RESEARCH IN MATHEMATICS EDUCATION

Editor
Athanasios Gagatsis

CONFERENCE OF FIVE CITIES: NICOSIA, RHODES, BOLOGNA, PALERMO, LOCARNO

University of Cyprus

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"Research in Mathematics Education"

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A Book on the Occasion of the Conference of Five Cities

This book was published on the occasion of the "Conference of Five Cities" which was the first in a series of conferences on Research in Mathematics Education. In the following years the conferences will be organized in turn, in four Greek- and Italian-speaking cities: Rhodes, Bologna, Palermo and Locarno.

The first conference took place at the University of Cyprus in Nicosia (Cyprus), from the 13th until the 14th of September, 2008. It was organized with great success by the University of Cyprus, University of the Aegean, University of Bologna, University of Palermo and ASP Pedagogical High School, Locarno, in cooperation with the Cyprus Mathematical Society.

The core of this book is based on a selection of papers presented at the "Conference of Five Cities" while some of the articles included were not presented at the conference. The main aim of this publication is to promote research in mathematics education. The contributions in this book are interesting, stimulation and thoughtful and will give the reader insight into research in the area of mathematics education.

New directions in research are presented in the six chapters of the book which cover a variety of topics such as, representations and visualization, teaching and learning of geometry, proportionality and pseudo-proportionality, problem solving, the history and philosophy of mathematics and teaching and learning in mathematics.

This book provides avenues to engage mathematics education researchers, teachers and prospective teachers in critical and productive discussions about specific mathematical topics not only in the five cities mentioned above but in all Europe.

Athanasios Gagatsis
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CHAPTER 1

Representations and Visualization in Mathematics Education

University of Cyprus

Conference of Five Cities: Nicosia, Rhodes, Bologna, Palermo, Locarno
"Research in Mathematics Education", 2008, 3 -12

The stability of students' approaches in function problem solving: A coordinated and an algebraic approach

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Abstract

The aim of this study was twofold. First to contribute to the understanding of the algebraic and "coordinated" approaches teachers develop in solving function tasks and to examine which approach is more correlated with teachers' ability in problem solving. Secondly, to investigate the stability of these approaches and to examine the impact teachers' mathematical background has on them. The study was conducted in two phases. Participants were 288 pre service teachers. Results were similar in both phases, indicating the stability of teachers' approaches and providing support for their intention to use the algebraic approach. Teachers who were able to use the coordinated approach had better results in problem solving. Teachers who dealt with mathematics systematically used more often the coordinated approach.

Introduction and theoretical framework

The concept of function is central in mathematics and its applications. The understanding of functions does not appear to be easy. Students of secondary or even tertiary education, in any country, have difficulties in conceptualizing the notion of function. A factor that influences the learning of functions is the diversity of representations related to this concept (Hitt, 1998). An important educational objective in mathematics is for pupils to identify and use efficiently various forms of representation of the same mathematical concept and move flexibly from one system of representation of the concept to another.

The use of multiple representations has been strongly connected with the complex process of learning in mathematics, and more particularly, with the seeking of students' better understanding of important mathematical concepts (Greeno & Hall, 1997), such as function. The ability to identify and represent the same concept through different representations is considered as a prerequisite for the understanding of the particular concept (Duval, 2002; Even, 1998). Some researchers interpret students' errors as either a product of a deficient handling of representations or a lack of coordination between representations (Greeno & Hall, 1997). The standard representational forms of some mathematical concepts, such as the concept of function, are not enough for students to construct the whole meaning and grasp the whole range of their applications. Mathematics instructors, at the secondary level, traditionally have focused their teaching on the use of the algebraic representation of functions (Eisenberg & Dreyfus, 1991). Sfard (1992) showed that students were unable to bridge the algebraic and graphical representations of functions, while Markovits, Eylon and Bruckheimer (1986) observed that the translation from graphical to algebraic form was more difficult than the reverse.

The theoretical perspective used in this study is mainly based on the studies of Even (1998) and Mousoulides and Gagatsis (2004). Even (1998) focused on the intertwining between the flexibility in moving from one representation to another and other aspects of knowledge and understanding. This study indicated that subjects had difficulties when they needed to flexibly link different representations of functions. An important finding was that many students deal with functions pointwise (they can plot and read points) but cannot think of a function in a global way. The data also suggested that subjects who can easily and freely use a global analysis of changes in the graphical representation have a better and more powerful understanding of the relationships between graphical and symbolic representations than people who prefer to check some local and specific characteristics.

Mousoulides and Gagatsis (2004) investigated students' performance in mathematical activities that involved principally the conversion between systems of representation of the same function, and concentrated on students' approaches as regards the use of representations of functions and their connection with students' problem solving processes. The most important finding of this study was that two distinct groups were formatted with consistency, that is, the algebraic and the geometric approach group. The majority of students' work with functions was restricted to the domain of algebraic approach. Only a few students used an object perspective and approached a function holistically, as an entity. Students who had a coherent understanding of the concept of functions (geometric approach) could easily understand the relationships between symbolic and graphical representations in problems.

In this study the concept of function is viewed from two different perspectives, the algebraic and the coordinated perspective. The algebraic perspective is similar to the pointwise approach described by Even (1998) and the one described by Mousoulides and Gagatsis (2004). In this perspective, a function is perceived of as linking x and y values. The coordinated perspective combines the algebraic and the graphical approach. In this perspective, the function is thought from a local and a global point of view at the same time. The students' can "coordinate" (flexibly manipulate) two systems of representation, the algebraic and the graphical one.

The purpose of this study is to contribute to the understanding of the algebraic and coordinated approaches teachers develop and use in solving function tasks and to examine which approach is more correlated with teachers' ability in solving complex problems. Furthermore, this research study aims to investigate the stability of teachers' approaches and to examine the impact teachers' mathematical education in high school has on them.

Method

The study was conducted in two phases. The first phase was conducted in 2005 with 135 participants and the second phase was conducted two years later, in 2007, with 153 participants. The participants of the second phase graduated from a slightly different type of high school with different textbooks and different procedures for the selection of lessons as a result of the major changes happened in the educational system, at high school. The participants, in both phases, were pre service teachers. The subjects were for the most part students of high academic performance admitted to the University of Cyprus on the basis of competitive examination scores. Nevertheless there are big differences among them concerning their mathematical education in high school. More specifically, 122 of them dealt with mathematics systematically in high school (Mathematics group). In contrast, the other 166 teachers did not have a special interest or specialization on mathematics and in high school they dealt systematically with theoretical lessons such as history (Theoretical group).

A test was administrated to all the participants. The test consisted of seven tasks. The first four tasks were simple tasks with functions (T1a, T1c, T2a, T2c, T3a, T3c, T4a, T4c). In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions (e.g. $f(x)= 2x$, $g(x)= 2x+1$). The participants were asked to interpret graphically the second function. The other three tasks were complex problems. The first problem consisted of textual information about a tank containing an initial amount of petrol (600 L) and a tank car filling the tank with petrol. The tank car contains 2000 L of petrol and the rate of filling is 100 L per minute. Students were asked to use the information in order to give the two equations (Pr1a), to draw the graphs of the two linear functions (Pr1b) and to find when the amounts of petrol in the tank and in the car would be equal (Pr1c). The second problem consisted of textual and algebraic information about an ant colony. The number of ants (A) increases according to the function: $A=t^2+1000$ (t= the number of days). The amount of seeds, the ants save in the colony, increases according to the function $S=3t+3000$ (t= the number of days). Students were asked to use the information in order to draw the graphs (Pr2a) of the quadratic and linear functions and to find when the number of ants in the colony and the number of seeds would be equal (Pr2b). The third problem consisted of a function in a general form of $f(x) = ax^2+bx+c$. Numbers a, b and c were real numbers and the f(x) was equal to 4 when x=2 and f(x) was equal to -6 when x=7. Students were asked to find how many real solutions the equation ax^2+bx+c had and explain their answer (Pr3). The test was administered to students in a 60 minutes session.

The results concerning students' answers to the four tasks were codified by an uppercase T (task), followed by the number indicating the exercise number. Following is the letter that signifies the way students solved the task: (a) "a" was used to represent "algebraic approach – function as a process" to the tasks, (b) "c" stands for students who adopted a "coordinated approach – function as an entity". A solution was coded as "algebraic" if students did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as coordinated if students observed and used the relation between the two functions in constructing the graph of the second function. In this case students used and coordinated two systems of representation. They noticed the relationship between the two equations given and they interpreted this relationship graphically by manipulating the function as an entity. The following symbols were used to represent students' solutions to the problems: Pr1a, Pr1b, Pr1c, Pr2a, Pr2b and Pr3. Right and wrong answers to the problems were scored as 1 and 0, respectively.

For the analysis of the collected data the similarity statistical method was conducted using a computer software called C.H.I.C. Two similarity diagrams of teachers' responses, one for each phase, were constructed (Gras, Peter, Briand, & Philippe, 1997). In order to examine whether there are statistically significant differences between the teachers of phase A and B and to determine whether teachers' mathematical education in high school affect the approach they used and their performance in problem solving, multivariate analysis of variance (MANOVA) was performed by using SPSS.

Results

The main purpose of the present study was to examine the mode of approach pre service teachers, participating in phase A and B, used in solving simple tasks in functions and to investigate which approach is more correlated with solving complex mathematical problems. Table 1, shows teachers' responses to the first four tasks. According to Table 1, most of the teachers, participating in both phases, solved correctly Task 1 and 2.

Task 1 involved a linear function and Task 2 the simplest form of an equation of a parabola ($y=x^2$). Their achievement radically reduced in tasks involved "complex" quadratic functions (T3 and T4). More than half of the teachers chose an algebraic approach to solve the first three tasks. In Task 4 most of the teachers chose a coordinated approach. In this task a coordinated approach seemed easier and more efficient than the algebraic. The teachers participating in both phases gave quite similar responses to the four tasks. The only difference was that the teachers of phase B used less the coordinated approach and gave more incorrect responses than teachers of phase A.

In the case of Task 1 ($y=2x$, $y=2x+1$), some teachers who used an algebraic approach found the points of intersection with x and y axis and constructed the graph. Others constructed a table of values in order to help them construct the graph. The teachers who used a coordinated approach compared the two equations and mentioned that the slope was the same and the two functions are parallel. Then they referred to the fact that the points of the second function are "one more" than the points of the other. Some of them found a point in order to verify their assertion.

Table 1: Teachers' responses to the first four tasks (Phase A and B)

Tasks (%)		Algebraic approach with correct answer	Coordinated approach with correct answer	Incorrect answer
1	A	54.8	32.5	12.7
	B	56.2	22.2	21.6
2	A	54.8	31.1	14.1
	B	56.9	25.5	17.6
3	A	56.3	17.7	26
	B	43.8	15	41.2
4	A	24.4	48.1	27.5
	B	24.8	47.1	28.1

In the case of Tasks 2 ($y=x^2$, $y=x^2-1$) and 3 ($y=x^2+3x$, $y=x^2+3x+2$), teachers who used an algebraic approach found the real solutions of the second equation and the minimum point and constructed the graph without using the first graph. In contrast, teachers who used a coordinated approach first compared the two equations and realized that they are parallel. Then they mentioned that the minimum point in the first case is "one down" and in the second case "two above". Some of them found another point in order to draw the graph more precise. In the case of Task 4 ($y=3x^2+2x+1$, $y=-(3x^2+2x+1)$), the teachers who used an algebraic approach found the point of intersection with y-axis and the maximum point. The participants who used a coordinated approach compared the two equations and mentioned that the two functions are "opposite" and "symmetrical" to the x-axis. In this task, an algebraic approach was more complicated due to the fact that the equation does not have real solutions. Most of the teachers, after an unsuccessful effort to find the points of section with x-axis drew the graph using a coordinated approach.

