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Examining primary school students' operative apprehension of geometrical figures through a comparison between the hierarchical clustering of variables, implicative statistical analysis and confirmatory factor analysis [1]

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Abstract. This study aims to gain insight about the distinct features and advantages of three statistical methods of analysis, namely the hierarchical clustering of variables, the implicative method and the Confirmatory Factor Analysis, by comparing the outcomes of their application in the operative apprehension of the geometrical figure. Data were obtained from 125 students in grade 6. Using Confirmatory Factor Analysis, we developed and verified a model that provides information about the significant role of the mereologic, optic and the place way modification in operative apprehension of the geometrical figure. Using the hierarchical clustering of variables, evidence is provided to the phenomenon of compartmentalization among modifications in students' operative apprehension. In general, the outcomes of the three methods were found to coincide and to be open to complementary use in capturing the ways in which students use the different types of figure modification.

Résumé. Cette étude a comme objectifs de clarifier les caractéristiques et les avantages distincts de trois méthodes d'analyse statistique, à savoir la similarité hiérarchique des variables, la méthode implicative et l'analyse factorielle confirmatoire, en comparant les résultats de leur application dans l'appréhension fonctionnelle de la figure géométrique. Des données ont été obtenues à partir de 125 étudiants dans la classe 6 de l´Ecole Primaire (11-12 ans). En utilisant l'analyse factorielle confirmatoire, nous avons développé et avons vérifié un modèle qui fournit des informations au sujet

1 This paper is in the context of the research project *"Ability to use multiple representations in Functions and Geometry: The Transition from Middle to High school"* of the Research Promotion Foundation of Cyprus (ΙΠΕ) [ΑΝΘΡΩΠΙΣΤΙΚΕΣ/ΠΑΙΔΙ/0308(ΒΕ)/03].

du rôle significatif de la modification mereologique et optique et la modification de la place de la figure dans l'appréhension fonctionelle de la figure géométrique. En utilisant la similarité hiérarchique des variables, l'évidence est fournie au phénomène du compartmentalization parmi des modifications de la figure concernant l´ appréhension fonctionnelle dans étudiants. Généralement les résultats des trois méthodes se sont avérés pour coïncider et être ouverts d'utilisation complémentaire en capturant les manières que les étudiants emploient les différents types de la modification de la figure. (MSC97G10)

Περίληψη Η μελέτη αυτή έχει ως στόχο να εντοπιστούν οι διαφορές σχετικά με τα διαφορετικά χαρακτηριστικά και τα πλεονεκτήματα των τριών στατιστικών μεθόδων ανάλυσης, δηλαδή της ανάλυσης ομοιότητας, της συνεπαγωγικής στατιστικής ανάλυσης και της επιβεβαιωτικής παραγοντικής ανάλυσης, συγκρίνοντας τα αποτελέσματα της εφαρμογής τους ως προς τη λειτουργική κατανόηση του γεωμετρικού σχήματος. Τα δεδομένα συλλέχθηκαν από 125 μαθητές έκτης τάξης Δημοτικού σχολείου. Χρησιμοποιώντας την επιβεβαιωτική παραγοντική ανάλυση αναπτύξαμε και επαληθεύσαμε ένα μοντέλο, το οποίο παρέχει πληροφορίες για το σημαντικό ρόλο των μερολογικών, οπτικών και των τροποποιήσεων αλλαγής θέσης στη λειτουργική κατανόηση του γεωμετρικού σχήματος. Μέσα από την ανάλυση ομοιότητας προέκυψε το φαινόμενο της στεγανοποίησης μεταξύ των τριών διαφορετικών τροποποιήσεων στη λειτουργική κατανόηση του γεωμετρικού σχήματος. Σε γενικές γραμμές, διαπιστώθηκε ότι τα αποτελέσματα των τριών μεθόδων συμπίπτουν και μπορούν να χρησιμοποιηθούν συμπληρωματικά, ώστε να σχηματίσουμε μια πιο ολοκληρωμένη εικόνα για τη χρήση των διαφορετικών τρόπων τροποποίησης του γεωμετρικού σχήματος από τους μαθητές.

1. Theoretical framework

1.1. The geometrical figure

A figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one, because in a figure various relations of an object with other objects are depicted (Mesquita, 1996).

