WORKING GROUP 3B / GROUP DE TRAVAIL 3B

Classroom practices and learning spaces (from grade 9) / Pratiques en classe et autres espaces d’apprentissage (à partir du college)
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Working Group 3B was focused on secondary and higher education. The work began with a review and reformulation of questions from the conference discussion paper, which led to the following initial topics for discussion:

1. How ICT are useful and how can we have evidences that they are useful?
2. Multiple and different kinds of representations and the relationship with visualization.
3. Role of discussion in a mathematics teaching/learning environment.

The discussion around the utility of ICT has been enriched by research studies in different contexts and with different kinds of technology. Therefore, first of all, we realized the importance of specifying for whom or what they can be useful: For the teacher? For the student? For the teaching-learning processes?

We explored the potentialities of a teaching approach blending an e-learning platform, online tasks with a well-argued solution, online feedback and discussion meetings on the online work (Albano & Pierri). We investigated the use of the WebQuests method to lead students in their researches on the web and the use of online collaborative documents to process the collected information within an extra-curricular course on fractals (Alfieri). The analysis of an experience of graphing motion with WiiGraph for fostering students’ reasoning on spatio-temporal mathematical relationships (Ferrara & Ferrari) contributed with interesting issues on the students’ engagement and learning through “playing” mathematical instruments. Three studies on the use of GeoGebra combined with paper and pencil have been presented (López & Guzmán; Serpe & Frassia; Guzmán & Zambramo) and led us to discuss other questions: Is there no new questions introduced because of ICT? Are the pedagogical, didactical and research concerns the same as paper and pencil or the same as physical objects manipulation?

The debates around technology, and in particular the potentiality offered in terms of visualization, raised another important issue: multiple and different representations and their relationship with visualization. We recognized that in mathematics we do not work directly on the object, but we actually work on one (or more) of its possible representations. Does the representation co-construct the mathematical objects with the mathematical learner/thinker? Different contributions in the group allowed us to investigate this question in-depth. Olsher & Hershkowitz proposed to analyze the process of solving a “visual-pattern-problem” starting from the algebraic expression used to solve it. They analyzed how the visual strategies that are behind the algebraic components of the given expression are detected and reconstructed. Discussion on representation and visualization led us to ask a constellation of other questions we have to cope with as mathematicians, as teachers, as teacher educators, and as researchers: Is one of our goals in mathematics education (at all levels) to help students as they make transitions among different representations? Do mathematicians generalize from collections of concrete representations to make conjectures and then to prove these conjectures about the “concrete” specific cases? If different students use different representations, do they have different understandings? We wondered whether visualization is a process or a
product, and how the links among mathematical experiences with representations of the mathematical object(s)-representations in turn affect the experiences. We finally decided to consider two possibilities: (a) visualization as a complex set of tools with which one manages representations; and, (b) visualization as a form of representation at the same level as language, i.e., concrete, contextual, abstract symbolism. Managing these different points of view can be related to the teacher’s objective in terms of how the teacher wants students to visualize, and leads to a non-linear combination of interactions and relationships rather than direct empirical causations.

This debate fostered further discussion about the utility of technology in mathematics education. When we work on/with/through a mathematical object with technology, we have to ask what kind of representations the software is producing. Different issues have been addressed:

- Although we usually assume the student/mathematician is creating a representation, might it be more useful to see the representation as enunciating simultaneously the student/mathematician and the mathematics?
- To say that the technology represents a concept or relationship that we do not have access to is nonsense, because how would we represent something that we did not beforehand have access to?

Discussion around representation led us to assume the importance of the transition between mathematics formulas and procedures and common language. Orozco Vaca and Zazueta proposed a teaching strategy based on writing as a metacognitive tool, by providing students with a grid of questions for leading their problem solving activity. Thus, a strategy can be, for instance, that the students describe what they do in the exercise, while they are doing it. For the teacher, this is extremely important; but it is also often difficult to understand the meaning of what a student is saying. There can be many meanings when we talk, and we ourselves are often not even certain about what we are intending to mean until later, when we have looked back upon what we have said. Moreover, we want students to understand what the other students are saying, so they can discuss with each other; once we consider the complexity of co-constructed meanings in a constantly shifting and mediated environment, we need also to reconsider what we are even studying in the first place.

This issue led us to explore the role of discussion in a mathematics teaching/learning environment: What about students as a resource for their peers and for the teacher? But in a class there are several students; this multiplicity introduces complexities and problems. Kotarinou et al. presented a remarkable example of a year-long, interdisciplinary, didactical intervention giving to students the possibility of expanding the boundaries of official school mathematics discourse while actively involved in (re)negotiating their own learning processes.

Regarding the interaction between a teacher and students, we focused also on the specific role of the teacher in guiding the construction of knowledge and practices in the classroom. We particularly analyzed how the semiotic resources, and especially the gestures, can contribute to the mathematical discourse (Panero) as well as the physical discourse (Salinas & Guzmán) around the theme of variations. Do the gestures help at a theoretical level, but also at a practical one in the construction of techniques? And, to come back on the second point of discussion, are the gestures part of the visualization processes, or are they representations?

On one point we all agreed: the mathematical object emerges from all of the semiotic activity, and all the used semiotic resources and representations (mediated by the technology or not) contribute to disclose it.
From rote procedures to meaningful ones: a blended semiotic approach

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Abstract: This paper proposed a blended learning approach for supporting undergraduate students to overcome rote learning practices and for helping them to reflect on the rules and relations of the mathematics they are doing. To this aim a learning activity has been set up: it consists in time-restricted online tasks (to solve problems and write argumentation to justify what done), online individual feedbacks from teacher/tutor, weekly face-to-face meeting to discuss the online work. Among the added values of such blended setting we highlight the chance of achieving practices focusing on crucial topic such as argumentation, which are not allowed in traditional lectures based on one-to-hundred/s communication. With respect to written argumentation, here we have investigated students’ protocols, according to the frame of the functional linguistics.