Table 2 shows teachers' responses to complex problems. Teachers' performance was moderate. In Problem 1 only 38.5% of the phase A teachers and 22.9% of the phase B teachers managed to use the information given in order to give the two equations. A larger percentage constructed the two graphs correctly (59.2% and 45.8% respectively) and found their point of intersection (70.4% and 55.6%). Many teachers were unable to give the equations but manage to construct the graphs by constructing a table of values for x and y. Some of the teachers did not construct the graphs but found their point of intersection by using the table of values. In Problem 2 only 46.6% of the phase A and

35.3% of the phase B teachers managed to construct the graphs. A smaller percentage (35.5% and 27.5%) found their point of intersection. In this problem in order to find the point of intersection the teachers had to solve a second degree equation and that caused difficulties. Problem 3 was quite difficult for the teachers of both phases since only 37% and 20.3% respectively managed to solve it correctly. The teachers participating in phase A performed better than the teachers of phase B.

Table 2: Teachers' responses to the complex problems (Phase A and B)

Problems (%)		Correct answer	Incorrect answer
1a	A	38.5	61.5
	B	22.9	77.1
1b	A	59.2	40.8
	B	45.8	54.2
1c	A	70.4	29.6
	B	55.6	44.4
2a	A	46.6	53.4
	B	35.3	64.7
2b	A	35.5	64.5
	B	27.5	72.5
3	A	37	63
	B	20.3	79.7

In order to examine whether there are statistically significant differences between the teachers of phase A and B concerning the approach they used and their problem solving ability, a multivariate analysis of variance (MANOVA) was performed. Overall, the effects of teachers' phase were significant (Pillai's $F(3, 284) = 3.66$, $p<0.05$). Particularly, there were significant differences between the two phases concerning the effectiveness in problem solving ($F(1, 284) = 10.11$, $p<0.05$). There were not statistically significant differences between the teachers of phase A and B concerning the algebraic ($F(1, 284) = 0.25$, $p=0.62$) and coordinated approach ($F(1, 284) = 1.43$, $p=0.23$). Specifically, the teachers of phase A ($\overline{X}=2.87$, $SD=2.24$) performed better than the teachers of phase B in problem solving ($\overline{X}=2.07$, $SD=2.04$).

Teachers' (participating in phases A and B) correct responses to the tasks and problems are presented in the similarity diagrams in Figure 1 and 2 respectively. The two similarity diagrams are quite similar. More specifically in both diagrams, two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables "T1c", T2c", "T3c", "T4c", "Pr1a", "Pr1b", "Pr1c", "Pr3", "Pr2a" and "Pr2b" and refers to the use of the coordinated approach and the solving of problems. The second cluster consists of the variables “T1a”, “T2a”, “T3a” and “T4a” which represent the use of algebraic approach.

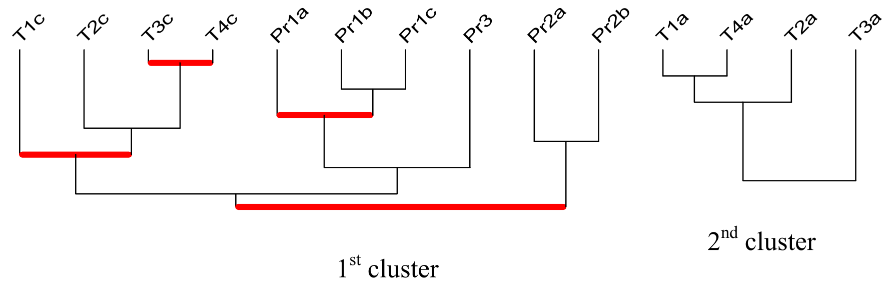

Figure 1: Similarity diagram of teachers’ participating in phase A responses

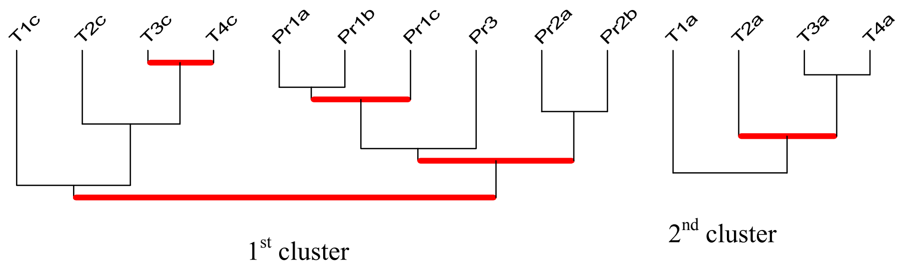

Figure 2: Similarity diagram of teachers’ participating in phase B responses

From the similarity diagrams it can be observed that the first cluster includes the variables corresponding to the solution of the complex problems with the variables representing the coordinated approach. More specifically, teachers’ coordinated approach to simple tasks in functions is closely related with effectiveness in solving problems. This close connection may indicate that teachers, who can use effectively different types of representation- in this situation both algebraic and graphical representation- are able to observe the connections and relations in problems, and are more capable in problem solving. It is noteworthy the fact that the similarity clusters presented in the two diagrams are almost the same indicating that the connections and relationships between the approaches and problem solving are very strong and long-lasting. The first cluster of both groups is exactly the same, while the second cluster although it contains the same variables it presents small differences concerning the relations between the tasks. Thus, the highest similarity in 2nd cluster, in phase B, concerns the tasks T3a and T4a that are the most complex as it has been noticed previously.

In order to determine whether there are significant differences between the two groups (Mathematics and Theoretical Group) concerning the approach they used and their performance in problem solving, a multivariate analysis of variance (MANOVA) was performed. Overall, the effects of teachers' mathematical education in high school were significant (Pillai's F (3, 284) = 65.78, p<0.001). Particularly, the mean value of the Mathematics group concerning the coordinated approach ($\overline{X}$= 1.93, SD= 1.54) was statistically significant higher (F (1, 284) = 116.99, p<0.001) than the mean value of the Theoretical group ($\overline{X}$=0.64, SD=0.98). In contrast, the mean value of the Mathematics group concerning the algebraic approach ($\overline{X}$= 1.83, SD= 1.46) was lower than the mean value of the Theoretical group ($\overline{X}$=1.88, SD=1.47) but this difference was not statistically significant (F (1, 284) = 0.087, p=0.78). As far as the problem solving concerns the Mathematics group ($\overline{X}$=3.97, SD=1.94) outperformed the Theoretical group ($\overline{X}$=1.33, SD=1.57) and this difference was statistically significant (F (1, 284) = 488.57, p<0.001). The Mathematics group used more often the coordinated approach and had also better results in problem solving.

Discussion

A main question of this study referred to the approach teachers use in order to solve simple function tasks. It is important to know whether teachers are flexible in using algebraic and graphical representations in function problems. Most of the teachers, participating in phase A and B, used an algebraic approach in order to solve the simple function tasks. A coordinated approach is fundamental in solving problems even though many students have not mastered even the fundamentals of this approach. This finding is in line with the results of other studies that suggest that many students deal with functions pointwise, although a global approach is more powerful (Even 1998). Students who can easily and freely use a global approach have a better and more powerful understanding of the relationships between graphical and algebraic representations and are more successful in problem solving. Students' preference in the algebraic solution is probably the curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation.

Teachers' performance in problem solving was moderate. Teachers participating in phase A performed better than teachers of phase B. Although problems used in this study are some of those taught at school, subjects had difficulties. This finding suggests that in order to give a correct solution to a complex function problem the students must be able to handle different representations of function flexibly and move easily from one representation to the other. Furthermore, an important finding of this study is the relation between the coordinated approach and the problem solving. The data from both phases suggest that students who have a coherent understanding of the concept of function (coordinated approach) can easily understand the relationships between symbolic and graphical representations and therefore are able to provide successful solutions to complex problems. Furthermore, it is noteworthy that this close relationship between the coordinated approach and problem solving ability is strong and stable. Although the second phase conducted two years later and major changes have happened in the educational system, teachers' approaches were the same and a strong relationship between the coordinated approach and problem solving ability still existed. The only difference between the two phases was the effectiveness in problem solving. The teachers participating in phase A performed better than teachers participating in phase B. This difference is probably the result of the major changes happened in high school.

Although all the participants of this study were pre service teachers they had many differences concerning their mathematical education in high school. Some of the students had dealt with mathematics systematically in high school (Mathematics group). The Mathematics group used more often the coordinated approach to solve the simple tasks. Furthermore, they were able to use an algebraic and a graphical representation at the same time and therefore were very successful in problem solving. It's obvious that the students who dealt with mathematics systematically in high school had developed a conceptual understanding of the concept of function. They were able to handle different representations of the concept, easy translate one representation to the other and as a consequence they were more successful in problem solving.

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The phenomenon of change of the meaning of mathematical objects due to the passage between their different representations: How other disciplines can be useful to the analysis

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Abstract

In this paper we will demonstrate a consequence at times manifest in the semiotic transformations involving the treatment and conversion of a semiotic representation whose sense derives from a shared practice. The shift from one representation of a mathematical object to another via transformations maintains the meaning of the object itself on the one hand, but on the other hand it can change its sense. This is demonstrated in detail through a specific example, while at the same time it is collocated within a broad theoretical framework that poses fundamental questions concerning mathematical objects, their meanings and their representations.

Episodes

In D'Amore (2006), D'Amore and Fandiño Pinilla (2007a, b), we have reported and discussed, exclusively from a semiotic structural point of view, episodes taken from classroom situations in which students are mathematics teachers in their initial training, engaged in facing representations problems. Some examples of the phenomenon have been given orally in Rhodes, on April 13th 2006, during a general conference (How the treatment or conversion changes the sense of mathematical objects) at the 5th MEDCONF2007 (Mediterranean Conference on Mathematics Education), 13-15 April 2007, Rhodes, Greece (D'Amore, 2007).

The task consisted in this: working in small groups the trainee teachers received a text written in natural language; such texts had to be transformed into algebraic language. Once they had come to the algebraic formulation, this was explained by the group and collectively discussed. Our duty as university teachers was to suggest the further transformation of the obtained algebraic expressions into other algebraic *expressions*, to face collective discussions on their meaning.

We present three examples below:

Example 1
[We omit the original linguistic formulation which, in this case, is not relevant];

The final algebraic formulation proposed by group 1 is: $x^2+y^2+2xy-1=0$, which in natural language is interpreted as follows: «A circumference» [the interpretation error is evident, but we decide to pass over]; we carry out the transformation which leads to: $x+y=\frac{1}{x+y}$ that after a few attempts is interpreted as «A sum that has the same value of its reciprocal»;

question: «But $x+y=\frac{1}{x+y}$ is it or not the "circumference" we started with?»;

student A: «Absolutely no, a circumference must have x^2+y^2»;

student B: «If we simplify, yes».

One can ask whether or not it is the transformation that gives a *sense*: from the episode it seems that if one would perform the inverse passages, then one would return to a "circumference". But it could also instead be that the meanings are attributed to the specific representations, without links between them, as if the transformation that makes sense for the teacher it does not make sense for the person who performs it.

Example 2

The text written in natural language requires the algebraic writing of the sum of three consecutive natural numbers and the proposal of group II is: $(n-1)+n+(n+1)$ [obviously the doubt remains in the case of $n=0$, but we decide to pass over]; we carry out the transformation that leads to the following writing: $3n$ that is interpreted as: «The triple of a natural number»;

question: «But $3n$ can be thought as the sum of three consecutive natural numbers?»;

student C: «No, *like this* no, *like this* it is the sum of three equal numbers, that is n».