Geometrical figures are simultaneously concepts and spatial representations. In this symbiosis, the figural facet is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). The double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (Mesquita, 1998). According to Duval (1995), the usefulness of the geometrical shape in the analysis of a geometrical problem is considered to be unquestionable, since it provides an intuitive presentation of the components and relationships in a geometrical situation. However, students are often not helped by the figure, in order to reach the solution of the problem.

1.2.. Discriminating the apprehensions of geometrical figures

Duval (1995) distinguishes four apprehensions for a geometrical figure. *Perceptual* apprehension refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential* apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive* apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g., denomination, definition, primitive commands in a menu). However, it is through *operative* apprehension that we can get an insight to a problem solution when looking at a figure.

1.3. Operative apprehension: Visualization and figural processing

A fundamental component of visualization is visual processing. Visual processing includes the following functions-processes of mental images: change in the position of the represented object (e.g. object rotation), change in the structure of the represented object, combination of the above changes (Yakimanskaya, 1991). Operative apprehension is a form of visual processing that concerns geometrical figures. It depends on the various ways of modifying a given figure: the *mereologic* way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the *op-*

tic way is when one makes the figure larger or narrower, while the *place* way refers to its position or orientation variation.

Each of these different modifications can be performed *mentally* or *physically*, through various operations. These operations constitute a specific figural processing which provides figures with a *heuristic function*. In operative apprehension the given figure becomes a starting point to explore other configurations that stem from the applications of these visual operations. A configuration may give insight into the solution of a problem. The ability to draw some units on a given figure is an indication of operative apprehension.

Previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Mesquita, 1996). Recent studies have used different methods of analysis to investigate students' geometrical figure understanding. A research by Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2009, in press) confirmed the role of perceptual, operative and discursive apprehension in geometrical figure understanding using Confirmatory Factor Analysis (CFA). Moving a step forward, Elia, Gagatsis, Deliyianni, Monoyiou and Michael (2009) investigated the role the mereologic, the optic and the place way modifications exert on operative figure understanding and they verified a model. This study has attained its outcomes by using CFA and the implicative method. Another relative study has used the hierarchical classification in combination to the implicative method (Michael, Gagatsis, Deliyianni, Elia, & Monoyiou, 2009).

2. Aim and research questions

An important aspect that arises is which aspects of a study each statistical analysis serves better. In an attempt to answer these questions, the purpose of this study, which concentrates on students' operative figure understanding, is to apply the three aforementioned statistical methods of analysis on the same sample data and compare their outcomes. The study focuses on the comparison and combination of the results of CFA, hierarchical clustering of variables and implicative method on the same sample data.

We use the CFA trying to understand the multiple dimensions of operative apprehension, we discuss the implicative analysis graph that provides additional lighting for a better structuring of the relations between the types of modifications and finally we examine these relationships in the light of hierarchical consolidation. Our main consideration is to form a more comprehensive picture for each statistical method, regarding their special characteristics, their advantages and their limitations when dealing with the important issue of operative figure understanding. We are, also, focussing on whether the outcomes of the three statistical methods used are complimentary or even overlapped, based on the students' performances that we observed.

Concerning those mentioned above, we state our research questions:

– Which are the common characteristics of the results of the three statistical methods? Is there a consistency between the outcomes arisen from each method?

– Which statistical method is more appropriate and complementary use open for particular aspects of the study?

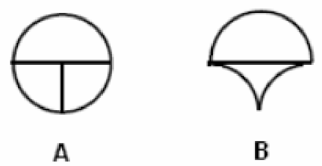
3. . Method

3.1.. Participants, instrument and variables

The study was conducted among 125 students, aged 11 to 12, from primary schools in Cyprus (Grade 6). The test consisted of three groups of tasks:

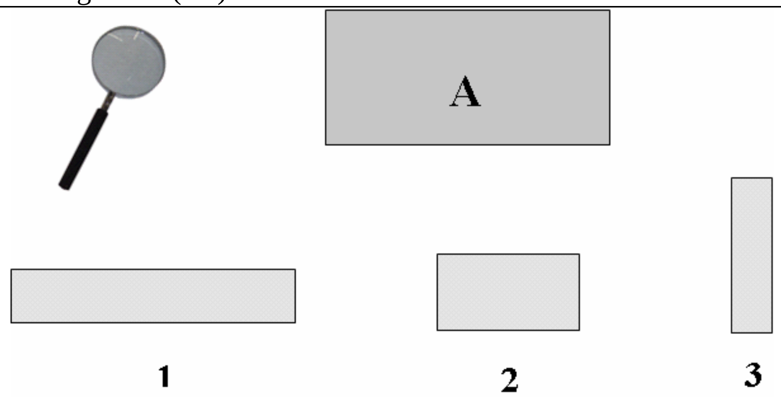
1. The first group of tasks includes task 1 (M1), 2 (M2) and 3 (M3) concerning students' **mereologic** way of modifying a given figure.

Underline the right sentence: (M1)
a) Fig. A has bigger perimeter than Fig. B
b) Fig. A has equal perimeter with Fig. B
c) Fig. A has smaller perimeter than Fig. B



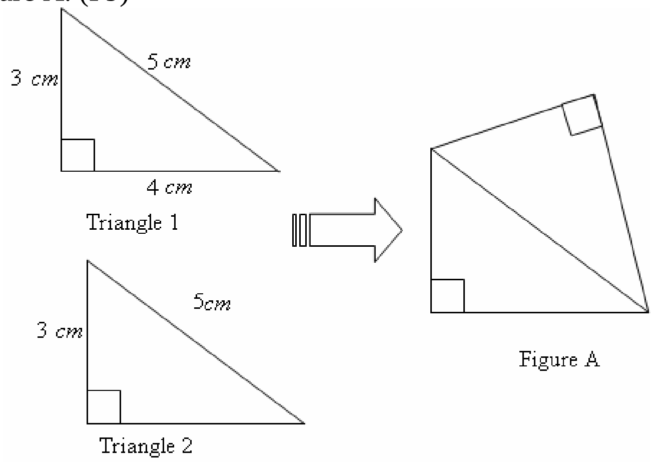
2. The second group of tasks includes task 4 (O4), 5 (O5) and 6 (O6). These tasks examine students' **optic** way of modifying a given figure.

Vassilis constructed a rectangle in his writing book. Shape A is the rectangle as it looks through a magnifier. Circle the picture that shows the rectangle, as it is in Vassilis writing book. (O4)



3. The third group of tasks includes task 7 (P7), 8 (P8), 9 (P9) and 10 (P10) that correspond to the place way of modifying a given figure.

Theodosis combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (P9)



3.2. . Data Analysis

3.2.1. . Structural Equation Modceling and CFA

"Structural equation modeling (SEM) is a statistical methodology that takes a hypothesis testing (i.e. confirmatory) approach to the multivariate analysis of a structural theory bearing on some phenomenon" (Byrne, 1994). This theory concerns "causal" relations among multiple variables (Bentler, 1988). These relations are represented by structural, namely regression equations, which can be modeled in a pictorial way to allow a better conceptualization of the involved theory.

SEM differs from the more traditional multivariate statistical techniques in at least three dimensions: First, with the use of SEM the analysis of the data is approached in a confirmatory manner rather than in an exploratory way, making hypothesis testing more accessible and easier, compared with other multivariate procedures. Second, whereas SEM gives the estimates of measurement errors, the "conventional" multivariate methods cannot assess or correct for these parameters. Third, SEM involves not only observed but also latent (unobserved) variables, whereas the older techniques incorporate only observed measurements.

Factor analysis is a well known statistical technique for examining associations between observed and latent variables. The covariation among a set of observed variables is investigated to get information on their underlying latent factors. Of primary interest is the strength of the regression paths from the factors to the observed variables. Jolliffe (2002) claims that there is a confusion between factor analysis and the principal component analysis (PCA) that possibly arose due to Hotelling’s (1933) original paper, in which principal components were introduced in the context of providing a small number of ‘more fundamental’ variables that determine the values of the *p* original variables.

Principal component analysis has often been dealt with in textbooks as a special case of factor analysis, and this practice is continued by some widely used computer packages, which treat PCA as one option in a program for factor analysis. This view is misguided since PCA and factor analysis, as usually defined, are really quite distinct techniques. Both PCA and factor analysis aim to reduce the dimensionality of a set of data, but the approaches taken to do so are different for the two techniques. Principal component analysis has been extensively used as part of factor analysis, but this involves ‘bending the rules’ that govern factor analysis and there is much confusion in the literature over the similarities and differences between the techniques. A major distinction between factor analysis and PCA is that there is a definite model underlying factor analysis, but for most purposes no model is assumed in PCA (Jolliffe, 2002).