Résumé: Cet article propose une approche d'apprentissage mixte dont l'objectif est de soutenir les étudiants de premier cycle dans les pratiques de l'apprentissage par cœur ou rote learning et les aider à réfléchir sur les règles, les relations et les domaines mathématiques abordés. Dans ce but, une activité d'apprentissage a été mis en place, qui comprend: des tâches en ligne en un temps limité (résoudre des problèmes et écrire l'argumentation expliquant les solutions), des évaluations individuelles en ligne par l’enseignant / tuteur, des rencontres hebdomadaires en face-à-face pour discuter du travail effectué en ligne. Parmi les valeurs ajoutées de cette configuration mixte, il est mis en évidence la possibilité de parvenir à une pratique centrée sur des aspects cruciaux tels que l’argumentation, pas fournie dans les classes traditionnelles basée sur une communication un à cent. En ce qui concerne l’argumentation écrite, cet article étudie les protocoles des étudiants, selon le cadre de la linguistique fonctionnelle.

Introduction

In this paper, we discuss a learning activity in blended setting implemented as support in an undergraduate linear algebra course for freshmen students in engineering. Experience shows that students in such faculty where mathematics is, with no doubt, a “service” domain, sometimes confuse such “service” domain with “procedural” mathematics vs “conceptual” mathematics (Hiebert&Lefevre, 1986). This means that many students do not care for understanding, but just for “operating” in some ways by rote learning, without being aware of what they are doing and why they do that way. Moreover, Italian setting of undergraduate courses, which consists in plenary lectures to one hundreds of students or more, just allows the teacher to state theoretical results (i.e. definitions, theorems and proofs) and to show some examples of solution techniques of typical problems. In this academic setting, deeper and actual conceptual understanding, required to pass the exam, is committed to the personal responsibility of each student. E-learning platforms allow to support and engage students out lecture time by meaningful online activities, designed and implemented according to the outcomes of research in mathematics education (Albano&Ferrari, 2008, 2013, Albano, 2011). The activity we refer to here consists in online time-restricted tasks and post face-to-face discussions among students and teacher/tutor. The online tasks require the students to solve some of previous final exam problems and to give suitable argumentations to support their solving procedures. Individual online feedbacks are provided on the students’ products and a face-to-face meeting are followed to discuss on that.

The benefits of writing-to-learn are widely recognized (Morgan, 1998). In particular, we assume that forcing written argumentations through our online task can raise the student’s reflections on the
rules and relations of the mathematics he/she is doing. This can promote a shift from procedural to conceptual knowledge, fostering understanding how and why the procedures work and connecting procedures with their conceptual underpinnings (theorems, definitions and their characterizations, properties of mathematics objects, and so on). Moreover, written language can be used as a semiotic mean of objectification that is the student’s intentional use of language in a social process of meaning production in order to achieve awareness of what he/she is doing (Radford, 2002).

An argumentation is first a piece of written (verbal, figural, symbolic) text and is strongly affected by both the student’s competence in language and in the specific mathematical contents of the task (Ferrari, 2004). Here we do not adopt a specific theory of argumentation, but regard argumentation as a piece of text, written for more or less explicit purposes. We will investigate students’ arguments in the frame of Halliday’s Systemic Functional Linguistics (Halliday, 1985; O’Halloran, 2005). The importance of focusing on the linguistics functions is due to two opposite aims of the communication in mathematics education: on one hand, the need for effective representation of mathematical concepts and procedures and, on the other hand, the need for effective communication with other people. The context affects communication: the context of situation, which is space, time, the participants as individuals, the context of culture, concerning beliefs and knowing related to the participants and to the topics of the communication. Circumstances of communication can suggest the individuals to use different registers (intended as a linguistic variety): colloquial registers, mainly used in spoken everyday communication, and literate registers, mainly used in written communication among educated people. More details about such themes in mathematics education can be found in Ferrari (2015).

**Methodology**

Universities are aware that, due to the increasing use of the Internet by people, introduce the technology in education may help reduce the gap between students’ out-of-school and educational experience. Sometimes, the use of technology in universities has been limited to considering an online platform as a repository of resources (course notes, exam results, technical information...). The eLearning modalities may offer the possibility to plan the personalization of individual paths as well as activities involving collaboration among students but also between student and teacher. In the context of Linear Algebra course, we created and followed up a blended course, which was piloted with almost 70 first year engineering students in the 2014/2015 academic year.

Linear Algebra course is composed by 60 hours, (5 per week), split into 3-hours of theoretical lecture and 2-hours of exercises. Examination consists in two parts: first, a written is required to be passed in order to access to an oral examination (discussion), in which the student is required to master definitions and theorems (including understanding of the proofs). The written examination, to be carried out over three hours, is composed by five or six exercises related to the topics illustrated during the course asking to provide for each of them appropriated explanations. The final mark depends on both the written and oral examination.

Often, students in faculties such as engineering where mathematics is, with no doubt, a “service” domain, confuse such “service” domain with “operational” mathematics. This means that they do not care of understanding, but just of “operating” in some ways, without being aware of what they are doing and why they do that way. Our purpose is to try to overcome these modes of procedure and refer students to justify what they do in the resolution of the exercises. Generally, they deliver a set of counts without any justification, leaving the reader the task of understanding what they are doing, why and how those operations are linked to the question of the exercise/problem.