Example 3

We consider the sum of the first 100 natural positive numbers: $1+2+\ldots+99+100$; we perform Gauss classical transformation; 101×50; this representation is recognized as the solution of the problem but not as the representation of the starting object; the presence of the multiplication sign compels all the students to look for a sense in mathematical objects in which the "multiplication" term (or similar terms) appears;

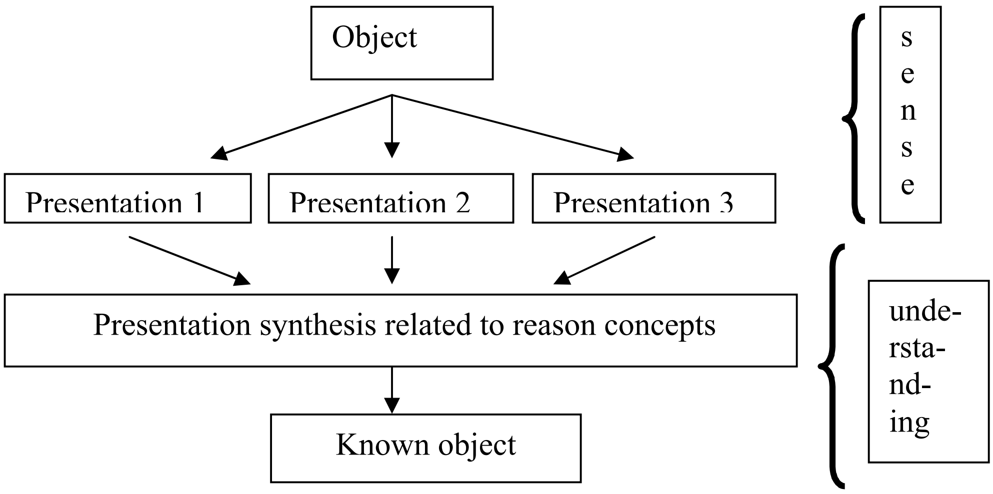
question: «But 101×50 is it or not the sum of the first 100 positive natural numbers?»;

student D: «That one, is not a sum, that is a multiplication; it corresponds to the sum, but it is not the sum».

In these episodes we witness a constant change of meaning during the transformations: each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner.

The causes of the changes of meaning

What are the causes of the changes of meaning, what origin do they have?
We can start from this diagram that we appreciate a lot because of its attempt to put in the right place the ideas of *sense* and *understanding* (Radford, 2004a).



The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in *Example 2,* the pure passage from $(n\text{-}1)+n+(n+1)$ to $3n$ should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, starting from single specific meanings culturally endowed to the signs of the algebraic language ($3n$ is the triple of something; 101×50 is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From an, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equally significant since they are obtainable through semiotic treatment, but from inside this picture is almost impossible, bound as it is to the culture constructed by the individual in time. In other words, we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the cultural processes THROUGH which it has been introduced.

In Luis Radford's semiotic anthropological approach (ASA) mathematical knowledge is seen as the product of a reflexive cognitive mediated praxis. «Knowledge as cognitive praxis (*praxis cogitans*) underlines the fact that what we know and the way we come to know it are underpinned by ontological positions and by cultural processes of meaning production that give form to a certain way of rationality within which certain types of questions and problems are posed. The *reflexive* nature of knowledge must be understood in Ilyenkov's sense, that is, as a distinctive component that makes cognition an intellectual reflexion of the external world in accordance with the forms of individuals' activity (Ilyenkov, 1977, page 252). The mediated nature of knowledge refers to the role played by tools and signs as means of knowledge objectification and as instruments that allow to bring to a conclusion the cognitive *praxis*» (Radford, 2004b, page 17).

On the other hand, «the object of knowledge is not filtered only by our senses, as it appears in Kant, but overall by the cultural modes of signification (...). (...) the object of knowledge is filtered by the technology of the semiotic activity. (...) knowledge is culturally mediated» (Radford, 2004b, page 20). «(…) These terms are the semiotic means of objectification. Thanks to these means, the general object that always remains directly inaccessible starts to take form: it starts to become an "object of consciousness" for the pupils. Although general, these objects however remain *contextual*» (Radford, 2004b, page 23).

The approach to the object and its appropriation on the part of the individual who learns, are the result of personal intentions with which individuals express themselves through experiences that see the objects used in suitable contexts: «Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorical experiences accounting for the way objects become attended and disclosed (Husserl, 1931)» (Radford, 2002, page. 82).

In the cases we presented above, and in mathematics in general, it is clear that the objects are attended from the first moment in their formal expression, in our case in the algebraic language; the individual learns to formally handle these signs, but what happens to the initial mathematical object? What happens to the initial meanings? We suppose that these meanings are tightly bound to the arithmetic experience of the pupil and overall to the way in which such an experience becomes objective through its objective transposition into ordinary language. Deep understanding of algebraic or, in general, formal manipulation, holds a prominent position.

Through an interesting comparison, Radford expresses himself on this point as follows: «While Russell (1976, page 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1961, page 79), which led him talk about signs having a *game signification* (...)» (Radford, 2002, page 88).

After having shown the broad and complex significance of the phenomenon, we must refer to other disciplines in order to understand better and better the issue of the different meanings of algebraic expressions, that is, in order to give a significant contribution to this aspect of mathematics education.

Analysis of the phenomenon thanks to theories "external" of mathematics education

We believe that some theories "external" of mathematics education can have, and in fact they already have, a strong influence on the analyses of various phenomena, like the ones described here, therefore giving a contribution to changing the theoretical frame of our discipline in its future research developments.

Philosophy. In section 2, we have seen how philosophy (Husserl's phenomenology) can have remarkable contribution and we will not repeat ourselves.

Learning is taking consciousness of a general object in accordance with the modes of rationality of the culture one belongs to.

More importantly we must face here the issue of the philosophical dilemma on concept and object, and even more the problem of the need of a previous choice between realist and pragmatist positions (D'Amore, Fandiño Pinilla, 2001; D'Amore, 2003; D'Amore, 2007).

In **realist theories** the meaning is a «conventional relationship between signs and ideal or concrete entities that exist independently of linguistic signs; they therefore suppose a conceptual realism» (Godino and Batanero, 1994). As Kutschera (1979) already claimed: «According to this conception the meaning of a linguistic expression does not depend on its use in concrete situations, but it happens that the use holds on meaning, since a clear distinction between pragmatics and semantics is possible».

In the realist semantic that it derives, we attribute to linguistic expressions purely semantic functions; the meaning of a proper name (as: 'Bertrand Russell') is the object that such proper name indicates (in such a case: Bertrand Russell); the individual statements (as: 'A is a river') express facts that describe reality (in such a case; A is the name of a river); the binary predicates (as: 'A reads B') designate attributes, those indicated by the phrase that expresses them (in this case: person A reads thing B). Therefore every linguistic expression is an attribute of certain entities: the nominal relationship that derives is the only semantic function of expressions.

We recognise here the bases of Frege's, Carnap's and Wittgenstein's (*Tractatus)* positions.
A consequence of this position is the acknowledgement of a "scientific" observation (at the same time therefore, empiric and subjective or intersubjective) as it could be, at a first level, a statement and predicate logic.

From the point of view we are mostly interested in, if we apply to Mathematics the ontological assumption of realist semantics, we necessarily draw a platonic picture of mathematical objects: notions, structures, etc. have a real existence that does not depend on human being, as they belong to an ideal domain; "to know" from a mathematical point of view means "to discover" in such domain entities and relationships between them. It is also obvious that such picture implies an absolutism of mathematical knowledge, since it is thought as a system of external certain truths that cannot be modified by human experience because they precede or, at least, are extraneous and independent from it.

Akin positions, although with different nuances, were sustained by Frege, Russell, Cantor, Bernays, Goedel,…; they also encountered violent criticisms [Wittgensteins' *Conventionalism* and Lakatos' *quasi-empirism* : see Ernest (1991) and Speranza (1997)].

In **pragmatic theories** linguistic expressions have different meanings according to the context in which they are used and therefore any scientific observation is impossible, since the only possible analysis is a "personal" and subjective one, anyway circumstantial and not generalizable. We cannot but analyse the different "uses": the set of "uses" in fact determines the meaning of objects.

We recognize here Wittgenstein's positions of the *Philosophical Investigations,* when he admits that the significance of a word depends on its function in a "linguistic game", since in such game it has a way of 'use' and a concrete purpose for which it has been precisely used: therefore the word does not have a meaning *per se*, but nevertheless, it can be meaningful.

Mathematical objects are therefore symbols of cultural units that emerge from a system of uses that characterise human pragmatics (or at least of individuals' homogeneous groups) and that continuously modify in time, also according to needs. In fact, mathematical objects and the meaning of such objects depend on the problems that we face in Mathematics and on their solution processes.

	"REALIST" THEORIES	**"PRAGMATIC" THEORIES**
meaning	conventional relationship between signs and concrete or ideal entities independent of linguistic signs	depends on the context and use
semantics Vs pragmatics	clear distinction	no distinction or faded distinction
objectivity or intersubjectivity	complete	missing or questionable
semantics	linguistic expressions have purely semantic functions	linguistic expressions and words have "personal" meanings, are meaningful in suitable contexts, but they don't have absolute meanings *per se*
analysis	possible and licit: logic for example	only a "personal" or subjective analysis is possible, not generalizable, not absolute
consequent epistemological picture	platonic conception of mathematical objects	problematic conception of mathematical objects
to know	to discover	to use in suitable contexts.
knowledge	is an absolute	is relative to circumstance and specific use
examples	Wittgenstein in *Tractatus*, Frege, Carnap [Russell, Cantor, Bernays, Gödel]	Wittgenstein in *Philosophical Invesigations* [Lakatos]

It is obvious and it would be easy to prove with philosophical examples, that the two fields are not fully complementary and clearly separated even if, for reasons of clarity, we preferred giving this "strong" impression.

With regard to the philosophical bases of mathematics education, we have decided to stay in the pragmatic domain that seems much closer to the reality of the empiric process of Mathematics teaching/learning. It seems that each specification that appears in the right column, cell by cell, is part of the same process and of its explicitation. It seems that focusing didactic activity (and therefore research) on learning and consequently on epistemology of the domain that has the student as a protagonist, we are obliged to interpret each step of knowledge construction as responding to the *language game,* therefore admitting that the semantics blur the use pragmatics.

Sociology. In D'Amore (2005) and D'Amore and Godino (2007), we show how the results of the analyses relative to the behaviours of individuals engaged in an activity of conceptual learning of mathematical objects, their transformations of the descriptions of such objects from ordinary language to formal language, the manipulations of such formalizations can be framed within a sociological interpretation key: the learning environment is framed within a sociological interpretation key and the individuals' behaviours are interpreted through the notion of "practice" and its "meta-practice" evolution. Essentially the individuals shift from a shared practice, recognized as characteristic of the social group they belong to, to a meta-practice that modifies such characteristic; the interpretative behaviour therefore ceases to be global and social and becomes local and personal; the notions that come into play in such interpretations are specific of the circumstance and not of the situation in its entirety.

We pass over this point, referring back to the quoted texts.