Exploratory Factor Analysis (EFA) and Confirmatory Factor Analysis (CFA) are two basic types of factor analysis. EFA is employed to determine how the observed variables are connected to their underlying constructs in situations where these links are unknown. By contrast, CFA, which is the type of analysis employed here, is used in situations where the researcher aims to test statistically whether a hypothesized linkage pattern between the observed variables and their underlying factors exists. This a priori hypothesis draws on knowledge of related theory and past empirical work in the area of the study.

CFA allows the researcher to test the hypothesis that a relationship between the observed variables and their underlying latent construct(s) exists. The researcher uses knowledge of the theory, empirical research, or both, postulates the relationship pattern a priori and then tests the hypothesis statistically. Traditional statistical methods normally utilize one statistical test to determine the significance of the analysis. However, Structural Equation Modeling, CFA specifically, relies on several statistical tests to determine the adequacy of model fit to the data. The chi-square test indicates the amount of difference between expected and observed covariance matrices. A chi-square value close to zero indicates little difference between the expected and observed covariance matrices. In addition, the probability level must be greater than 0.05 when chi-square is close to zero (Suhr, 2006).

The basic steps that a researcher follows for carrying out CFA are described below: The model is specified based on knowledge of relevant theory and previous empirical research. Using a model-fitting program, such as EQS, the model is analyzed so that the estimates of the model's parameters with the data are derived. Then the tenability of the model is tested based on data that involve all the observed variables of the model. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > 0.9, RMSEA < 0.06. If the hypothesized model is not consistent with the data the model is respecified and the fit of the revised model with the same data is evaluated (Byrne, 1994; Kline, 1998).

The number of levels that the latent factors are away from the observed variables determines whether a factor model is called a first-order, a second-order or a higher order model. Correspondingly, factors one level removed from the observed variables are labeled first-order factors while higher-order factors which are hypothesized to account for the variance and co-variance related to the first-order factors are termed second-order factors. A second or a higher order factor does not have its own set of measured variables. In this study a second-order model will be considered.

A structural equation model involves two basic types of components: the variables and the processes or relations among the variables. A schematic representation of a model, which is termed path diagram, provides a visual interpretation of the relations that are hypothesized to hold among the variables under study. The observed or measured variables, which constitute the actual data of the study, are often designated as Vs and are shown in rectangles. The unmeasured variables, which are hypothetical and represent the structural organization of the phenomenon under study, are designated as F and represented in the path diagram in ellipses or circles.

One type of the relations involved in a model is the structural regression coefficients indicating the impact of one variable on another. They are represented by one-way arrows. For example, the unidirectional arrows leading from the Factor "F1" (Figure 1) to the three observed variables (M1, M2, M3) indicate that the scores on the latter variables are "caused" by the factor "F1". These relations are called "factor loadings". Similarly, unidirectional arrows from one factor to another imply that a factor causes or predicts another factor, e.g. in Figure 1 the arrows starting from "Operative Apprehension" and pointing toward "F1", "F2" and "F3" imply that "Operative Apprehension" predicts "F1", "F2" and "F3".

3.2.2. Implicative Statistical Analysis and Hierarchical Clustering of Variables

For the analysis of the collected data, the hierarchical clustering of variables and Gras's implicative statistical method has been also conducted using the computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Couturier, & Gras, 2000). These methods of analysis determine the hierarchical similarity connections and the implicative relations of the variables respectively (Gras 1992; Gras et al., 1996). For the needs of this study, similarity and implicative diagrams have been produced from the application of the analyses on the sample of students.

The hierarchical clustering of variables (Lerman, 1981) is a classification method which aims to identify in a set V of variables, sections of V, less and less subtle, established in an ascending manner. These sections are represented in a hierarchically constructed diagram using a similarity statistical criterion among the variables. The similarity stems from the intersection of the set V of variables with a set E of subjects (or objects). This kind of analysis allows the researcher to study and interpret clusters of variables in terms of typology and decreasing resemblance. The clusters are established in particular levels of the diagram and can be compared with others. This aggregation may be indebted to the conceptual character of every group of variables.