The students were enrolled in the blended course from the beginning, part of the first face-to-face lesson was dedicated to explaining to them how to use the platform, and what activities were available on it. The learning resources that they can delivery through the platform include theoretical materials for deepening topics from a theoretical point of view (including videos reproducing explanations of theorems and their proofs) and practical point of view (worked-out
exercises and problems), and self-assessment quizzes (close-ended questions). They also had access to a private email service where they could exchange comments and questions about the activities among students or with the teacher.

Besides these, we have exploited the open-ended questions (tasks – “compito a casa”), suitably temporized, in order to set the online tasks previously described. The students have been split into two groups, which have worked on tasks of the same type during two different periods of the term. We have assigned the participants of the two groups distributing them homogeneously according to the entry test mark.

Each three days a week, we chose a problem among the ones of the previous written examinations and related to the topic just seen at the lectures. It has been assigned to the students as web tasks (using the platform) by using the functionality “compito a casa”. The temporization of the activity induces students to carry out the assigned task in a fixed set time and at the same time, give them the possibility to work wherever they want and at any moment of the day. An example of the assigned web tasks is the following:

Solve the question 2) c) of the exam problems of 14 January 2014, and justify little by little the procedures you apply in the solution of the problem as you were explaining to a friend that cannot do this exercise and you wanted to make him clear how and why it is solved in the way you're doing.

We have chosen previous written examinations to motivate them, as students’ interest is closely related to passing the written examination.

The students were required to make them in the three days. After that, we checked the students’ products and sent them feedbacks with the evidence of the errors together with some sketch of explanations of the errors. We don’t say them exactly which errors they made but we provide them with our feedback on their product consisting in some specific questions to provoke the student’s reflections on his/her own errors.

After they received our feedback, we invite them to come in our room during “reception time” (at least 2 hours per week). In this “informal session” the single student can express his/her own difficulties and/or we can address them specifically, and in individually learning path. This latter depends on the interactions between us and the students and it is guided by the difficulties encountered during interaction which are made clear also by means of punctual questions we pose to the students depending on what we observe or we supposed to be the trouble at stake.

**Sample of blended interactions**

In the following we will show samples of the interaction online between the students and the teacher and its relation with the face-to-face one

Let us see a protocol containing the computation of the determinant of a matrix, together with the student’s explanations (through the application of Laplace theorem).
Teacher’s comment: Is the Laplace theorem applied to an element, for example $a_{1,3}$? In the Laplace development that you consider, are you using a row?

Table 1. The Laplace’s theorem application.

If you look exclusively to the computations, it seems that everything is fine. But looking at the writings (red box), we have “I apply the Laplace theorem to the element $a_{1,3}$”. At the same time, the writings in the green box refer to the correct statement of the Laplace theorem with respect to a row. Face-to-face discussion has made evident that this student, as others, has learnt by rote to use elementary operations on the rows so that to have a column with only one non-zero element. Then he/she knew that applying Laplace theorem to that column consists in considering the only product of that element and its algebraic complement. It came out that he/she was not able to apply Laplace theorem to a column with more than one non-zero element, he/she always considers just one element in the chosen column: he/she did not understand Laplace theorem indeed! Analogously, he/she learnt by rote what he/she wrote in the green box, without being aware of its procedural meaning.

This is an example of the usefulness of both written and spoken texts to detect and fix some fundamental bugs in the students’ knowledge.

In the following, we show the computations of the algebraic complements of elements in a matrix.

Table 2. Computations of the algebraic complements
The student’s definition (green box) misses “multiply by \((-1)^{\frac{3}{3}}\), but he/she computes correctly (red box). When the student came to the face-to-face meeting, he/she could not yet see the difference between the computation performed and the definition s/he wrote. Then the teacher proposed to the student to read the his/her definition while s/he (the teacher) would act as a performer. As the student expresses words, the teacher converted them into operations. It was at this point that the student realized the incongruence between what he/she has in mind (corresponding to what he/she did correctly) and what he/she wrote. Thus, he/she grasped the difference between the object “algebraic complement” and the process of its computation and the link between them.

Let us consider the request of the inverse image of a vector, given a homomorphism. Here some students’ writings and teacher’s comments:

**Teacher’s comment:** Which is the difference between \(f^{-1}\) and \(f^{-1}(-1,-3,-1)\)?

### Table 3. Computations of the algebraic complements

The teacher would ask the student focus on his/her use of the symbol \(f^{-1}\) he/she used referred to a set. During the face-to-face discussion, the teacher can investigate if in such use the student implies \(f^{-1}(-1,-3,-1)\) or if it is not clear to him/her the difference between the inverse function and the inverse image of a vector. Some students have troubles in distinguishing conceptually a function from its value in a point. The same occurs in case of inverse function and inverse image of a point. Thus, they learn some procedures, even correct, without being aware of what they are doing, as well shown by the following protocol.

**Teacher’s comment:** Which is the difference between “inverse image of \(f(x,y,z)\)” and \(f^{-1}\)? What do you mean as \(f^{-1}(-1,-3,-1)\)? Is it an element of the domain or of the codomain of the function? And \(f(x,y,z)\)? What means the equality in the red box? How do you obtain the linear system from the equality you wrote (formula)? What happens to “\(f^{-1}\)”?

### Table 4. Inverse function and inverse image

The student says by words and in symbols (red box) that in order to compute the inverse image of \((-1,-3,-1)\), it is needed to equal \(f^{-1}(-1,-3,-1)\) and \(f(x,y,z)\). Then, he/she says that this equality is made concrete by a linear system (green box) corresponding to \((-1,-3,-1)=f(x,y,z)\). The student does not care of the symbol \(f^{-1}\) previously written in the red box! From oral discussion, it came out that such symbol was void of meaning for him/her!