Anthropology. In D'Amore and Godino (2006, 2007) we go into strongly anthropological details in order to explain the nature of the choices of the individual who learns mathematics. In such articles we highlight how «Having obliged the researcher to point all his attention to the activities of human beings who have to do with mathematics (not only solving problems, but also communicating mathematics) is one of the merits of the anthropological point of view, inspiring other points of view, amongst which the one that today we call "anthropological" in the proper sense: the ATD, anthropological theory of didactics (of mathematics) (Chevallard, 1999; page 221). Why this adjective "anthropological"? It is not an exclusiveness of the approach created by Chevallard in 80s, as he himself declares (Chevallard, 1999), but an "effect of the language" (page 222); it distinguishes the theory, identifies it, but it is not peculiar to such theory in a univocal way» (D'Amore and Godino, 2006, page 15). The ATD is almost exclusively centred on the institutional dimension of mathematical knowledge, as a development of the research program started with fundamental didactics. The crucial point is that «ATD places the *mathematical* activity, and therefore *the study* in mathematics activity, *in the set of human activities and of social institutions*» (Chevallard, 1999).
This kind of analyses, although subjected to criticisms in D'Amore and Godino (2006, 2007), has opened the way to the use of anthropology as a critical instrument, as a new theoretical frame at research into mathematics education, in accordance with what has been already highlighted in the above quoted articles. It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and- now that must be coherent with that history.

We pass over this point, referring back to the quoted texts.

Psychology. In D'Amore and Godino (2006) we show how the shift from the anthropological picture to the onto-semiotic one is made necessary (amongst other things) by the need of not trivializing the presence of psychology in the study of learning and, in general, classroom situations. In D'Amore (1999) we show, for example, how ideas on representation drawn from psychology, regarding the explanation of the passage from image (weak) to model (stable) of concepts (Paivio, 1971; Kosslyn, 1980; Johnson-Laird, 1983; Vecchio, 1992), can be placed as a unitary basis of the explanation of several didactic phenomena, as intuitive models, the shift from internal to external models, the figural concepts, up to misconceptions, studied mainly in the 80s. Also the ideas of frame and script (Bateson, 1972; Schank and Abelson, 1977) have been used for the same purpose.

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The structure of fraction addition understanding: A comparison between the hierarchical clustering of variables, implicative statistical analysis and confirmatory factor analysis[1]

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Abstract

The aim of the study was to combine and compare the outcomes of confirmatory factor analysis (CFA), hierarchical clustering of variables and implicative method concerning 5th and 6th graders fraction addition understanding. CFA affirmed the existence of seven first-order factors indicating the differential effect of task modes of representation, representation functions and required cognitive processes, two second-order factors representing multiple representation flexibility and problem solving ability and a third-order factor that corresponded to the fraction addition understanding. Using hierarchical and implicative analysis, evidence was provided of students' attempt to overcome compartmentalized thinking. However, primary students did not construct the whole meaning of the concept of fraction addition yet. The outcomes of the three methods were found to coincide and complement.

Introduction

There is a basic difference between mathematics and other domains of scientific knowledge as the only way to access mathematical objects and deal with them is by using signs and semiotic representations. Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2006).

Nowadays the centrality of different types of external representations in teaching and learning mathematics seems to become widely acknowledged by the mathematics education community (e.g. Elia & Gagatsis, 2006). Furthermore, the NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations and stresses the importance of the use of multiple representations in mathematical learning. Duval (2006) maintains that

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mathematical activity can be analyzed based on two types of transformations of semiotic representations, i.e. treatments and conversions. Treatments are transformations of representations, which take place within the same register that they have been formed. Conversions are transformations of representations that involve the change of the register in which the totality or a part of the meaning of the initial representation is conserved, without changing the objects being denoted. In fact, recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998). Moving a step forward, Hitt (1998) identified different levels in the construction of a concept, which are strongly linked with its semiotic representations. The particular levels are as follow: 1) incoherent mixture of different representations of the concept, 2) identification of different representations of a concept, 3) conversion with preservation of meaning from one system of representation to another, 4) coherent articulation between two systems of representations, 5) coherent articulation between two systems of representations in the solution of a problem.

Lack of competence in coordinating multiple representations of the same concept can be seen as an indication of the existence of compartmentalization, which may result in inconsistencies and delays in mathematics learning at school. The particular phenomenon reveals a cognitive difficulty that arises from the need to accomplish flexible and competent translation back and forth between different modes of mathematical representations (Duval, 2002).

Aim and research predictions

The aim of the study was to combine and compare the outcomes of confirmatory factor analysis (CFA), hierarchical clustering of variables and implicative method on the same sample data concerning student multiple representation flexibility and problem solving ability as far as fraction addition understanding was concerned. In fact, a main concern was to gain insight into the distinct features, advantages and limitations of each of the three statistical methods in a significant topic of mathematics education, namely the understanding of the concept of fraction addition, and to examine whether they coincided or even complemented each other.

Method

The study was conducted among 829 pupils aged 10 to 12 of different primary schools in Cyprus (414 5th graders, 415 6th graders). The test that was constructed in order to examine the hypothesis of this study included:

1. Recognition tasks in which the pupils were asked to identify similar (RELa, RECa, RERa, RELb, RERb) and dissimilar (RELc, RERc, RECc) fraction addition in number line, rectangular and circular area diagrams.
2. Conversion tasks having the diagrammatic and the symbolic representation as the initial and the target representation, respectively. Similar fraction additions were presented in number line (COLSs) and circular area diagram (COCSs), whereas

dissimilar fraction additions were presented in number line (COLSd) and rectangular area diagram (CORSd).
3. Symbolic treatment tasks of similar (TRSa) and dissimilar (TRSb, TRSc) fraction addition.
4. Conversion tasks having the symbolic and the diagrammatic representation as the initial and the target representation, respectively. Pupils were asked to present the similar fraction addition in circular area diagram (COSCs) and in number line (COSLs), whereas they were asked to present the dissimilar fraction additions in rectangular area diagram (COSRd).
5. Diagrammatic addition problem in which the unknown quantity was the summands (PD).
6. Verbal problem that was accompanied by auxiliary diagrammatic representation and the unknown quantity is the summands (PVD).
7. Verbal problem whose solution required not only fraction addition but also the knowledge of the ratio meaning of fraction (PV).
8. Justification task that was presented verbally and was related to similar or dissimilar fraction addition (JV).

Representative samples of the tasks used in the test appear in the Appendix. It should be noted, that not any diagrammatic representation treatment tasks are included in the test since the students' ability to manipulate diagrammatic representations was examined through conversion tasks in which the target representation is a diagram.

Results

Confirmatory factor analysis outcomes

CFA was used to test statistically whether a hypothesized connection pattern between the observed variables and the underlying factors exist. Our first prediction dealt with the structure of the processes underlying fraction addition understanding. Specifically, keeping in mind the classic difference between "exercise" and "problem" (Polya, 1945; Dunker, 1945; D' Amore & Zan, 1996), we expected that fraction addition multiple representation flexibility and problem solving ability would differentially affect the fraction addition understanding, since they activate different mental processes.

We also assumed that fraction addition multiple representations flexibility would constitute a multifaceted construct in which other variables in addition to functions (recognition, treatment, conversion, according to Lesh et al., 1987) the representations fulfilled would be involved. These variables would be the modes of representations and relative concepts of similar and dissimilar fraction addition. To be specific, primarily we expected that the ability to recognize fraction addition in various diagrammatic representations, the ability to manipulate symbolic fraction addition equations and to convert from one fraction addition representation to another would come out as distinct dimensions of performance. We also assumed that the concept of similar and dissimilar fraction addition would affect the ability to recognize fraction addition in multiple diagrammatic representations. In fact we pointed out that a student who recognizes a similar fraction addition in a diagrammatic representation should bear in mind that the summands are represented on the same diagram which has the same number of subdivisions (or a multiple) as the denominator. On the other hand, when a student recognizes dissimilar fraction addition in a diagrammatic representation he/she should bear in mind that each summand were represented on a different diagram. Each of these diagrams has the same number of subdivisions as the denominator of the corresponding fraction. Then, the student identifies a diagram in which the number of subdivisions is the least common multiple of the two denominators. Taking this process into account the high association of the fraction equivalence with dissimilar fraction addition understanding was indicated, as well. Thus, in CFA the ability to recognize similar and dissimilar fraction addition would come out as distinct dimensions of performance. On the other hand, we assumed that the ability to solve symbolic similar and dissimilar fraction addition would be one factor since students were familiar with both of them. In fact, symbolic similar and dissimilar fraction addition treatments based heavily on basic algorithms and the specific processes automated by the age group students involved here.

Furthermore, we expected that the different types of representation would differentially affect the solution process, because they activated different mental processes when processing the tasks. Demetriou, Efklides and Platsidou (1993) showed that the nature of representation and symbol system used to express information is an independent dimension organizing cognitive performance in addition to the mental operations and types of relations involved. Therefore, in confirmatory factor analysis, the ability to convert flexibly from diagrammatic to symbolic equation would come out as a dimension of performance distinct from the ability to convert flexibly from a fraction addition equation to a diagrammatic representation. Furthermore, we expected that the presence of a diagrammatic representation would differentially influence fraction addition problem solving.

In order to explore the structure of the various fraction addition understanding dimensions a third-order CFA model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > .9, RMSEA < .06. The a priori model hypothesized that the variables of all the measurements would be explained by a specific number of factors and each item would have a nonzero loading on the factor it was supposed to measure. The model was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal.

Figure 1 presents the results of the elaborated model, which fits the data reasonably well (x^2/df=1.911, CFI=0.968, RMSEA=0.033). In fact, the third-order model which was considered appropriate for interpreting fraction addition understanding, involved seven first-order factors. The first-order factors F1 to F5 regressed on a second-order factor that stood for the multiple representations flexibility. The first-order factor F1 referred to the similar fraction addition recognition tasks, while the first-order factor F2 to the dissimilar fraction addition recognition tasks in a variety of diagrammatic representations. The first-order factor F3 consisted of the similar and dissimilar fraction addition treatment tasks. Conversion tasks in which the initial and the target representation was similar and dissimilar fraction equation and diagrammatic representation, respectively, constituted the first-order factor F4, while the first-order factor F5 referred to the similar and dissimilar fraction addition conversion tasks from a diagrammatic to a symbolic representation.

The factor loadings indicated that conversion from a diagrammatic to a symbolic representation was more closely associated with multiple representations flexibility than the other first-order factors were. Nevertheless, the first-order factor F1 to F4 loadings strength revealed that the flexibility in multiple representations of similar and dissimilar fraction addition constituted a multifaceted construct in which relations between: a) modes of representation (symbolic, diagrammatic), b) functions (recognition, treatment, conversion) fulfilled by representations and c) relative concepts (similar and dissimilar fractions, equivalence) arose.

The majority of tasks which involved number line had higher loadings than the other tasks, suggesting that the number line model was more strongly related to multiple representations flexibility than the circular and rectangular diagrams. Furthermore, dissimilar fraction tasks loadings were higher than the respective similar fraction addition loadings, indicating that in order to be solved different mental processes were required since the fraction equivalence understanding was involved, as well. The specific knowledge was also needed to solve similar fraction addition recognition tasks in which the number of subdivision was double that of the denominator (e.g. RERa). As a result, higher loadings were observed in these tasks relative to other similar fraction addition tasks.

The other two first-order factor F6 and F5 regressed on a second-order factor that represents problem solving ability. The first-order factor F6 consisted of problems having a diagram as an autonomous or an auxiliary representation. Both of them had a common mathematical structure since they had the summands as the unknown quantity. On the other hand, the verbal problem whose solution required the knowledge of the ratio meaning of fraction and the justification task formed the first-order factor F7, since in order to be solved different cognitive processes were needed. The two second-order factors that correspond to the multiple representations flexibility and to the problem solving ability regressed on a third-order factor that stood for the fraction addition concept understanding. Their loadings values were almost the same revealing that pupils' fraction addition understanding is predicted from both multiple representations flexibility and problem solving ability.