In particular, the method used here is the 'likelihood linkage analysis' (LLA) (Lerman, 1991). LLA is a methodology for grouping data into significant classes and subclasses, using an algorithm of hierarchical classification. This method introduces a most original notion of statistics for measuring statistical relationships and proximities, namely the "likelihood" concept. Lerman (1991) sets up the "likelihood" notion as part of the "resemblance" notion. The flexibility of this method enables us to take into account any combinatorial and logical structure of which the modality set of a given descriptive variable is provided.

The construction of the hierarchical similarity diagram is based on the following process: Two of the variables that are most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables gets very weak. In this study the similarity diagrams allow for the arrangement of the variables, which correspond to students' responses in the tasks of the tests, into groups according to their homogeneity.

The implicative statistical analysis (Gras et al., 1996; Gras, Peter, Briand & Philippe, 1997) aims at giving a statistical meaning to expressions like: *"if we observe the variable A in a subject, then in general we observe the variable B in the same subject"*. Thus the underlying principle of the implicative analysis is based on the quasi-implication: *"if A is true then B is more or less true"*. An implicative diagram represents graphically the network of the quasi-implicative relations among the variables of the set V. In this study the implicative diagrams contain implicative relations, which indicate whether success to a specific task implies success to another task related to the former one.

It should be noted that the present paper is related to the ones of Michael, Gagatsis, Deliyianni, Elia and Monoyiou (2009) and Elia, Gagatsis, Deliyianni, Monoyiou and Michael (2009), whose basic findings are included in the theoretical section (2).

4. Results

4.1.. Outcomes of CFA

A main concern of this study was to validate a CFA model that could capture the structural organization underlying the processes of the students' operative figure understanding. Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [x^2 (32) = 35.228, CFI = 0.961, RMSEA =0.029]. The coefficients of each factor were statistically significant. The errors of variables are omitted.

The second-order model which is considered appropriate for interpreting operative apprehension, involves three first-order factors and one second-order factor. On the second-order factor that stands for operative apprehension the first-order factors F1, F2 and F3 are regressed. The first-order factor F1 refers to the tasks which correspond to the mereologic way of modifying a given figure, the first-order factor F2 refers to the optic modification tasks and the first-order factor F3 refers to the place modification tasks. The factor loadings reveal that the mereologic and place types of modification are the primary source explaining students' operative apprehension of geometrical figures. That is, they are highly related to the operative apprehension. However, the results indicate that all three ways of modifying geometrical figures have a significant effect on operative figure understanding.

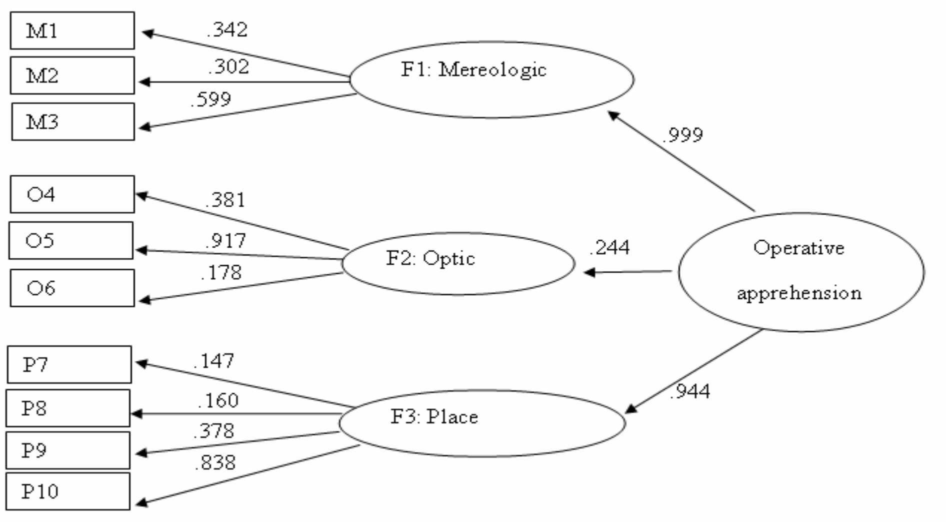


Figure 1.The CFA model of operative apprehension.