A further critical issue in general concerns the relation between the posed question and the outcomes of the performed process. Let us consider, for instance, the following conclusion given by a student:

**Teacher’s comment:** Which is the connection between the founded solution and the exercise’s request? What happens for \(h=0,-1\)?

### Table 5. Relation between the posed question and the computations outcomes

Some students make some computations but, at the end of the process, they are not able to come back to the question. In this case, the student writes “the solution” of the system, but there is no more reference to the required inverse image. Moreover, the question concerns all the real values of
the parameter $h$, but the student misses the cases where the system has not a single unique solution.

**Written argumentations’ analysis**

In this section, we want to analyse the written argumentations of the students, under the lens of multisemioticity and multivariety and their use in the context of the task.

First, we can say that the students use both symbolic and verbal language to perform the task, as well as they use both colloquial and literate registers. We have focused our attention to their intertwined use with respect to the specific requirement of “explaining what they do”.

From the analysis of the protocols, we can outline mainly three type of verbal argumentations:

- Theoretical recalls, such as definitions, with incoherent operational procedures;
- Theoretical recalls, such as characterizations or properties, which give reasons of the performed procedures;
- Description of an algorithm, such as a verbal sequence of steps for solving the problem and performing the exercise.

We report some protocols, which seem to be meaningful and representative, and how we analyzed them, using the above described categories.

![Fig. 1: Protocol PE1.D17](image1)

**PE1.D17**

A basis $B=\{u_1,\ldots,u_n\}$ of a vector space $V$ is a set of linearly independent vectors, that is $h_1u_1+\ldots+h_nu_n=0 \iff h_1=\ldots=h_n=0$ and it is a set of generators for the space $V$, that is $\forall v \in V \exists t_1,\ldots,t_n \in \mathbb{K}: v = t_1u_1+\ldots+t_nu_n$. Being $V$ by definition, a set of generators, in order to calculate its basis we must only verify that its vectors are linearly independent. To verify this, we build a matrix $A_v$ whose rows are the vectors $u_1,\ldots,u_n$ generators of $V$. We calculate the rank of matrix by row-echelon reducing.

In Fig. 1, we can find theoretical recalls that are not linked to operational procedures: the student refers to the definition of linear independence but does not use it for performing the exercise; he/she uses the characterization in terms of rank of a matrix, and in particular computes such rank by means of the row-echelon reduction.

![Fig. 2: Protocol PE1.D06](image2)
In order to compute a Cartesian representation we must impose that a generic vector \((x,y,z,t)\) \(\in\mathbb{R}^4\) belongs to \(V\). This means that it is linear dependent, which is equivalent to say that posed as last row in the matrix with the bases of \(V\) it will become zero. [...] Now that the row-echelon reduction is completed the last two components of the last row remain to be posed to 0, contemporarily and then with a system.

In Fig. 2 we can see how the student justifies how he/she constructs a certain matrix and why he/she reduce it in row-echelon form and pose to zero the last row in order to obtain a homogeneous linear system. Each step is suitably linked to the theoretical underpinnings, such as the belonging to a vector space, the linear dependence, its characterization and so on.

![Fig. 2: Protocol PE1.D18](image)

Fig. 3: Protocol PE1.D18

Given a vector subspace, for calculating a Cartesian representation we must follow some steps: 1) To calculate a basis \(B\) of \(V\); 2) To build a matrix whose rows are the vectors of \(B\) and in the last row we put the generic vector \((x,y,z,t)\); 3) To row-echelon reduce the matrix \(M_w\); 4) to impose that the last row of reduced matrix (corresponding to vectors \((x_1,\ldots,x_n)\)) is zero.

In Fig. 3 the student simply gives an ordered verbal list of procedures to be performed in order to calculate a Cartesian representation. No comments are given to justify why to do in such a way in order to solve the posed question. Sometimes, the analysis of the verbal part has made evident some difficulties, such as confusion among various known results (Fig. 4) or actual gaps in knowing (Fig. 5). Let us see in more details.

![Fig. 4: Protocol PE1.D19](image)

Fig. 4: Protocol PE1.D19

We determine a basis of \(V\) known a set of vectors generators extracting a minimal set of linearly independent vectors. The basis \(V\) is defined as a set of linear independent vectors. The linear independence is provided by the rank that is the maximum number of non-zero rows or columns.
In Fig. 4 we can see that the student seems to recall two facts: on one hand, he/she links the linear independence with the rank of a matrix; on the other hand, he/she gives as characterization of the rank the one referred to a row-echelon matrix, not to a generic matrix.

![Fig. 4: Protocol PE1.D22](image)

PE1.D22 To calculate a Cartesian representation means to find a homogeneous linear system (the column of the known terms is equal to zero) whose solutions are the vectors of any basis of the subspace.

Fig. 5 shows that the student considers as solutions of the Cartesian representation of a vector space only vectors belonging to a basis.

Verbal argumentations accompany operational procedures, which are added with “argumentation” in symbolic language. Comparing verbal and symbolic statements, we can distinguish various cases:

- Discrepancy between what the student verbally affirms to do and what he/she actually says in symbols (Fig. 6);
- Confirmation in symbolic language of errors made evident by verbal argumentation (Fig. 7);
- Correct use of symbols and symbolic argumentation, but accompanying by short verbal argumentation, which can highlight some misconceptions to be verified in face-to-face meetings (Fig. 8).