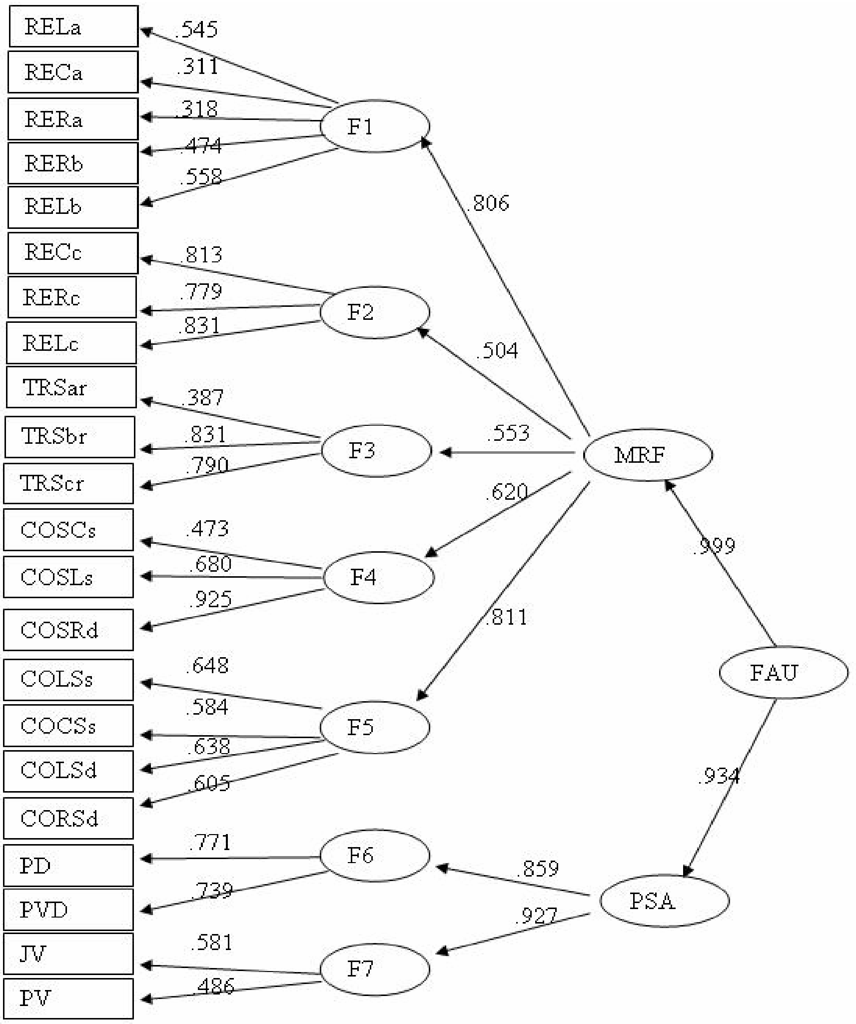

Figure 1: *The CFA model of the fraction addition understanding*

Note: 1. Errors of the variables were omitted. 2. MRF=multiple representation flexibility, PSA= problem solving ability, FAU=fraction addition understanding

The outcomes of the hierarchical clustering of variables and the implicative method of analysis

The hierarchical clustering of variables aimed at bringing to light the consistency among student responses to the various tasks in a hierarchical manner. The implicative method gave information about whether success on one task implied success at another task and about the relative difficulty of the tasks based on student performance. In fact, we expected that similarity and implicative relationships would be primarily established among the variables corresponding to the functions the representation fulfilled, namely recognition, treatment and conversion, and secondly among the variables corresponding to the conversions of the same starting representation, namely symbolic and diagrammatic representation. This hypothesis was based on findings suggesting the fragmentary way of student thinking when dealing with different types of representation (Duval, 2006; Gagatsis, Elia, & Mougi, 2002) and the lack of flexibility between different ways of approaching concepts (Elia, Panaoura, Eracleous, & Gagatsis, 2006). A second prediction was that distinctly close relationships will be formed among variables standing for similar or dissimilar fraction addition recognition tasks. Third, we assumed that success on similar fraction addition tasks would entail success on dissimilar fraction addition tasks. In fact, we considered the understanding of similar fraction addition and fraction equivalence, as well, as the prerequisite for the understanding of dissimilar fraction addition concept.

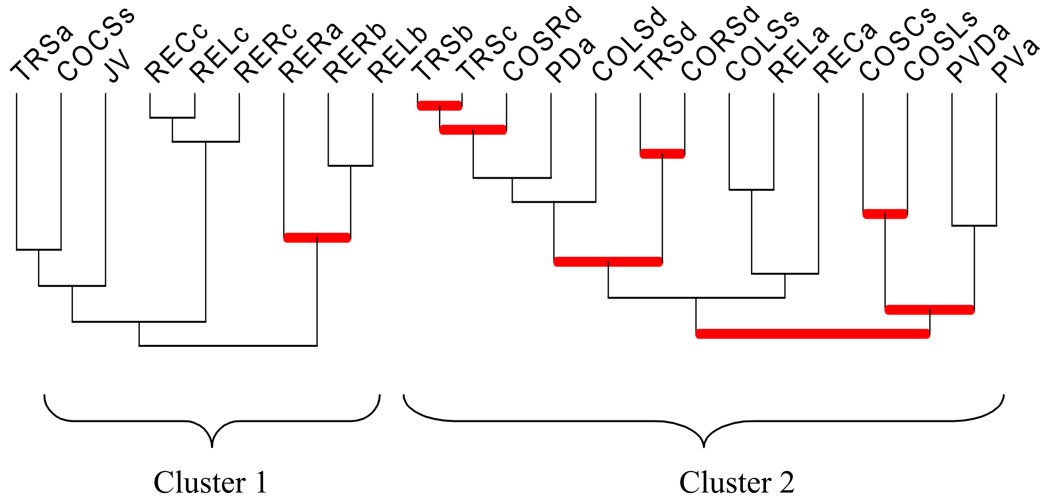

Figure 2: The hierarchical similarity diagram among the responses of primary school students to fraction addition tasks

Figure 2 illustrates the similarity relations among the variables corresponding to grade 5 and 6 student responses to the tasks of the test. Two distinct clusters of variables were established in the hierarchical similarity diagram. The first cluster involved three similarity groups. The first group included a symbolic similar fraction addition treatment task, the conversion having circular area diagram and a similar fraction addition equation as the source and the target representation, respectively, and the justification fraction addition problem (TRSa, COCSs, JV). The second group involved the dissimilar fraction addition recognition tasks in various diagrammatic representations (RECc, RELc, RERc), while similar fraction addition recognition tasks in which the number of subdivision was double that of the denominator (RERa, RERb, RELb) formed the third similarity group. The connection between treatment and conversion from diagrammatic to symbolic representation fraction addition tasks implied that students carried out these tasks in a similar way since even though these tasks fulfilled different functions they referred to the similar fraction addition concept. Similar fraction addition concept influence also arose in justification problem solving. Furthermore, the second and third group formation indicated that in order to solve similar and dissimilar fraction addition recognition tasks different cognitive processes were required. However, their similarity connection provided further support for the assertion that equivalent concept knowledge was needed so as to develop similar and dissimilar fraction addition recognition ability.

The second cluster involved three similarity groups, as well. The first group mainly included dissimilar fraction addition treatment and conversion tasks as well as the diagrammatic fraction addition problem (TRSb, TRSc, COSRd, PDa, COLSd, TRSd, CORSd). Thus, the formation of the first group underlined the differential role similar and dissimilar fraction addition exerted on multiple representation flexibility. The similarity connection among these variables indicated also that the students tackled diagrammatic fraction addition problem solving and dissimilar fraction addition treatment and conversion tasks, using similar processes. The second and the third group included mainly similar fraction addition tasks. Specifically, conversion task having the number line and similar fraction addition equation as the source and the target representations were linked together in the second group (COLSs, RELa, RECa). On the other hand, the third group included two conversion tasks having similar fraction addition equation and diagrammatic representation as the source and the target representations, respectively, the verbal problem with diagrammatic auxiliary representation and the verbal problem task (COSCs, COSLs, PVDa, PVa). The similarity relationships were established among the two group variables corresponding to similar and dissimilar fraction concept and the initial mode of representation. In fact, conversion tasks having a diagram and an equation as the source representation were involved in the second and the third group, respectively. Even though, similar fraction addition tasks were included in distinct groups, the similarity relationship between them revealed that the students tackled them almost in a “de-compartmentalized” way.

Nevertheless, the phenomenon of compartmentalization still exists since the tasks included in two clusters were differentially approached. In fact, the 5th and 6th graders did not yet understand that even though the various representations fulfilled different functions they referred to the same concept. It is also worth mentioning that the students did not approach problem solving tasks in a different way from multiple representation flexibility tasks. As a result, the interaction of both multiple representations flexibility and problem solving ability as far as fraction addition conceptual understanding was concerned revealed.

Figure 3 shows the implicative relations among the variables corresponding to 5th and 6th graders responses to the tasks of the test.

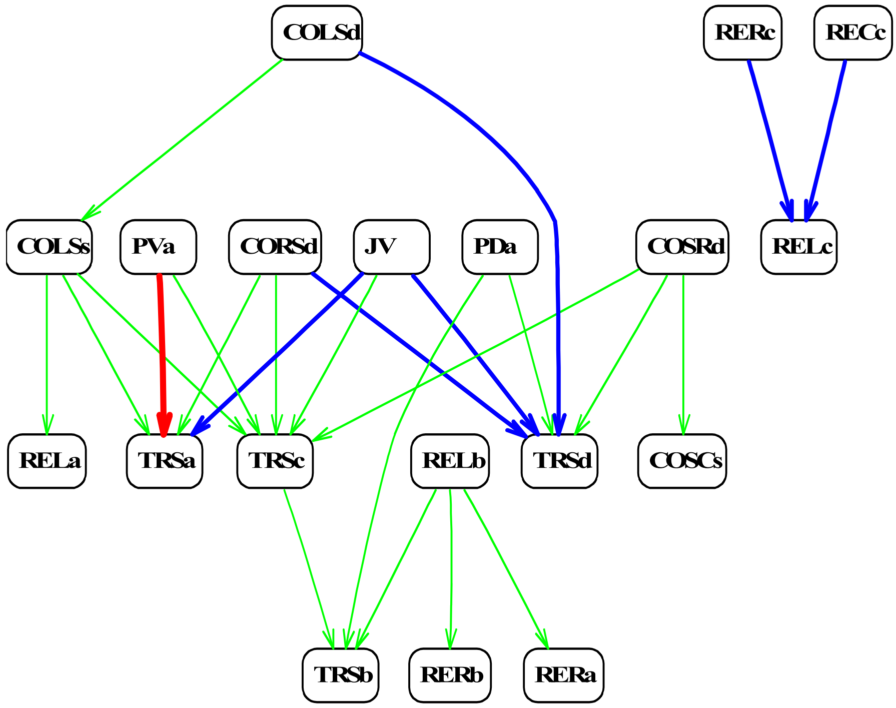

Figure 3: *The implicative diagram among the responses of primary school students to the test tasks*

The establishment of different implicative chains between similar and dissimilar fraction addition recognition tasks gave further support to the differential role similar and dissimilar fractions exerted on recognition ability. As far as conversion tasks are concerned, carrying out the conversion task having the dissimilar fraction addition equation and rectangular area diagram as the source and the target representation, respectively, implied success in the conversion task having the similar fraction addition equation and circular area diagram as the source and the target representation, respectively. Furthermore, carrying out the conversion task having number line and dissimilar fraction equation, as the source and target representation, respectively, implied success in the conversion task having number line and similar fraction addition equation as the source and the target representation, respectively, which in turn entailed correct performance in the similar fraction addition recognition in a number line task. In fact, the results indicated that implicative relationships primarily formed among variables corresponding to the conversions of the same starting representation. Furthermore, the students' difficulties in carrying out dissimilar fraction addition tasks were underlined. The fact that similar and dissimilar fraction addition treatment tasks were found in the chain endings implied that the specific processes automated by the primary school students were involved here. It should be also mentioned that problem solving tasks entailed success in symbolic similar and dissimilar fraction addition treatment tasks, indicating that 5^{th} and 6^{th} graders depended primarily on symbolic manipulations in order to solve them.

