FRACTAL MODELS IN ARCHITECTURE: A CASE OF STUDY

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ABSTRACT The word "fractal" was coined by Benoit Mandelbrot in the late 1970's, but object now defined as fractal in form have been known to artists and mathematicians for centuries [8]. Mandelbrot's definition "*a set whose Hausdorff dimension is not an integer*" is clear now in mathematical terms. The fractal geometry is quite young (the first studies are the works of G. Julia (1893 - 1978) at the beginning of this century) but, only with the mathematical power of computers it is become possible to obtain the beautiful and colourful images derived by the arid formulas. A fractal object is self - similar in that subsections of the object are similar in some sense to the whole object. No matter how small a subdivision is taken, the subsection contains no less detail than the whole. A typical example of fractal object is the "Snowflake Curve" (devised by Helge von Koch (1870 - 1924) in 1904 [20]). There are many relationships between architecture, arts and mathematics for example the symmetry, the platonic solids, the polyhedra, the golden ratio, the spirals, the Fibonacci's sequence [1, 7, 11, 12, 13, 15, 18], but it is difficult to find some interconnections between fractals and architecture. This paper investigates some relationships between architecture and fractal theory.

1. FRACTALS IN ARCHITECTURE

Architectural forms are handmade and thus very much based in Euclidean geometry, but we can find some fractals components in architecture, too.

We can divide the fractal analysis in architecture in two stages [19]:

- **little scale analysis** (e.g, an analysis of a single building)
- **large scale analysis** (e.g., the urban growth).

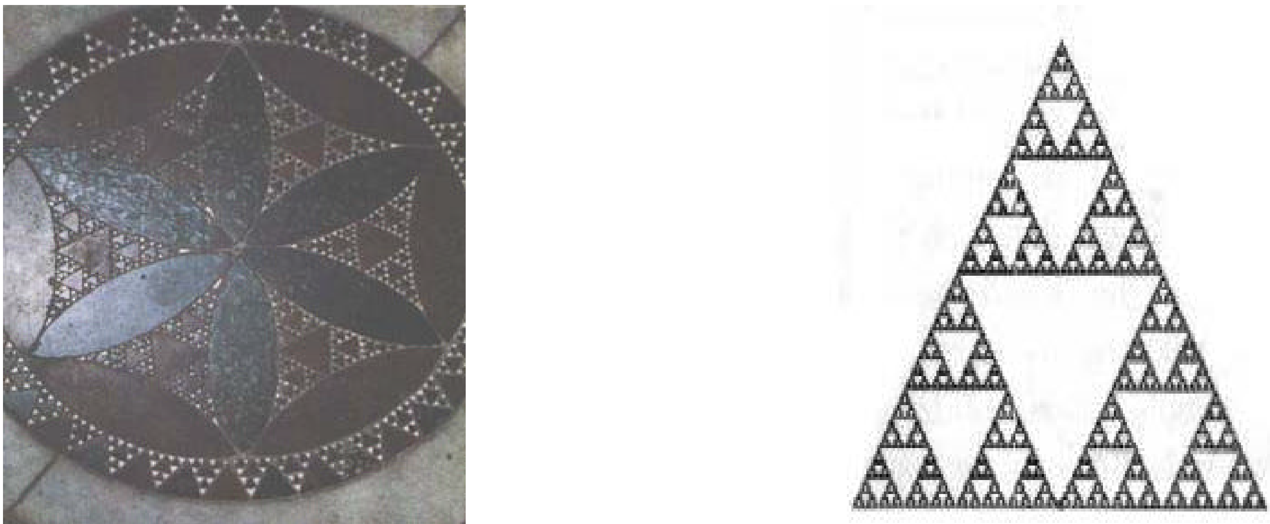
The little scale analysis comprises [4, 19]:

- **the building's self-similarity** (e.g. a building's component which repeats itself in different scales)
- **the box-counting dimension** (to determine the fractal dimension of a building) [5].

2. LITTLE SCALE ANALYSIS: THE BUILDING'S SELF-SIMILARITY

We can start the research of the buildings' self-similarity by the 1104. In fact in the cathedral of Anagni (Italy) there is a floor which is adorned with dozens of mosaics, each in the form of a Sierpinski gasket fractal. In the Figure 1 there is a part of one mosaic, showing the fractal at its fourth stage of iteration. The cathedral and its floor were built in the year 1104 and is possibly the oldest handmade fractal object [20].

Reims' Cathedral (France) has a rising fractal structure which represents the elevation. Every tower has: a big arcade, two windows, four ogives and their dimensions decrease in each floor [11]. In Saint Paul church in Strasbourg (France), we have observed the presence of self-similar shapes on each two towers. In Venice there are many palaces that have a rising fractal



(a) (b)

Figure 1. Floor of the cathedral of Anagni (a) and the Sierpinski gasket (b)

structure (e.g., Ca' Foscari, Ca' d'Oro, Duke Palace, Giustinian Palace), and for this reason we can talk about "fractal Venice" (see Figure 2).