![Fig. 5: Protocol PE1.D22](image)

PE1.B20 We build a minor of order 2:

\[
\begin{vmatrix}
1 & \frac{1}{2} & \frac{1}{3} \\
2 & \frac{1}{2} & \frac{1}{3} \\
3 & \frac{1}{2} & \frac{1}{3}
\end{vmatrix} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\]

After having found a non-zero minor, we build some 3-minors that including rows and columns of the non-zero 2-minor:

\[
\begin{vmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
2 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
3 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4}
\end{vmatrix} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\]

As we can see, the student declares to do something but he/she does something else, in fact, the latter 3-minor considered does not satisfy the condition concerning the inclusion stated before.
To calculate a Cartesian representation of the subspace \( V \) of \( \mathbb{R}^4 \)

We start by building a matrix whose vectors are the subspace \( V \)

\[
\begin{pmatrix}
1 & -2 & 3 & 1 \\
-1 & 0 & 0 & 2 \\
-1 & 4 & -6 & 0
\end{pmatrix}
\]

Now we calculate the rank of this matrix and so we reduce in row-echelon form through the Gauss method.

In the case of Fig. 7, we first observe that the student rewrites the given question and, in doing this, substitutes the original symbols \(<_>\) by brackets. Looking just at the symbols, we can doubt on the awareness of semantic of different kind of brackets, but looking at the verbal argumentation, we can state that he/she actually thinks that only the vectors listed constitute the vector space.

Fig. 7: Protocol PE1.D23

**PE1.D23**

To calculate a Cartesian representation of the subspace \( V \) of \( \mathbb{R}^4 \)

We start by building a matrix whose vectors are the subspace \( V \)

\[
\begin{pmatrix}
1 & -2 & 3 & 1 \\
-1 & 0 & 0 & 2 \\
-1 & 4 & -6 & 0
\end{pmatrix}
\]

Now we calculate the rank of this matrix and so we reduce in row-echelon form through the Gauss method.

In the case of Fig. 7, we first observe that the student rewrites the given question and, in doing this, substitutes the original symbols \(<_>\) by brackets. Looking just at the symbols, we can doubt on the awareness of semantic of different kind of brackets, but looking at the verbal argumentation, we can state that he/she actually thinks that only the vectors listed constitute the vector space.

Fig. 8: Protocol PE1.C26

Fig. 8 shows a student’ work where symbolic language is mainly used, sometimes referring to the underlying process (see the two cross lines in the computation of the determinant). The calculations made are correct and so the symbolic language seems, but the student writes “I use Laplace on the place \( a_{22} \)”. This suggests the teacher to further investigation on the correct understanding of the Laplace theorem in face-to-face meeting (see the above discussion concerning Table 1).

Some pieces of the performed tasks are equipped with only symbolic argumentations. These can be correctly used and referred to operational explanations, such as in Fig. 9.
Some cases show a correct use without complete awareness of the semantic underpinning some sequence of symbols and interrelations. This is the case of students who are not able to use what they have found symbolically, even if correctly discussed (Fig. 10).

Some other cases seems to use correct use of symbolic language, such as in Fig. 11 if you look at the last row. Actually, verbal description highlights an erroneous meaning, different from the conventional one, of the symbol “\( \min\{m,n\} \)”, which denotes “the minor”. According to the student, the last symbolic row has no sense!
The rank of a matrix $A$ in $M_{m,n}(K)$ is the maximum order of a non-zero minor of $A$. The minor is in turn the determinant of a submatrix of $A$, that is a matrix of $p$ row indices and $p$ column indices. The minor will be denoted by $\min\{m,n\}$, then the rank:

$$rk(A) \leq \min\{m,n\}$$

Finally, we note that, in some other cases, the symbolic language is used in a colloquial register: from the strict grammatical view of point, we have incorrect writings, but if we look at them under the Radford objectification, we can found that they refer to and reproduce a process in mind. In Fig. 12, the student writes the matrix and rewrites the first two columns, then traces some lines with different colors (probably indicating the difference between the sign of the product to take into account), such lines have also a versus (probably indicating the order the student uses for the product).

![Fig. 12: Protocol PE1.A16](image)

**Conclusions**

In this paper, we proposed a blended learning approach for trying to overcome the operational modes that the students adopt related to mathematics for helping them to achieve some awareness what they are doing and why they do that way.

E-learning environments have a great potential in the context of higher education. They provided opportunities to use multiple representation systems, to create activities of construction and treatment of semiotic representation, to develop several new forms of communication (regardless of time and distance) and to foster self and peer assessment processes. We exploited the potential of e-learning to pursue an important educational goals: the devolution of awareness into the learner in his/her interaction with mathematics. To this aim we used a specific open-ended questions of the learning platform (tasks – “compito a casa”), that requires the students to solve samples of exam problems, with the additional request of written argumentation to justify what they do. The added value to use this functionality is its temporization that have a double purpose: on one hand it induces students to carry out the assigned task in a fixed set time, keeping pace with the face-to-face lectures, and, at the same time, give them the possibility to work wherever they want and at any moment of the day. Experience shows that nowadays freshmen students need to be supported in individual extra-lectures work and such kind of time-restricted online activities can be a useful tool to this aim. The blended setting, online tasks and teacher/tutor’s feedback on the students’ products, and post face-to-face deeper discussions during extra weekly sessions has allowed to make realistic a template of learning activity which is not realizable for large group of students in traditional setting. Moreover, it allows keeping the focus on a crucial topic for meaningful understanding, such as argumentation, which is not foreseen in communication one-to-hundred/s, that is the case of the traditional courses.

In brief, from the analysis of the protocols we have found the use of verbal and symbolic languages, which can be correct or incorrect. When both on hand, referred to a single process, they can be coherent or not each other. Moreover, both of them can have been used in a colloquial or literate register. What we have considered important for learning is not the register used, but the coherence of the text showing a certain level of understanding. From the analysis, it seems us that in some students, who seems to use literate registers, simply rewrite definitions or theorems from books, but
the theoretical references they do are not appropriate to justify the process of solution made. We can argue that maybe the aim of their communication here is not to answer to the task requirement (that is to justify what they do) but to make the teacher convinced that they “know” the theory underpinning, where “to know” is interpreted as reproduction of books.