Discussion

This study investigated students' fraction addition understanding as far as multiple representation flexibility and problem solving ability were concerned. The data were analyzed from different perspectives using three distinct statistical methods, each of which was based on a different rationale. A major concern of this study was to compare in detail the findings of the hierarchical clustering of variables, implicative statistical analysis and confirmatory factor analysis so as to learn whether their outcomes on the same sample data were congruent and complement to each other.

The results provided a strong case for the important role of the multiple representations flexibility and problem solving ability in 5^{th} and 6^{th} graders fraction addition understanding. Specifically, CFA showed that two second-order factors were needed to account for the flexibility in multiple representations and the problem solving ability. Both of these second-order factors were highly associated with a third-order factor representing the fraction addition understanding. The outcomes of the other two methods were in line with CFA findings. In fact, the similarity connection between problem solving and multiple representation flexibility tasks and implicative relations between problem and treatment tasks suggested the interaction of both multiple representation flexibility and problem solving ability in fraction addition understanding.

CFA also showed that five first-order factors were required to account for the second-order factor that stood for the flexibility in multiple representations and two first-order factors were needed to explain the second-order factor that represented the problem solving ability. Thus, the results indicated the varying effect of both problem modes of representation and required cognitive processes on problem solving ability. Furthermore, the findings provided evidence to Duval's (2006) view that changing modes of representation is the threshold of mathematical comprehension for learners at each stage of the curriculum since the conversion from a diagrammatic to a symbolic representation dimension was more strongly related to multiple representations flexibility than the other dimensions were. Nevertheless, the factors loadings of the proposed three-order model suggested that the flexibility in multiple representations constituted a multifaceted construct in which representations, functions of representations and relative concepts were involved. In fact, the ability to recognize similar and dissimilar fraction addition in a variety of diagrammatic representations, manipulate similar and dissimilar fraction addition equations and converse flexibly from diagrammatic to symbolic representation standing for similar and dissimilar fraction addition, and vice versa, were necessary for multiple fraction addition representation flexibility. As a result, the separate grouping of the responses to multiple representation flexibility tasks in hierarchical clustering of variables analysis revealed student inconsistencies when dealing with them. In fact, the students tackled in a distinct way relative to the other multiple representation flexibility tasks, the similar and dissimilar fraction addition recognition tasks in which the number of subdivisions was a multiple of the denominator, a similar fraction addition equation and a conversion task having circular area diagram and similar fraction equation as a source and target representation, respectively.

On the other hand, the students carried out similar and dissimilar fraction addition treatment and conversion tasks in an almost consistent way taking into account the underlying concept. Furthermore, similarity connection among conversion tasks with different initial representation indicated 5th and 6th graders' attempt to breach the "compartmentalization" phenomenon. However, implicative relations established primarily among conversion tasks with the same starting representation indicated that students did not construct the whole meaning of the concept of fraction addition yet.

Regarding CFA findings it is worth mentioning that the high factor loadings in tasks involving number line revealed the specific model's importance in fraction addition and the different cognitive processes which were activated in order to handle it relative to other diagrammatic representations. In fact the number line is a geometrical model, which involves a continuous interchange between a geometrical and an arithmetic representation. Operations on real number are represented as operations on segments on the line (e.g. Michaelidou, Gagatsis, & Pitta- Pantazi, 2004). That is, the number line has been acknowledged as a suitable representational tool for assessing the extent to which students have developed the measure interpretation of fractions and for reaching fractions additive operations (e.g. Keijzer & Terwel, 2003).

Furthermore, the strength of factor loadings in dissimilar fraction addition tasks confirmed that different mental processes relatively to the corresponding similar fraction addition were required so as to be solved since the knowledge of fraction equivalence was also needed. The fact that recognition of similar and dissimilar fraction recognition tasks were found to have considerable autonomy between them and the other tasks in implicative chains confirmed the CFA findings. Implicative relations revealed also that dissimilar fraction addition tasks increased difficulty in relation to the corresponding similar fraction addition tasks. In fact, the high association of the fraction equivalence with fraction addition understanding was highlighted by all the three analyses. As Smith (2002) points out in order to develop fully the measure personality of fractions pupils need to master the equivalence of fractions.

On the other hand, the fact that success in similar and dissimilar fraction addition treatment tasks entailed success in problem solving and conversion tasks provided evidence that the treatment processes were automatically carried out by the 5th and 6th graders. This is in line with CFA results that similar and dissimilar fraction did not differentially affect fraction addition symbolic manipulation ability.

In general, the application of all the analyses yielded congruent results. However, at the same time given that these statistical processes approached the data from different perspectives, they emphasized different aspects of student outcomes. This differentiation allowed for the accumulation of a number of new distinctive elements in each analysis that contributed to the unravelling and making sense of student performance, the structure of abilities, difficulties and inconsistencies on the particular subject. The findings of the study suggested that the three statistical methods were open to complementary use and each one did not operate at the expense of the other. CFA provided a means of making sense of the structure of student multiple representation flexibility and problem solving ability as far as fraction addition understanding was concerned. The hierarchical clustering of variables provided a means of classifying student responses, of identifying student consistencies and inconsistencies among different abilities and for investigating the factors influencing this behaviour. The implicative method provided a means of examining the implicative relations among the responses to the tasks and the relative difficulty of the fraction addition tasks on the basis of student performance. Provided that applying these methods of analysis is consistent with the objectives of a study, their combination on the same sample data could contribute to the overcoming some significant limitations of each analysis employed separately, and consequently could enrich and deepen the outcomes of the investigation.

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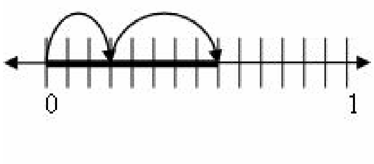
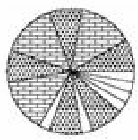
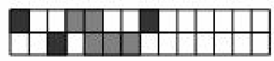
Appendix

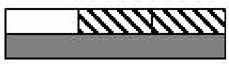
1. Circle the diagram or the diagrams whose shaded part corresponds to the equation 3/14 + 5/14.



(RELa) (RECa) (RERa)

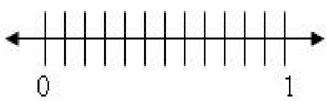
2. 1/6 + 4/12 = ….. (TRSb)
3. Write the fraction equation that corresponds to the shaded part of the following diagram:



Equation: .... (CORSd)

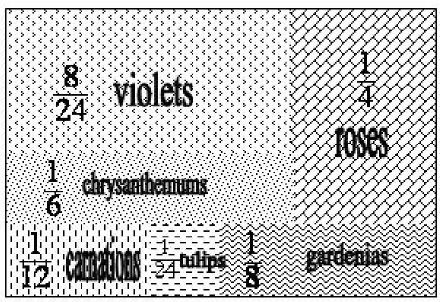
4. Present the following equation on the diagram:

1/12 + 7/12=….



(COfSLs)

5. In the addition of two fractions whose numerator is smaller than the denominator, the sum may be bigger than the unit. Do you agree with this view? Explain. (JV)
6. Each kind of flower is planted in a part of the rectangular garden as it appears in the diagram below:

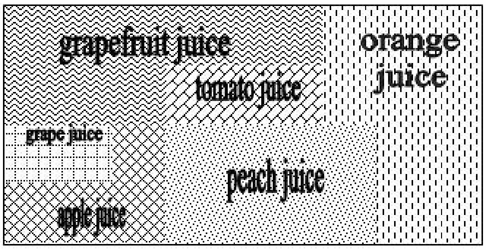


Which three kinds of flowers are planted in the 3/4 of the garden?(PD)

7. A juice factory produces the following kinds of natural juice:

- 1/4 of the production is grapefruit juice.
- 5/18 of the production is orange juice.
- 3/36 of the production is tomato juice.
- 2/9 of the production is peach juice.

- 1/18 of the production is grapes juice.
- 4/36 of the production is apple juice.



Which four kinds of juice make up 1/2 of the production? (PVD)

8. The manager of a circus is preparing the performance that will be given in a few days. He wrote the duration of each program in his notes: Clowns: 1/2 hour, Dancers: 1/3 hours, Animals: 1 hour, Acrobats: 1/6 hour, Jugglers: 2/1 hour

Write as a fraction, what part of the total duration of the performance corresponds to the jugglers' program (PV, Evapmib, 2007).

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The role of verbal description, representational and decorative picture in mathematical problem solving

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Abstract

This study investigates the role of representational and decorative picture in solving one-step mathematical problems of the additive structure with the unknown in the first part (a) by primary second grade students. It also investigates pupils' attitude towards the use and the role of pictures. For the purposes of the study, 125 pupils were asked to complete a questionnaire with two verbal problems, two problems accompanied by a representational picture and two by a decorative one. The results indicate that the presence of both the representational and the decorative picture did not have a significant impact in pupil's performance, even though pupils' attitude towards them is positive.

Theoretical and empirical background

Introduction

Last decades a great attention has been given on the concept of representation and its role in the learning of mathematics. A basic reason for this emphasis is that representations are considered "integrated" with mathematics (Kaput, 1987). This study aims to shed light on the influence of two types of representation on additive problems. Specifically, we investigate the role of representational and decorative pictures. These are contrasted to each other and to the use of plain verbal description (written text) for the solution of one-step addition problems presented in a number of different structures to be described below. Specifically, below we first discuss the nature and possible effects of different types of representation of arithmetic problems and then the different structures in which these problems may be presented.

Representations in mathematics learning

A representation is defined as any configuration of characters, images and concrete objects that can symbolize or "represent" something else (Kaput, 1985; Goldin, 1998; DeWindt-King & Goldin, 2003). Kaput (1987) suggested that the concept of representation involves the following five components: A representational entity, the entity that it represents, particular aspects of the representational entity, the particular aspects of the entity that it represents that form the representation and finally the correspondence between the two entities.

A basic discrimination that is pointed out in the region of representations is between internal/mental and external/semiotic representations (Dufour – Janvier et al., 1987). Internal/mental representations are mental schemes constructed by individuals in order to represent, explain and understand reality. External/semiotic representations are external symbolic carriers, such as symbols, shapes and diagrams, which aim at representing a specific reality, for example mathematics. Goldin and Kaput (1996) suggest that there is a dual, two-way relationship between external/semiotic and internal/mental representations.

A type of external representation that is used extensively in mathematics textbooks and is considered to enhance problem solving in all the phases of the certain process is visual representations (Larkin & Simon, 1987). Schnotz (2002) suggests that text and visual displays belong to different classes of representations, namely descriptive and depictive representations, respectively. Descriptive representations consist of symbols that have an arbitrary structure and are associated with the content they represent simply by means of a convention. Depictive representations include iconic signs that are associated with the content they represent through common structural features on either a concrete or a more abstract level.