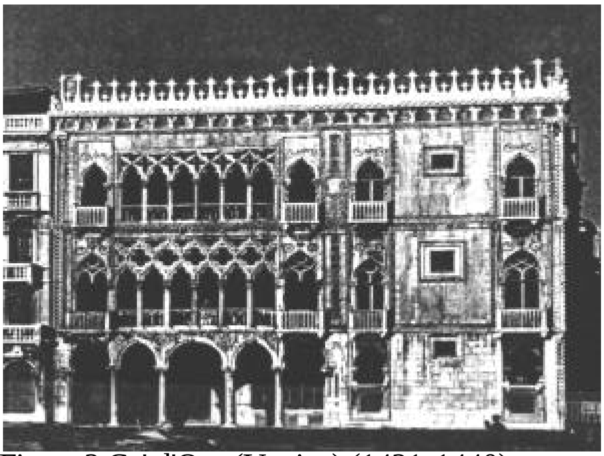


Figure 2 Ca' d'Oro (Venice) (1421-1440).

Other example of fractal architecture is "Castel del Monte" (Andria, Apulia, Southern Italy) raised by Federico II (1194 - 1250). The outer shape is an octagon, as is the inner courtyard. Even the eight small towers have octagonal symmetry (Figure 3 (a)).

Some researchers of the University of Innsbruck have found a connection between the building shape of Castel del Monte and a Mandelbrot set (Figure 3 (b)) [19].

Castel del Monte has other interesting implications with geography, astronomy, and mathematics (e.g. the presence of the golden ratio in the main portal [17]). We can affirm that Castel del Monte is: "*expression of a mysterious Middle Ages…*" [16].

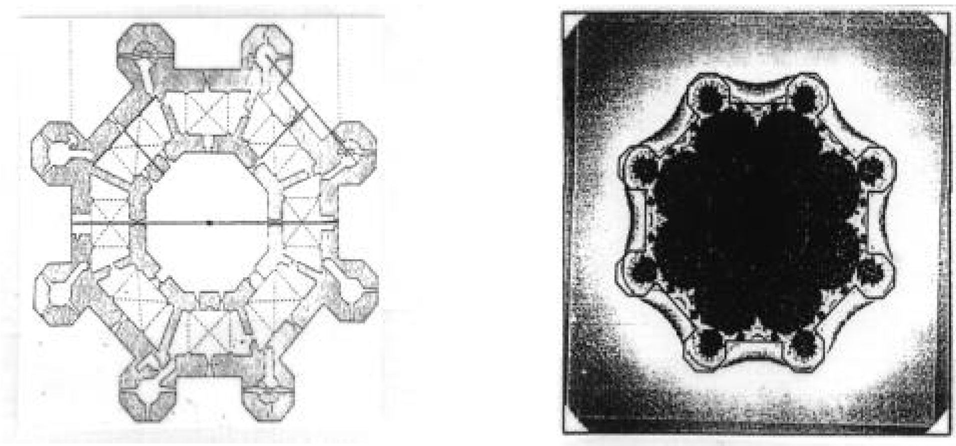


Figure 3 Castel del Monte, Apulia (Italy) (a) and its comparison with a Mandelbrot set (b).

We can find in Oriental architecture the self-similarity, too.

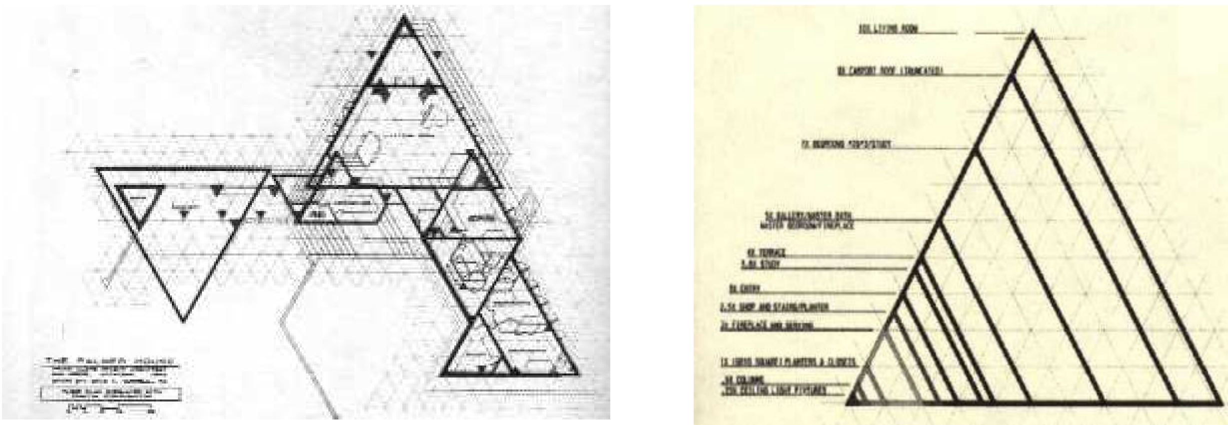
Eighteenth century Shiva Shrine is in the heart of India, and it is made in red sandstone. The Shiva Linga resides in the temple, while in the lntel above the door is a figure of Lord Ganesha. The Shikhara is typical of North Indian temple architecture [19, 20].