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L-system Fractals: an educational approach by new technologies

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Abstract: Living in a technological era it is essential that teachers and pupils have regular access to technologies. They support and advance mathematical sense making, reasoning, problem solving and communication. L-system fractals is an interesting topic to show the use of new technologies in an educational approach.

Résumé: Vivre dans une ère technologique, il est essentiel que les enseignants et élèves ont régulièrement accès aux technologies. Ils favorisent et promouvoir la création d'un sens mathématique du terme, raisonnement, résolution de problèmes et de la communication. Fractales de L-System est un sujet intéressant pour illustrer l'utilisation des nouvelles technologies dans l'approche pédagogique.

Introduction

Digital technologies impact more and more all aspects of the personal, social and professional people’s life, pupils and teachers included. New technologies are spreading at an incredible speed. Social networks numbers exponentially increase and these tools become more and more influential; they deeply impact our communication, as well as our relationships with authorities and institutions. All these changes impact learning processes and teacher’s work at school and outside, the relationships between teachers and pupils. They create new challenges and responsibilities. Artigue M. (2013) suggests interesting and shared questions:
- What is the advantage in mathematics education of the incredible amount of accessible information and resources?
- What is the advantage of the change of social communication and Internet facilities for creating and supporting communities of mathematics teachers and pupils?
- How to use the new affordances to give pupils more autonomy, to develop mathematical topics?
- How to develop a mathematics education more open on the outside world and still faithful to the epistemology of the discipline?
- How to use these new opportunities to better address the specific needs our pupils may have?

The geometrical topic about L-system fractals is an example of an educational activity in which the use of new technologies is essential firstly to manipulate geometric objects, to create new conjecture and demonstrate new hypotheses; secondly it is necessary to communicate outside the classroom and to share experiences.

L-System Fractals.

Fractals are a “good” geometrical topic for the application of new technologies, because after the study of affinity transformations by pencil and sheet, it is impossible to plot a fractal imagine or make conjecture to think new fractals, without using a software.
**What a fractal is**

Benoit Mandelbrot is the father of fractal theory. In the past fractals were regarded as mathematical monsters, because of their unusual properties. In 1975, Mandelbrot called them fractals, from the Latin word *fractus*, meaning *fraction*. Fractal Geometry is around us, into the clouds, rivers, nature (Mandelbrot B.B. 1998):

“What Euclidean geometry is unable to describe the complexity of nature, (because) by observing the nature, we are able to see that clouds are not spheres, mountains are not cones, coastlines are not circles but they are complex geometrically objects.”

Fractals are geometrical figures, that are characterized by unlimited repetition of the same shapes of a more lowered sequence. Fractal’s properties are: self-similarity, scaling laws and not integer dimension. There are different types of fractal: IFS (iterated function systems) and L-system.

**What an L-systems fractal is**

L-systems fractals have been conceived as a mathematical theory of plants development. After the incorporation of geometric features, plant models, expressed using L-systems, become detailed enough to allow the use of computer graphics for realistic visualization of plant structures and their development processes. Aritid Lindenmayer (1925-1989) was an Hungarian biologist who developed a formal languages called Lindenmayer Systems or L-systems to generate fractals. They were introduced as a theoretical framework to study the development of a simple multicellular organisms and subsequently applied to investigate higher plants and plant’s organs. The central concept of L-systems is rewriting. In general, rewriting is a technique to define complex objects by successively replacing parts of a simple initial object using a set of rewriting rules or productions.

In 1968, Aristid Lindenmayer, introduced a type of string-rewriting mechanism, subsequently termed L-systems. In L-systems we can define a string as an ordered triplet $G = (V, \omega, P)$ in which:

1. $V$ is a finite set of symbols called alphabet,
2. $\omega \in V^+$ is a non-empty word called axiom ($V^+$ is the set of all non-empty words over $V$),
3. $P$ is a finite set of production: $P \subset V \times V^*$, $V^*$ is the set of all words over $V$.

$P$ defines how the variables can be replaced with combinations of constants and other variables. A production $(a, \omega) \in P$ is written as $a \rightarrow \omega$. The letter $a$ and the word $\omega$ are called the predecessor and the successor of this production, respectively. It is assumed that for any letter $a \in V$, there is at least one word $\omega \in V^*$ such that $a \rightarrow \omega$. If no production is explicitly specified for a given predecessor $a \in V$, the identity production $a \rightarrow a$ is assumed to belong to the set of productions $P$ (Lindenmayer, Prusinkiewicz, 1990).

The formal language is not enough for building images or for creating geometric patterns. At this point it is essential working by multimedia tools, choosing a software to convert some numerical and symbolic codes in images.

**L-system Fractals: what technology?**

One of the geometric systems that computer graphics use for the L-system’s generation is called Turtle Geometry. The basic idea of turtle interpretation is given below.

A state of the turtle is defined as a triplet $(x,y,\alpha)$ where the Cartesian coordinates $(x, y)$ represent the turtle’s *position*, and the angle $\alpha$, called the *heading*, is the direction in which the turtle is facing. Given the step size $d$ and the angle increment $\delta$, the turtle can respond to commands represented by the following symbols:
1. \textbf{F} (it moves forward a step of length \( d \) the state changes to \((x' = x + d \cos \alpha, y' = y + d \sin \alpha, \alpha)\). A line segment between points \((x, y)\) and \((x', y')\) is drawn;

2. \textbf{f} (it moves forward a step of length \( d \) without drawing a line);

3. \textbf{+} (it turns left by angle \( \delta \) the state changes to \((x, y, \alpha + \delta)\));

4. \textbf{–} (it turn right by angle \( \delta \), the state changes to \((x, y, \alpha - \delta)\)).