In mathematics education, visual representations play an important role both as an aid for supporting reflection and as a means for communicating mathematical ideas. Therefore, many researchers consider imagistic representations as a fundamental cognitive system for mathematical learning (DeLoache, 1991) and problem solving (De Windt-King & Goldin, 2003; Diezmann & English, 2001), while experts mathematicians as well as mathematics students perceive visual representations as a useful tool in Mathematical Problem Solving (MPS) and frequently attempt to use them (Stylianou, 2001).

However, the use of pictorial representations may not have the intended effects due to obstacles they may cause to mathematics learning and problem solving (Bishop, 1989). For instance, these representations may divert attention to irrelevant details and they may highlight some aspects of the problem at the expense of others, more relevant to the task requirements (Colin, Chauvet, & Viennot, 2002; Presmeg, 1986). Moreover, a pictorial representation may fail to help in an educational setting, such as mathematical problem solving, when students do not understand how the representation is related to its referent (DeLoache, Uttal, & Pierroutsakos, 1998).

Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002). Three presuppositions for the mastery of a concept in mathematics are the following: First, the ability to identify the concept in multiple systems of representation; second, the ability to handle flexibly the concept within the particular systems of representation; and third, the ability to "translate" the concept from one system of representation to another (Lesh, Post & Behr, 1987). Principles and Standards for School Mathematics (NCTM, 2000) include a standard referring exclusively to representations and stress the importance of the use of multiple representations in mathematics learning. Ainsworth, Bibby, and Wood (1997) suggest that the use of multiple representations can help students develop different ideas and processes, constrain meanings and promote deeper understanding. By combining representations students are no longer limited by the strengths and weaknesses of one particular representation. For example, we use pictures in mathematics textbooks to increase the ‘‘readability’’ of standard mathematical expressions. However, interacting with multiple representations requires the understanding of the relationship between them. This is a complex process. Research shows that students encounter difficulty in integrating information from different sources (Case & Okamoto, 1996; Demetriou, Christou, Spanoudis, & Platsidou, 2002) or in moving from one representation of a mathematical object to another. As a result, they tend to use representations in isolation (Ainsworth, 2006; Duval, 2002).

Although, the mental processes, and particularly the visual-spatial images, used in MPS or mathematics learning have received extensive research in the field of mathematics education (e.g., Presmeg, 1992; Gusev, & Safuanov, 2003), the role of pictorial representations or number line in MPS, has received much less attention (Gagatsis & Elia, 2004). An effort to study the function of pictorial representations was made by Carney and Levin (2002) who proposed five functions that pictures serve in text processing – decorative, representational, organizational, interpretational and transformational. Given Carney and Levin’s (2002) five functions that pictures serve in text, Theodoulou, Gagatsis & Theodoulou (2003) proposed a similar categorization for the functions of pictures in MPS. Specifically, they suggested that pictures have the following four functions in MPS: (a) decorative, (b) auxiliary-representational, (c) auxiliary-organizational and (d) informational.

Decorative pictures do not provide any actual information concerning the solution of the problem, but simply decorate the page bearing little or no relationship to the problem content. Auxiliary-representational pictures represent part or all of the problem content, but are not necessary to be used in order to solve the problem. Auxiliary-organizational pictures help the students to solve the problem by guiding them to organize the given statements of the problem. Finally, informational pictures provide information that is essential for the solution of the problem; in other words, the problem is based on the picture.

Recent researches tried to examine the role of specific types of pictures in MPS. A first research by Gagatsis and Markou (2002) showed that the incorporation of decorative pictorial representations in unused verbal problems did not lead to a change in students’ behavior towards these problems, thus not breaching of the didactical contract. Pupils ignored the existence of pictures and their attention was detracted by the numerical data in the problem statement.

Theodoulou, Gagatsis and Theodoulou (2004) examined the role of the four different types of pictures according to their function in MPS. The results showed that the presence of decorative and informational pictures had no significant effect on students’ problem solving performance, whereas auxiliary-organizational pictures had a significant positive effect. Auxiliary-representational pictures had a significant positive effect in some cases, according to the mathematical operations needed in order to solve the problem. It was also found that the kind of mathematical operation needed in order to solve the problem had a more significant effect on students' problem solving performance than the kind of picture that accompanied the problem. In many cases, although the children used the picture in order to solve a problem, they claimed that the picture was not useful for solving the problem.

Elia and Philippou (2004) explored the role of pictures based on their function, in the solution of non-routine problems by pupils of Grade 6, in the context of an experimental model of communication. Findings of the particular study revealed that the representational, informational and organizational picture, but not the decorative one, had a significant effect on MPS.

Elia, Gagatsis and Demetriou (2007) investigated the role of the four different types of representation in MPS and developed a model, which provided information about the structural organization underlying students' processes in the solution of one-step additive problems in multiple representations. This model involved four first-order representation-specific factors indicating the differential effects of each particular type of representation and a second-order factor representing the general ability to solve additive problems. The size of the factor coefficients of the proposed model indicated that pupils' general problem solving ability was highly associated with the abilities in solving problems in verbal form, with decorative pictures and number line. This finding suggested that students used similar processes to solve the problems in the three modes of representation, indicating that pupils overlooked the presence of the line or the decorative picture and gave attention only to the text of the problem. The decorative pictures had no impact on students' behavior in MPS. The informational pictures had a rather complex role in problem solving compared to the use of the other modes of representation. It is possible that it required extra and more complex mental processes relative to the other modes of representation, since it involved not only pictorial but verbal information as well (Pyke, 2003).

According to the studies of Deliyianni, Gagatsis and Koukkoufis (2003), and Gagatsis and Andronicou (2004), similar results occurred in the case of representational pictures. To be specific, it appeared that pupils certain times did not take them into consideration since their use was not essential for MPS. Thus, representational pictures were very often tackled in a similar way the verbal problems were tackled, presenting, as a consequence, the same degree of difficulty. On the other hand, the problems accompanied with organizational pictures seemed to be solved more easily by the pupils. Nevertheless, the presence of organizational pictures in the context of some problems made the solution of them more complicated. That is because the pupils could resolve these problems successfully without any picture. Certain times organizational pictures provided pupils with unnecessary directions for drawing or written work.

Besides, even though some pupils partly drew and wrote correctly what was asked for in the organizational picture, they were still unable to go on to the correct mathematical equation. A likely explanation is that no one can guarantee that pupils will conceive the symbolic relation between the representation and the entity in which it corresponds (DeLoache, Uttal & Pierroutsakos, 1998).

To sum up, in a comparative article of a number of studies related to the contribution of pictures and number line in MPS, Gagatsis and Deliyianni (2004), provided evidence for the non-significant role of the decorative picture, the negative effect of the informational picture, the ambiguous role of the representational picture and the positive influence of the organizational picture on students' performance in MPS.

Considering the research studies reported here, due to the fact that they were conducted in different settings, with various age samples, using distinct research methods, some of their findings are congruent, whereas others are incompatible. However, these investigations seem to concur with an important assertion: that apart from the nature of the notion involved in a mathematical task, such as the structure or the content of a problem, the different modes of representation do have an effect on students' performance. This suggests that problem solving, which is a major dimension of mathematical learning endeavor, and probably other mathematical activities as well, incorporate an important interaction between the mode of representation and the mathematical structure or the inherent mathematical properties involved (Monoyiou, Spagnolo, Elia & Gagatsis, 2007).

As regards the effects of visual representations, in some cases the presence of visual representations in addition to verbal ones was found to have a helpful role on students' performance. In other cases, visual representations were found not to differentiate at all students' performance or even to impede their solutions. This variation of the visual representations' impact is due to several factors. A number of these factors concern the types of visual representations and specifically their nature, structure and complexity; the mathematical concepts involved in the task; the relation or correspondence of the visual representations with the concepts or situations they represent; and students' features, such as their cognitive styles, familiarity with the representations and generally existing knowledge (Monoyiou, Spagnolo, Elia & Gagatsis, 2007). In the light of the above, the use of visual representations in mathematical teaching and learning is a multidimensional and complicated process and should be conducted with great attention (Seeger, 1998). Reading and using images constitute skills that should not be left to chance, but should be taught systematically (Dreyfus & Eisenberg, 1990) and not in isolation, but in association with linguistic representations.

Regarding the role of pupils' emotions and attitudes towards the use of pictures in MPS, De Bellis & Goldin (2006) supported that affect constitutes an internal representational system. According to their model, the person's ability to solve mathematical problems is based on five kinds of internal, mutually interacting systems of representation. One of these systems is the affective, which refers to the person's emotions, attitudes, beliefs, morals, values, and ethics. In the research they conducted, De Bellis & Goldin (2006) found that the affective domain can enhance or undermine pupils' performance in Mathematics.

Structures of addition problems

Researchers have analyzed the structure of one-step word problems and highlighted its role in the solution strategies employed by students (Christou & Philippou, 1998). Previous studies on one-step additive problems have identified three main types of semantic structures: change or transformation of a measure, combine or composition of two measures and compare two measures to each other (Nesher, Greeno, & Riley, 1982; Vergnaud, 1982). In the present study, we focused on one class of problems: one-step composition problems.

Empirical evidence suggests that problems within the same semantic category vary in difficulty, since the placement of the unknown influences students' strategies and performance (Carpenter & Moser, 1984; Nesher et al., 1982). In the present study, we explore the use of two other modes of representation in addition to the verbal description. Specifically, we examine the role of decorative and representational pictures, on additive problem solving.

The study

Purpose

The purpose of this study was to investigate the role of three different modes of representation (decorative picture, representational picture and verbal description – text) in MPS. More specifically the aim of the study was to explore and compare the effects of decorative and representational picture in the solution procedures of one-step problems of the additive structure. Furthermore, the study aimed to identify the pupils' attitudes towards the use and the role of pictures in MPS.

Methodology

Participants

The sample of the research consisted of 125 second grade (7 to 8 years old) students (70 boys and 55 girls) from four elementary schools in two districts of Cyprus. The sample was selected with convenience sampling method. The students were acquainted with one-step problems of the additive structure from the first grade of elementary school. Also, they were exposed to teaching using verbal, decorative and representational pictures before through their textbooks and school materials.

Data collection

In order to collect the data needed for this study, a questionnaire was constructed. The questionnaire consisted of 6 one–step grouping (part – part – whole) problems with additive structures (a+b=c), based on the classification of problems with additive structures proposed by Vergnaud (1982). More specifically, the focus was on situations with the placement of the unknown in the first part (a) because, in this case, the problem is considered to be more difficult. The problems were accompanied with or represented in different representational modes. All the categories of problems are presented in the following table.

Table 1: Specification Table of the problems included in the test

Type of representation	Problem	Example
Verbal	1, 4	Helens' classroom has some boys and 7 girls. All the children are 13. How many boys are there in Helens' classroom? ☐ boys
Representational picture	2, 5	Costas cut some roses and 5 marguerites. All the flowers he cut were 11. How many roses did he cut?  ☐ roses
Decorative picture 	3, 6	At the birthday party of Carina were some red and 9 yellow balloons. All the balloons were 13. How many red balloons were at the party? ☐ red balloons

Furthermore in the test, there were questions relative to the students' attitude towards the presence and the role of the different representational modes in MPS. Those questions were answered by choosing either "Yes" or "No".

Procedure

The written questionnaire was administered to the students in usual classroom conditions. Students were asked to solve all the items explaining their solution strategies. They were not obliged to use the pictures that accompanied the problems. Actually, they were instructed to use the representations if they believed that they could help them resolve the problems. Students were given 40 minutes to solve the problems. After the completion of the problem tasks, the questionnaires were collected.

Scoring of the tasks

Problem tasks were scored as follows: 0=wrong answer or no answer, 1=correct answer. Relatively to the questions, which concerned the use of the pictures by the students and their attitude towards them, affirmative answers were marked as 1 and negative answers were marked as 0.