4.2. The outcomes of the hierarchical clustering of variables and the implicative method of analysis

Each observed variable in the CFA model (Figure 1) represented students' "unified" score at the three types of modifying a geometrical figure, the mereologic, the optic and the place way. In order to be able to compare the three statistical methods we examine, we concluded that it would be useful to use the variables mentioned above in the implicative analysis and the hierarchical classification.

Figure 2 presents the similarity diagram of the sixth graders' responses to the tasks of the test. Two similarity clusters are identified. Cluster 1 involves students' responses to all the mereologic modification tasks (M1, M2, M3) and two of the place modification tasks (P9, P10). Cluster 2 is comprised of students' responses to all the optic modification tasks (O4, O5, O6) and the other two place modification tasks (P7, P8).

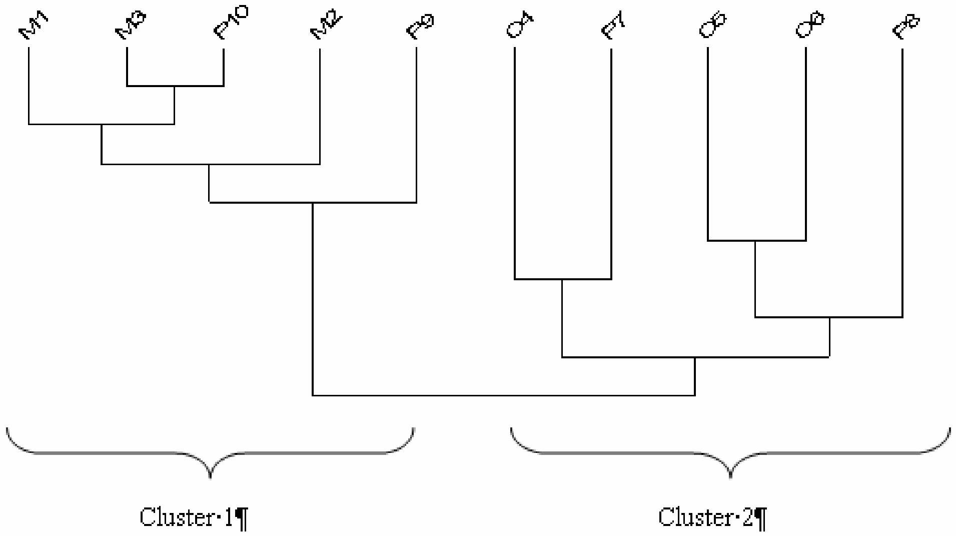


Figure 2.Similarity diagram of sixth graders’ responses to the tasks concerning the three types of geometrical figure modification.

The two clusters suggest that students displayed consistency in applying respectively the mereologic way and the optic way of modifying geometrical figures. This is not the case, though, for the place way of modifying geometrical figures, as the responses of the students to the four corresponding tasks are split into the two existing clusters. A number of the place modification tasks were approached similarly to the mereologic modification tasks, while the rest of the place modification tasks were tackled similarly to the optic modification tasks.

The following diagram presents the implications between the variables, according to students’ behaviour to the tasks of the test. Concerning Figure 3, a dual implicative chain is discriminated, which indicates the hierarchical ordering of the geometrical figure modification tasks with respect to their level of difficulty on the basis of students’ performance. Branch A involves the students’ responses to the concerning place way modification tasks. The other branch, named branch B, is comprised of tasks from the three types of modification of geometrical figures.

Both branches stem from the variable P10. This is a place way modification task that asked students to create a figure that corresponds to a given perimeter, by combining two triangles with the length of their sides given. It consists the most complex task of the test and students who provided a correct solution at it, succeeded at all of the other tasks of the test that appear in the implicative diagram. Students’ great difficulty in the particular task is also shown by their low success rate ($\overline{X}$ =30.4%).