Given a string \( \nu \), the initial state of the turtle \((x_0, y_0, \alpha_0)\) and fixed parameters \( d \) and \( \delta \), the \textit{turtle interpretation} of \( \nu \) is the figure drawn by the turtle in response to the string \( \nu \). Specifically, this method can be applied to interpret strings which are generated by L-systems (Prusinkiewicz P., 1999).

For example, in the following three approximations of the Koch snowflake. These figures are obtained by interpreting strings generated by the following L-system:

1. \textbf{axiom} \( \omega \): \textbf{f}++\textbf{f}++\textbf{f} start angle: 90°, turn angle 60° (it corresponds to the initiator or start figure of the fractal) (Fig. 1a).

2. \textbf{production} \( p \): \textbf{f}=\textbf{f}–\textbf{f}++\textbf{f}–\textbf{f} (it corresponds to the generator of the fractal) (Fig. 1b).

3. Koch snowflake at sixth generation (Fig. 1c).

The figures has been made by Fractal Grower. It is a Java software for Growing Lindenmayer Substitution Fractals (L-systems) created by Joel Castellanos, Department of Computer Science, University of New Mexico (http://www.cs.unm.edu/~joel/PaperFoldingFractal/paper.html). We use it just for educational purpose. The software is easy to use and above all, its versatility fosters the exploration of fractals and their properties. For example it is possible to change the start figure but the final fractal does not change. As you can see in the below figures (Fig. 2a, 2b, 2c) the start figure is an hexagon, not a triangle, the final fractal is that shown in Fig. 1c:

Fractal Grower, or any other similar software, allows to manipulate geometric objects graphically, for example translating, turning or reducing them, it helps to understand how the geometric transformations work or why the numerical codes change in dynamic figures, so that pupils can increase their curiosity, their imagination, their geometrical knowledge.
Below there are some pictures and codes of L-system fractals generated by Fractal Grower (Fig. 3, Fig. 4):

Axiom : f-f-f-f ; Production: f=ff-f-f-f-f-f+f at the sixth generation

Axiom α:af; Production: f= a![-f][++f]![--f][++]f at sixth generation.
**L-system Fractals: what educational approach?**

In our school a team of teachers (me included) have been participating since eight years to the project Mathematics & Reality, a national project managed by Prof. Primo Brandi and Prof. Anna Salvadori (University of Perugia). The project deals with making proposals to develop unsuspected and educational relationships between mathematics and the real world. Mathematics & Reality (M&R) is an innovative project of educational mathematics which central point is a dynamic interaction between the real world and the mathematical world, which is achieved by the language of mathematical models. The mathematical model of a real-world event is a process of rationalization and abstraction that allows to analyze the problem, to describe it as an object and create a simulation using the universal symbolic language of mathematics. Since 2006 we have been proposing many different mathematical laboratories attended by around 1,000 pupils. The M&R project is at the moment one of the most important extra-time activity of the school.

In M&R project, Fractal Geometry is the topic of my educational activity. It has three different phases: Fractal Class, Fractal Webquests, Fractal Cloud learning.

**First step: Fractals Class**

Fractal Class is an extra-time course about fractals. It takes 16 hours (one two-hour-lesson per week). Around 50 pupils per year on average attended this course (totally 300 pupils since 2006). During the first lesson goals and contents are proposed to the pupils.

Goals:
- modelling the world around us using affinity transformations;
- building the most famous fractals (Sierpinski’s Triangle, Koch’s snowflake, etc.);
- plotting fractals by software tools;
- making conjectures and simulations by using free software;
- mixing traditional teaching and new technologies.

Contents:
- Geometric transformations and matrices: composition of geometric transformations; inverse of a geometric transformation, affinity transformations: rotation, contraction, translations.
- Iterated Function System Fractal by genetic code: evolution of an iterative process of figures, attractors and fixed figures.

During this phase, the following educational approaches are applied:
- **interactive lesson:** between the teacher and pupils. The teacher proposes the open questions, and the class must answer or fill the missing parts of a theory by searching on Internet.
- **Cooperative learning:** it allows horizontal communication improving the cooperation among pupils, encouraging their participation and their involvement without inhibition, promoting their skills.

**Second step: Fractals WebQuests**

Lessons and school courses are not the only source of information for pupils. Living in the digital era means to exploit a flux of information, this is an advantage for making student more autonomous in his knowledge process. Before the Fractal Course ended, the teacher proposes some research topics about Fractal Geometry to whole class. It is a suggestion for a more deep study about fractal geometry after the end of the course and, at the same time, it is a new step of the activity, to which a few pupils joined spontaneously. The main aim of this additional activity in M&R project is aimed to the participation at M&R National Congress, held every year at the University of Perugia. During the conference pupils make a presentation about their research work and compete for the Best Presentation in Mathematics. Along these years experienced with pupils
many works have been produced such as:

1) “To make a tree… it takes an L-fractal-system” by Jessica Rubino, Chiara Punturiero, Ersilia Paparazzo.

2) “Fractal snowfall in Catanzaro” by Lucia Mazza, Stefano Caglioti, Sara Virgilio, Emmanuele Benedetto.

Both works have been presented by pupils at National Congress of Maths & Reality in Perugia in 2012 and 2014. The first topic was also presented in Gotheborg in Euromath (European Mathematical Congress for Students) in 2013. Both works represent an example of mixing affinity transformations, matrices and L-system fractals by using different free digital tools.