Variables of the study

The variables of the study were the following:

V1, V2:	Verbal problem
Iv1, Iv2:	Problem solving by drawing a representational picture
Sv1, Sv2:	Problem solving by using mathematical symbols
R1, R2:	Problem accompanied with representational picture
Ir1, Ir2:	Problem solving by using representational picture
D1, D2:	Problem accompanied with decorative picture
Id1, Id2:	Problem solving by using decorative picture
Idr1, Idr2:	Students solve the problem which is accompanied with decorative picture by drawing a representational picture
A1:	Students' attitude towards the pictures
Be1:	Students' opinion about the assistant role of the pictures in MPS

Method of analysis

Multiple methods of analysis were performed, using the collected data, including Descriptive Statistics Analysis by using the computer software SPSS and Gras's Implicative Analysis by using the computer software C.H.I.C (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier & Gras, 2000). Gras's Statistical Model is a method appropriate for collecting and analyzing data in order to reinforce or refute hypotheses and draw conclusions. This method determines the connections and the implicative relations of factors. The similarity diagram allows for the arrangement of tasks into groups according to their homogeneity. The implicative method gives the implicative graph, which represents the implicative relations among all variables which indicate whether success in one task entails success in another task related to the former one. The implications are valid at a level of significance of 99%, 95% or 90%.

Results

Students' performance on problem tasks accompanied with different representational modes.

A basic aim of the study was to examine whether different modes of representation (decorative picture, representational picture and verbal description – text) affect second grade students' performance in solving one-step problems of the additive structure. Table 2 shows the students' performance on the six problem tasks of the questionnaire. The highest percentage (95%) is observed when the problem is accompanied with representational picture (R1), whilst the lowest percentage (76%) refers to the verbal problem (V2) and to the problem which is accompanied with decorative picture (D2). The results show that the percentages are high in all the problem tasks. Therefore, students' performance is not altered by the mode of representation used.

Table 2: *Students' performance on addition problems accompanied with different representational modes*

Variables	V1	V2	R1	R2	D1	D2
Percentage of Success	82 %	76 %	95 %	78 %	77 %	76 %

Use of decorative and representational pictures

A second aim of the study was to examine whether students use the decorative and representational pictures when they solve one-step grouping (part – part – whole) problems of additive structure. Table 3 shows the percentages of students who declare that they used pictures to solve the mathematical problems. Students mention that they used more the representational pictures (Ir1 =67%, Ir2 = 62%) and less the decorative ones (Id1=31%, Id2=32%). From Table 3 it is also evident that few students draw a picture on verbal problems (Iv1= 2%, Iv2=3%) or solve problems which are accompanied with decorative picture by drawing a representational picture (Idr1= 3%, Idr2= 4%).

Table 3: Percentages of picture use.

Variables	Iv1	Iv2	Ir1	Ir2	Id1	Id2	Idr1	Idr2
Percentage of use	2 %	3 %	67 %	62 %	31 %	32 %	3 %	4 %

Students' attitude towards decorative and representational pictures

The study also aimed to investigate students' attitude towards decorative and representational pictures. As shown in Table 4, students seem to have a positive attitude to the presence and the role of decorative and representational pictures in MPS.

Table 4: Students' attitude towards decorative and representational pictures.

Variables	A1	Be1
Percentage	71 %	73 %

Similarity between the tasks

The Similarity Diagram (Figure 1) shows how tasks are grouped according to the similarity of the ways in which they have been solved. The similarities in bold color are important at level of significance 99%.

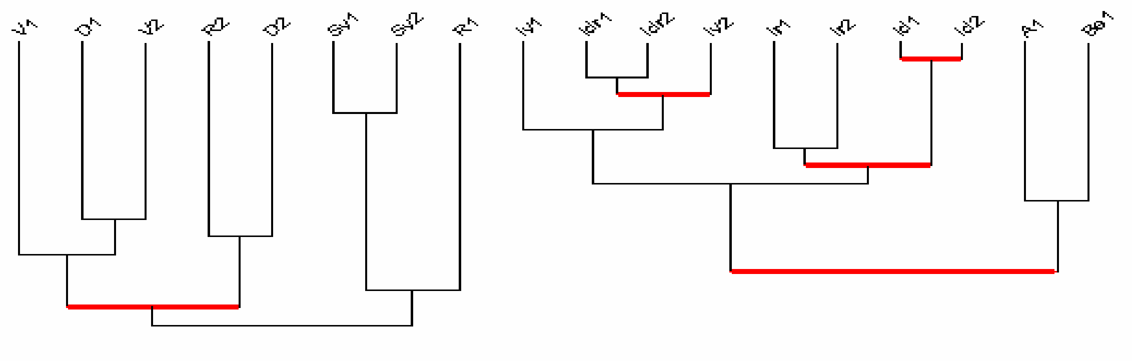

Figure 1: *Similarity diagram of students' responses to the tasks*

According to the Similarity Diagram (Figure 1), two groups are clearly distinguished. The first group consists of the variables V1, D1, V2, R2, D2, Sv1, Sv2 and R1, which represent students' efficiency in solving the problem tasks and using mathematical symbols. The variables Iv1, Idr1, Idr2, Iv2, Ir1, Ir2, Id1, Id2, A1 and Be1 are included in the second similarity group that concerns solving the problem tasks by using pictures and also students' attitudes and beliefs towards pictures.

In the first group, two subgroups are distinguished. The first subgroup consists of the variables V1, D1, V2, R2 and D2 which represent the students' efficiency in solving the addition problems. The second subgroup consists of the variables Sv1, Sv2 and R1. The connection between Sv1 and Sv2 is not surprising, as these variables represent students' tendency to solve verbal problems by using mathematical symbols and equations.

In the second group two subgroups are also distinguished. The first subgroup consists of all problem tasks which students solved by using decorative and representational pictures (Iv1, Idr1, Idr2, Iv2, Ir1, Ir2, Id1and Id2). The second subgroup consists of the variables which represent students' attitudes and beliefs towards pictures (A1, Be1).

According to the similarity diagram, the most significant similarity relationships can be observed between the variables of the second group. For example the variables Id1 and Id2 are connected with the most significant relationship. Thus, students justifiably behaved in a similar way when they solved the addition problems which were accompanied with a decorative picture. Furthermore, the variables Id1 and Id2 are significantly connected with the variables Ir1 and Ir2. This connection is not surprising because some of the students who declared that they used the decorative pictures to solve the problem tasks, they did the same for the representational pictures.

Moreover, significant similarity relationship can be observed between the variables Idr1, Idr2 and Iv2 which represent the cases in which students solved the verbal problems or the problems which were accompanied with decorative picture, by drawing a representational picture. This behavior can be considered as systematic because, students who need and draw representational pictures to solve verbal problems, do the same for the problems which are accompanied with decorative picture. However, as shown in Table 3, this behavior can be observed rarely.

Considering the similarity diagram, there is a similarity connection between the variables Iv1, Idr1, Idr2, Iv2, Ir1, Ir2, Id1, Id2 (group 1) and the variables A1, Be1 (group 2). This connection is expected, because students who use pictures to solve addition problems, have a positive attitude towards them.
Furthermore, as shown in the Similarity Figure, there is no similarity relationship between the two groups. This finding indicates that picture use and students attitudes towards them are not connected with students' performance in solving one-step problems of the additive structure.

Implicative Graph

The Implicative Graph (Figure 2) shows the implications between problem tasks, questions which referred to picture use and questions which referred to students' attitude towards pictures. The implications are important at level of significance 90%, 95% and 99%, according to the thickness of the line.

According to the Implicative Graph (Figure 2), variables which referred to the picture use on problem tasks which were accompanied with representational and decorative image (Id1, Id2, Ir1, Ir2), are connected with implicative relationships. From Figure 2 it can be concluded that students who use the decorative pictures (Id1, Id2), use also representational pictures (Ir1, Ir2) to solve the addition problems. This finding is in agreement with the percentages shown in Table 3 which indicate that more students use representational than decorative pictures.

It is also evident that picture use (Id1, Id2, Ir1, Ir2) entails positive attitude towards them (A1, Be1). Concretely, students who solve the problem tasks by using either decorative or representational picture, refer that they like pictures because they help them in problem solution.

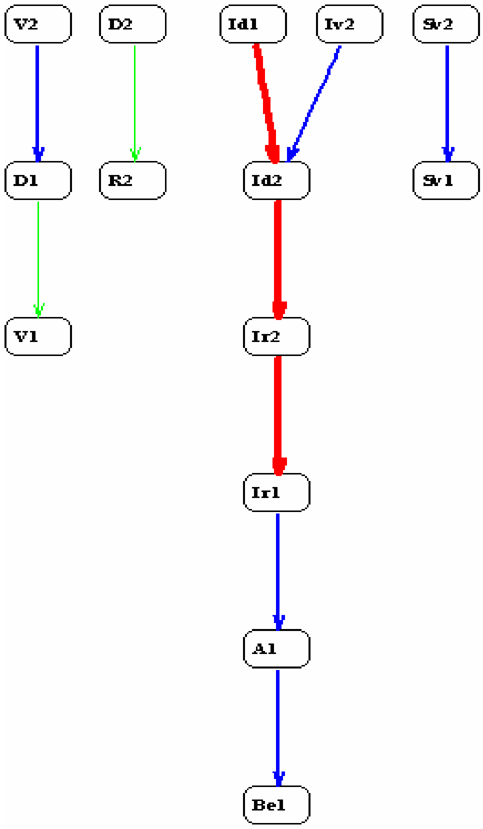

Figure 2: Implicative graph of students' responses to the tasks

Discussion

The gap in the research literature on the role of pictures in MPS (Gagatsis & Elia, 2004) contributed to conduct this study. Based on the functions that pictures serve in text processing, as proposed by Carney and Levin (2002), this study attempted to examine the role of decorative and representational picture in solving mathematical problems of the additive structure with the unknown in the first part (a). Moreover, considering that the affective domain can enhance or undermine pupils' performance in Mathematics (De Bellis & Goldin, 2006), we also focused on the role of pupils' emotions and attitudes towards the use of pictures in MPS.

A basic aim of the study was to investigate the role of decorative and representational pictures in solving one–step grouping (part – part – whole) problems with additive structures. As the results have shown, students' performance in MPS is not affected by the presence and use of decorative and representational pictures. This finding coincides with the findings of a previous study by Gagatsis et al (1999), which showed that different modes of representation, such as pictures, do not always assure successful overlapping of cognitive difficulties in Mathematics.

In this study we also examined the relation between students' performance and the nature of the representation (decorative or representational pictures) used. It is evident that the success percentages are high enough in all the problem tasks. Therefore students' performance is not altered according to the mode of representation used. This finding is in agreement with the results of a previous study by Gagatsis and Marcou (2002) which showed that decorative pictures did not lead to a change in students' behavior towards non-routine verbal problems. These results also support Theodoulou, Gagatsis and Theodoulou's (2004) conclusions that auxiliary-representational pictures had a significant positive effect only in some cases.

As regards the decorative picture, it seems that decorative pictures have no impact on pupils' behavior in MPS. It is also remarkable the fact that pupils sometimes draw a representational picture in order to solve a problem which is accompanied with decorative picture. Thus, Carney and Levin's (2002) opinion that decorative pictures do not enhance any understanding or application to the text appears to extend itself in the case of mathematical problems.

The results have also shown that students attitudes towards the role and use of pictures is not connected with their performance in MPS. Further research is needed to investigate the relationship between pictorial representations and affect, which constitutes an internal representational system according to De Bellis and Goldin (2006), in order to conclude that the affective domain can enhance or undermine pupils' performance in Mathematics.

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