It is also a fractal architecture. The central tower is repeated in different scale at the centre and at four corners [16, 17]. Humayun's Mausoleum at Dehli (India) (moghol art, 1557 – 1565) presents a descending fractal structure [11]. The Sacred Stupa Pha That Luang - Vientane (Laos) is another example of fractal architecture because the basic shape is repeated in different scales (Figure 4).

Figure 4 Sacred Stupa - Vientane (Laos)

In these examples the fractal components are unconscious, but some famous twentieth century architects have researched a conscious self-similarity; e.g. **Frank Lloyd Wright** (1867-1959), in his late work ("Palmer house" in Ann Arbor, Michigan, 1950-1951) has used some self-similar equilateral triangles in the plan (Figure 5 (a)). A kind of "nesting" of fractal forms can be observed at two point in the Palmer house: the entry way and the fireplace. At these places one encounters not only actual triangles but also implied (truncated) triangles (Figure 5 (b)). At the entrance there are not only the triangles composing the ceramic ornament, there is also triangular light fixture atop of triangular pier. There is a triangle jutting forward overhead. The fireplace hearth is a triangular cavity enclosed between triangular piers. In the Palmer house the fractal quality is in every case the result of a specific and conscious act of design [10].

Other Wright's example of fractal architecture is the Marion County Civic Center, San Rafael (1957) where the self similarity is present in the external arches.



(a) (b)

Figure 5 The plan of Palmer house (a) the self similar triangles inside the Palmer house (b).

3. LITTLE SCALE ANALYSIS: THE BOX-COUNTING DIMENSION

To understand fractal concepts, we have to familiar with two dimensions [5, 19]:

- self- similarity dimension (D_s);
- box-counting dimension (D_b).

All these dimensions are directly related to Mandelbrot's fractal dimension (D).
In all self-similar constructions there is a relationship between the scaling factor and the number of smaller pieces that the original construction is divided into [4].
This is true for fractal and nonfractal structures. The relationship is the law:

$$a = \frac{1}{(s)^D} = \left(\frac{1}{s}\right)^D$$

where a is the number of pieces and s is the reduction factor. For nonfractal structure the exponent D is an integer.
Solving for D:

$$D = \frac{\log(a)}{\log\left(\frac{1}{s}\right)}$$

The self-similarity dimension (D_S) is equivalent to the Mandelbrot's fractal dimension (D). In the Koch curve the scaling factor is 1/3 and the number of pieces is 4; the fractal dimension is: D_s= log(4) /log(3) =1.26… .
The box-counting dimension is presented with the problem of determining the fractal dimension of a complex two-dimensional image.
It is produced using this iterative procedure:

- superimpose a grid of square boxes over the image (the grid size as given as s_1);
- count the number of boxes that contain some of the image ($N(s_1)$);
- repeat this procedure, changing (s_1), to smaller grid size (s_2);
- count the resulting number of boxes that contain the image ($N(s_2)$);
- repeat this procedures changing s to smaller and smaller grid sizes.

Box-counting dimension is:

$$D_b = \frac{[\log(N(s_2)) - \log(N(s_1))]}{\left[\log\left(N\left(\frac{1}{s_2}\right)\right) - \log\left(N\left(\frac{1}{s_1}\right)\right)\right]}$$

Where $1/s$ is the number of boxes across the bottom of the grid. We can apply in architecture the box-counting dimension. It is calculated by counting the number of boxes that contain lines from the drawing inside them. Next figure 6 illustrates the box count of a Frank Llyod Wright's building (Robie house) [4]. Table 1 contains the number of boxes counted, the number of boxes across the bottom of the grid, and the grid size.