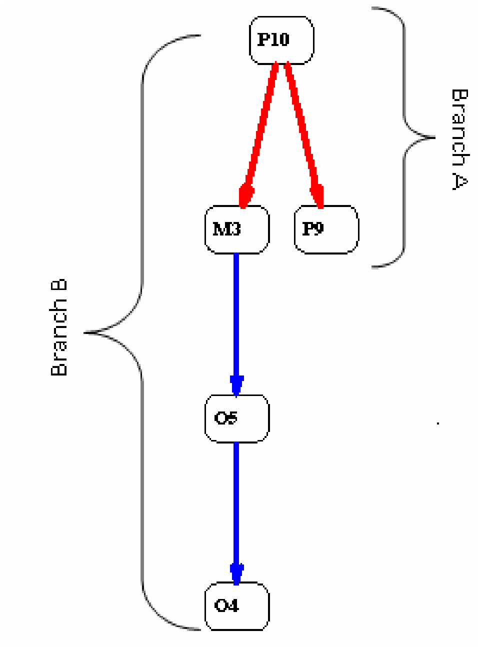


Figure 3. The implicative diagram among the responses of students to the geometrical figure modification tasks.

Referring to branch A, we see that students who accomplish the most difficult task of the test also succeed in the task P9. In task P9 students are asked to do the opposite think, related to what they are asked to accomplish in the place way modification task P10. In this task students were asked to calculate the perimeter of a figure that came up from the combination of two triangles with given sides. So from this branch it is obvious that students face difficulties when they have to change the orientation of the figures and combine them, compared to a situation when the figure is given to them. In other words, it is more difficult for students to perform these mental operations and visualize a figure, in order to manage to transform it.

Considering "Brunch B", it is indicated that the correct solution of the most difficult task of the test P10 leads to the accomplishment of the task M3. It seems that it is more difficult for students to face a place way modification task, than a reconfiguration of a shape in a mereologic task. Success in the mereologic task M3 entails success in the latter task, which is the optic modification task 5. Task O4 is situated in the bottom of the implicative "Brunch B" and it is concerned to be the easiest task among the tasks appearing in the diagram. Students perform easier in an optic task that requires making a figure smaller, while is more difficult for them to succeed in an optic task that involves enlargement of a figure. Looking at the whole diagram, we can conclude that the place modification is more difficult for students in comparison to the optic modification tasks.

5. . Discussion

In this study the students' geometrical figure understanding was examined. The special think about this study is that the data were analyzed with three different statistical methods, each one having its own perspective and rationale. CFA was used in order to verify the hypothesis based on Duval's (1995) theoretical framework, according to which the operative apprehension depends on the various ways of modifying a figure. The hierarchical clustering of variables aimed at tracking and presenting the consistency among students' responses to the modification tasks in a hierarchical manner, while the implicative method examined whether success at one task implies success at another task. A main focus of our study is the comparison of the findings of the statistical analyses performed, in order to gain an insight about the advantages of each method, as well as about whether their outcomes on the same sample data are congruent and can complement each other.

The summary and the comparison of the outcomes of the three statistical methods on the data of the study that concur or have a complementary role amongst them are presented in table 1. Whereas the implicative technique and the hierarchical clustering of variables incorporate only observed measurements, CFA allowed the development and validation of a model that involves not only observed but also latent (unobserved) variables, which cannot be observed or measured directly (Table 1, items 2). These constructs, which lied behind the corresponding observed measures, were the ability to modify a geometrical with the mereologic, the optic and the place way. Another abstract construct of a higher-order level was assumed to underlie these abilities, indicating that despite the discrepancy in students' performance among the different types of modification, all three types of modification are still basic components of a common construct, the operative apprehension of the geometrical figure. The structure of the CFA model provided a strong case for the role of the different types of modification of a geometrical figure.

The strength of relations of the three first – order factors to the second-order factor in the model showed that each type of modification has a different and an almost autonomous function in the operative apprehension of the geometrical figure. The compartmentalization of students' responses to modifications, concerning the optic and the mereologic type, was also revealed by the results of another method (Table

1, items 3). This finding lends support to Duval's (1995) conceptualization of the cognitive processes underlying operative figure understanding and suggests that, in order to develop operative apprehension during mathematics instruction in primary school, emphasis should be given on the three types of figure modification, which provides figures with a heuristic function. In specific, students' inconsistency when dealing with different types of modification was also evident from the similarity diagram, in which the responses to mereologic and optic modifications were separated. However, the implicative diagram indicated that success in one type of modification can lead to success in another type of modification.

