or in 5 tones and 2 not joinable semitones (according to the Pythagoreans).

What about ancient Greek music? The remains of Greek music are the following:

- a large collection of ancient music-theoretical texts, some of them based on a mathematical background (Pythagoreans, Euclid, Ptolemy, Nicomachus, Boethius, etc.), others (first and foremost Aristoxenus) with a more empirical Aristotelian flavour. However the earliest available texts are from the III century B.C., and also the mathematical systems (for example Pythagorean and Ptolemaic) are not completely coincident.
- few scores of ancient compositions, the earliest ones from the III century B.C., and including also symbols not easily reducible to sharp frequencies.
- fragments of literary texts incidentally reporting descriptions concerning musical performances, whose meaning is far from being always clear
- specimens of lyres and auloi. Musical analysis of the available auloi show for example that the ancient fourth was the Pythagorean (498 cents) and not the well-tempered one (500 cents) (Sachs, 1943, West, 1992). However aulos players were craftsmen without a mathematical and theoretical background. In addition, they were very skilful in producing different sounds employing the same hole, and hence the geometric intervals on the auloi did not mean necessarily proportional musical intervals. So the existence of exact Pythagorean consonances perhaps implies a sharp conditioning of mathematical music theory even on the actual performances.

Few decades ago well-tempered tuning was considered the right one and the others, ancient or not European, simply wrong or dissonant. Modern musicology and ethnomusicology revised this opinion and it is not infrequent today to listen not exactly well-tempered, but wonderful, musical pieces played by great singers, for example Billie Holiday.

It is likely that even the preference for well-tempered scales that can be empirically verified in modern musicians who play instruments with no predetermined tuning (violins and trombones) could be ascribed to a many-generations conditioning. Nonetheless our capacity of enjoying not well-tempered music shows that our musical perception is not sharply one-tuning conditioned.

Anthropologists today maintain that the musical world of a civilization is probably, even for primitive civilizations, very complex and not reducible to a scale. In addition they often point out that to understand a musical world means also to be able of listening and playing it (Sachs, 1943).

Now we can ask: how far is our understanding of ancient Greek civilization undermined by the fog surrounding our knowledge and our performances of its music? And how far was Greek mathematical theory of music correctly representing ancient Greek music?

It is worthwhile to remark the fact that the role of music in Greek civilization involved also Mathematics. It is known that the most reliable and surely ascribable part of Pythagorean mathematical tradition is the music-theoretical one (Burkert), and it has been claimed, first and foremost by Paul Tannery, that even the discovery of incommensurability had a musical genesis.

According to this opinion the original, crucial for the Pythagoreans because of the general role of music in the social and political establishment, problem was the "cutting of the tone" (the interval between the fourth and the fifth, i.e. the ratio 8:9) in two 'equal' parts. In other words the problem was to find x such that 8:x = x:9. Obviously 17/2 is to high, 33/4 to low, and so on. Archytas succeeded in proving that in general it was impossible to cut a superpaticular ratio, i.e. to find a x such that n:x = x:n+1 in the ancient Greek arithmetic (roughly we could say in rational arithmetic).

From this general result, for n=1, we get the irrationality of the square root of 2.

For more information about this topic we address the reader to a forthcoming paper (Borzacchini, 2001).

Most authors (Burkert, 1972, Knorr, 1975, Szabo, 1978) more or less agree with this hypothesis, ascribing to it however only a (more or less) minor role, "just a start".

Nevertheless, from an anthropological point of view, even a "start" seems very important if we consider that incommensurability seems almost the beginning of European mathematics. In other words it seems to us anthropologically very relevant to claim that European Mathematics did not stem as a by-effect of purely technical geometrical enquiries, but as the result of a socially and politically crucial musical problem.

The aim of our *Mathematica* notebook is to make available information, knowledge and tools to allow a more complete perception of ancient Greek music and its anthropological embedding, by the employment of the multimedia approach allowed by the language *Mathematica*. This notebook is an expanded release of an earlier prototype developed as a part of a degree thesis (Fascicolo, 2001).

It is just a start and probably our aim requires a much greater effort: for example a very relevant gap of our implementation is the lack of a part concerning metrics and rhythms.

A more 'aesthetical' limit is the simple sinusoidal shape of the played sounds, but it is possible with *Mathematica* to create more complex "tone colours".

The notebook at the moment includes the following parts:

1. Mathematical Theory of Music
2. Performance of Greek Music
3. The geometrical approach to Incommensurability
4. The musical approach to Incommensurability.

More in detail, the second part allows the choice of the instrument (tetrachord, octochord, the perfect system), of the basic tone and the starting frequence, of the length of time and of the mode (mesolydian, lydian, phrygian, doric, hypolidian, hypophrygian, hypodoric). In addition, for each mode, of the genus (diatonic, chromatic, enharmonic with their variants), and, for reach genus, of the specific theoretical tuning (Archytas, Aristoxenus, Eratosthenes, Didymus, Ptolemeus).

Then, by the numeric keys (1 to 4 for the tetrachord, 1 to 8 for the octachord, 1 to 15 for the perfect system), it is possible 'to play' in the selected scale, or it is possible to give a specific score and listen the performance: at the moment we have stored two scores: the first stasimo of Euripides' tragedy Orestes and the hymn to Helios of Mesomedes.

The same scores can be played also in the well-tempered tuning and, during the discussion of Fascicolo's thesis, the listeners recognized quite sharply the little difference between the well-tempered and the Pythagorean performance of Orestes' stasimo. Strange enough, the difference between two notes or two consonances is almost unperceivable, whereas the difference between the two executions is quite clear: the rational probably is in the 'differential' nature of our musical perception.

It is possible to export such played score in standard sound format (AIFF). Then we can store it elsewhere (for example on a CD) and play it independently on the computer and on *Mathematica*.

The third part includes the classical geometric examples of incommensurability (side and diagonal of the square and of the pentagon), with animations that show the infinite geometric series of decreasing intervals we can build on these examples.

The fourth part makes clear the argument of Archytas' theorem on the "cutting of the tone".

What about the perspectives?

It is crucial to understand how far such a system can foster our understanding of the Greek world. We underline that our notebook is not only a 'passive' multimedia system, but is embedded in a system that own the full power of an universal language of computation. Can a possible development of our system allow an effective simulation of the ancient musical world, even though we remember that ancient music was thoroughly contextual and that crucial aspects of that world are irremediably lost?

It is clear that an answer to this question and a relevant enhancement of the system both require an interdisciplinary effort: not only mathematics and computer science, but also ancient history, philology and musicology must play a greater role.

The reward could be that we could not only get a new tool for humanities, but also discover a new anthropological dimension for mathematics

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