Table 1

Box count	Grid size	Grid dimension
16	8	24 feet
50	16	12 feet
140	32	6 feet
380	64	3 feet

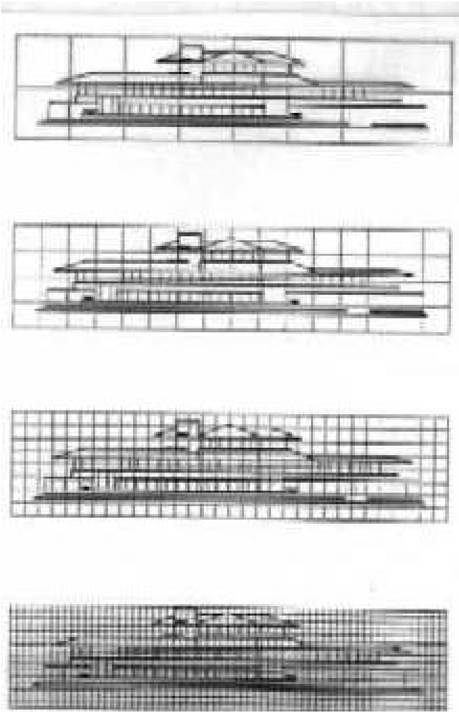


Figure 6 Box-counting grids places over elevations of the Robin House.

Three fractal dimensions can be calculated. The first is for the increase in number of boxes with lines in them from the grid with 8 boxes across the bottom (24 feet) to the grid with 16 boxes across the bottom (12 feet):

$$D_{(box, 24'-12')} = \frac{[\log(50) - \log(16)]}{[\log(16) - \log(8)]}$$

$$= \frac{(1.699 - 1.204)}{(1.204 - 0.903)} = \frac{0.495}{0.301} = 1.645$$

The next scanning range compares boxes that are 12 feet across with boxes that are 6 feet across:

$$D_{(box, 12'-6')} = \frac{[\log(140) - \log(50)]}{[\log(32) - \log(16)]}$$

$$= \frac{(2.146 - 1.699)}{(1.505 - 1.204)} = \frac{0.447}{0.301} = 1.485$$

The final scanning range compares boxes that are 6 feet across with boxes that are 3 feet across:

$$D_{(box, 6'-3')} = \frac{[\log(380) - \log(140)]}{[\log(64) - \log(32)]}$$

$$= \frac{(2.580 - 2.146)}{(1.806 - 1.505)} = \frac{0.434}{0.301} = 1.441$$

The last two calculations are in closer agreement than the first calculation.

4. CONCLUSIONS

The research of the building's self-similarity and the calculus of the building's fractal dimension are only two different approaches in the fractal little scale analysis.
Other fractal components are present in architecture:

- **In the fractal geometrical description of real built-up area** (e.g. using the self similarity or some particular fractal shape);
- **In the simulation of urban growth using fractal algorithm** (e.g. using the Diffusion Limited Aggregation (DLA) model [1]) [2, 3].

In the first approach there is present of an interesting syllogism [9, 19]:

- the house like a part of the town, and it is little image of the town, too;
- the town is constituted from copies of itself (this is an example of self-similarity) [9].

Other fractal approach in architecture is to find the connections between the structure of the towns and the fractal sets (e.g. Julia sets or Mandelbrot sets) [9].
The geographer Michael Batty suggested that the fractal geometry can describe the urban growth: "*The morphology of cities bears an uncanny resemblance to those dendritic clusters of particles which have been recently simulated as fractal growth processes*" [3] .
Fractal Cities is the title of Batty and Longley's (1994) book in which they show how Mandelbrot's (1983) theory of fractal geometry can be applied to the study of cities, their structure and evolution [14] . They show how complex geometries of urban form, growth and evolution, can be generated by means of Mandelbrot's type of fractal. This is an example of fractal approach in large scale [18].
Portugali introduces a new idea in the urban growth: "*Cities are self-organizing systems*"[2] [17]. To substantiate his revolutionary concept, he uses several interlinked methods. He employs in his argument theoretical tools developed in the interdisciplinary field of synergetics. He has performed detailed model calculations on cellular nets [17].
The aim of this work is to present how the fractal geometry is helping to newly define a new architectural models and an aesthetic that has always lain beneath the changing artistic ideas of different periods, schools and cultures [6, 7, 14, 19].

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[1] DLA is one of the most important models of fractal growth. It was invented by two physicists, Witten and Sander, in 1981. The growth rule is remarkably simple. We start with an immobile seed on the plane. A walker is then launched from a random position far away and is allowed to diffuse. If it touches the seed, it is immobilized instantly and becomes part of the aggregate. We then launch similar walkers one-by-one and each of them stops upon hitting the cluster. After launching a few hundred particles, a cluster with intricate branch structures results.

[2] Self-organization is the phenomena by which a system self-organizes its internal structure independent of external causes [17]. "*These systems exhibit also phenomena of nonlinearity, instability, fractal structures and chaos-phenomena which are related to the general sensation of life and urbanism at the end of the 20th century*" [17, p. 49].

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