Table 1: The congruent and complementary outcomes of the CFA, the hierarchical clustering of variables and the implicative method on the data of the study

CFA	Hierarchical clustering	Implicative method
1. Factorial structure of students' geometrical figure understanding.	1. Hierarchical classification and consistency of students' responses to the geometrical figure modifications.	1. Implicative relations between students' responses to the to the geometrical figure modifications, relative difficulty of the tasks.
2. Development of a model involving two latent (unobserved) factors for the effects of three types figure modification and a second-order factor standing for the operative apprehension of the geometrical figure.	2. Similarity groupings among observed measurements standing for students' responses to the three ways of modifying a geometrical figure.	2. Implications among observed variables standing for students' responses to the three types of modifications of the geometrical figures.
3. Difference in the strength of the relations of the three first-order factors to the second-order factor: each type of modification operates rather autonomously.	3. Compartmentalization in students' responses to the modifications with respect to the type of the modification.	3. Two implicative chains involving students' responses to the three types of modifications of the geometrical figures, lack of implications between the variables of the two chains.
4. Lower factor loadings of the optic modifications relatively to the mereologic and place modifications.	4. Separate grouping of the variables of the mereologic and the optic modification, significant role of the type of modification on students' consistency. Relatively weak similarity of students' place way modifications: distinct ways of approaching this type of modification.	4. A place way modification tasks was the most complex task than the tasks involving mereologic or optic way modification. The optic modification tasks were the easiest ones.

Comparing the three methods of analysis, the CFA did not verify the exact same groupings of tasks as the implicative and the hierarchical classification methods.

Despite this, the implicative and the hierarchical classification methods provided insight and a more analytic view about the construction and hierarchical structure of these groups and the implicative relations among students’ responses to the tasks. The hierarchical clustering of variables revealed some discrepancies in the ways students tackled particular tasks of the tests, which were not evident in the CFA model.

Particularly, the similarity diagram showed that students’ consistency varied across the three types of geometrical figure modification. Whereas students exhibited consistency in the mereologic modification tasks and the optic modification tasks respectively, they applied the place way of modifying geometrical figures in a rather fragmentary way. A number of the place modification tasks (P9, P10) were approached similarly to the mereologic modification tasks, and the rest of the place modification tasks (P7, P8), were tackled similarly to the optic modification tasks. This finding suggests that although it is the place modification that gives insight to the solution of the corresponding tasks (Duval, 1995) some additional operations need to take place so that students successfully reach the ultimate solution. These additional operations may have common characteristics with the figural processing which is required in either the mereologic modification tasks or the optic modification tasks. Furthermore, the implicative diagram revealed that the most difficult task was the place way modification task, in which students had to calculate the perimeter of a figure that came up from the combination of two triangles with given sides.

The outcomes of the three statistical processes uncovered how students dealt with different types of figure modification related to the operative apprehension (Table 1, items 4). The mereologic modification was found to have considerable autonomy from the optic modifications. The separate grouping of the former variables from the latter ones in the similarity diagrams revealed that students tackled the mereologic modifications differently from the optic ones. The implicative diagrams revealed additional information to the above findings, suggesting that the optic tasks were less complex than a mereologic task.

In general, the application of all of the analyses yielded congruent results. However, at the same time given that these statistical processes approached the data from different perspectives, they emphasized different aspects of students’ outcomes. This differentiation allowed for the accumulation of a number of new distinctive elements by each analysis that contributed to the unravelling and making sense of students' performance, structure of abilities, difficulties and inconsistencies on the particular subject (Elia & Gagatsis, 2008).

The findings of the study suggest that the three statistical methods are open to complementary use and each one does not operate at the expense of the other. CFA provided a means for making sense of the structure of students' operative apprehension of the geometrical figure. The hierarchical clustering of variables provided a means for classifying students' responses, for identifying students' consistencies and inconsistencies among different conversions and for investigating the factors influencing this behaviour. The implicative method provided a means for examining the implicative relations among the responses to the tasks and the relative difficulty of the different conversions on the basis of students' performance. This result is reinforced by Studer's and his colleagues' (2007) study in which the results of several exploratory statistical methods were compared. Specifically they compared statistical implicative analysis with multiple correspondence analysis and case clustering. They concluded that the implicative statistics advantageously complements more classical exploratory data analyses results. The application of these methods of analysis in combination may consist a way to overcome some limitations of each analysis employed separately. Therefore the results of an investigation can be enriched and deepened.

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