In this phase the webquests process is applied. This educational approach has the main goal to discover additional information on a particular topic and to provide some product using the gathered information. This teaching methodology was created by Bernie Dodge at University of San Diego (USA) in 1995 and today is recognized at international level. Dogde (2001) defined it as:

an inquiry-oriented activity in which most or all of the information used by learners is drawn from the Web. WebQuests are designed to use learners’ time well, to focus on using information rather than looking for it, and to support learners’ thinking at the levels of analysis, synthesis and evaluation (p. 6).

Internet is a chaotic space, conveying a myriad of information, endless realities, news, experiences; the cyberspace lives on exchanges, enriching overall product with multiple contribution. This chaotic knowledge must be deciphered, selected, structured, otherwise your search could seem sterile and superficial. Webquests permits to avoid getting lost into the network and to economize the time. To achieve that efficiency and clarity of purpose, WebQuests should contain at least the following parts (Dogde B. 2001):

1. An introduction that sets the stage and provides some background information;
2. A task that is doable and interesting;
3. A set of information sources needed to complete the task;
4. A description of the process the learners should go through in accomplishing the task;
5. A conclusion that brings closure to the quest.

A few lessons are needed to gather data and elaborate the structure of the research topic. During this step topic is assigned, the structure of the work is decided and the free software is chosen together by pupils and teacher. In the framework of both works there are four parts:

1. fractals in general (what a fractal is, the most important properties about fractals);
2. mathematical contents (affinity transformations and matrices);
3. L-system fractals (theoretical definitions about L-system fractals and their codes);
4. Logo (or Turtle)-language and free software for plotting imagines of fractals. Among them we used Fractal Grower, how it said before.

Thank to this activity, pupils acquire new skills. They learn to look for information on the web, to select the most relevant parts and to apply the most suitable among them. It is therefore a working strategy strongly characterized by cooperative work and problem-solving.

**Third step: Fractals Cloud-Learning**

After fractal course and fractal webquests, teacher works with groups of pupils to improve research and create multimedia presentations for M&R congress. When the course ended, pupils and teacher can not interact one each other at school, so technology offered new opportunities in improving teaching and learning. They enable individuals to customize the working/learning environment...
using a range of tools to meet personal interests and needs. This is the reason why it has been explored the educational potential of ‘cloud computing’. Web 2.0 tools give the choice to interact and cooperate each other in a social media dialogue as creators of user-generated content in a virtual community. Examples of Web 2.0 technologies include social networking sites, blogs, wikis, video-sharing sites. They also have the potential to promote sharing, openness, transparency and collective knowledge construction. During our experience we use Google Drive (as a storage of files) and Google Docs (as virtual learning environment). Inside this space pupils and teacher can exchange and share ideas and information and they can work at the same time – despite not being in the same classroom – in order to achieve the final version of the multimedia presentation. The advantages of using Google Docs are:
1. multiple people can work at the same time on the same document and everyone can see people’s changes as they make them, and every change is saved automatically;
2. everyone can collaborate in real time over chat. If more than one person has the document open, just click to open a group chat. Instant feedback is possible without leaving the document.

In this new learning environment teacher acts more systematically as advisor, guide and supervisor, as well as provider of the frameworks for the learning process of pupils. They have greater responsibility for their own learning in this learning environment, as they look for, find, synthesize and share their knowledge with others. During the cloud-learning phase, pupils and teacher work together and study the theoretical content of L-system fractals, they analyze and simplify theory to include it in the presentation.

Results

The most important goal of this work is to show new opportunities generated by ICT in learning and teaching geometric contents. L-system fractals demonstrate to be a good topic to get this achievement. In fact, Web2.0 tools are fundamental and essential in order to:
1. improve theoretical contents about L-system fractals theory;
2. create new virtual learning and teaching spaces, in which the student is left alone to choose the materials, to organize autonomously theoretical and technological topics.
Pupils are able to make new conjectures and create new fractals initially studying the appropriate geometric transformations and then checking the results with the help of the recommended software.

Results obtained after the activity are:
1. enhancement of mathematics skills (geometric transformations, iterative processes and functions, matrices and determinants);
2. enhancement of the mathematical competence in the sense of the definition given by PISA4:

Mathematical skill is the ability of an individual to identify and understand the role that mathematics plays in the real world, to operate based assessments and to use mathematics and confront it in ways that meet the needs of the life of that individual as citizen exercising a constructive role, committed and based on the reflection.

The educational activity applied in this project fosters the development of the personality and the attitudes of the pupils, supports them during their educational and emotional growing.
Also the teacher has advantages testing this activity:
- discovering new educational virtual environments where to share and cooperate with pupils and colleagues;
- exploring new methods to renew teaching and learning of mathematical topics.

New challenges have yet to be experienced about the educational approach, submitted in this paper, among them: to extend it to several mathematical topics and, even more difficult, to engage the whole class.
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Being Collaborative, Being Rivals: Playing wiigraph in the Mathematics Classroom

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Abstract: This paper describes an activity that involved a 9th grade class in graphing motion with WiiGraph, an interactive software application that uses the two Nintendo Wii remote controls to graphically display the position of users. Potential interactions with WiiGraph allow the students to experience spatio-temporal mathematical relationships through tasks in which they can be collaborating or competing with each other, offering kinaesthetic learning challenges. We analyse the ways in which these challenges affect the doing/doer of mathematics, as well as the mathematics itself, giving rise to new mathematical subjectivities. To do so, we pursue a participationist vision of learning with tool, which joins the view about playing mathematical instruments by Nemirovsky et al. (2013) and that about distributed agency by Rotman (2008) and de Freitas and Sinclair (2014), for which the human learner always reinscribes herself into a mathematics that is changed by its encounter with the technology in use.

Résumé: Cet article décrit une activité proposée à des élèves de 14-15 ans. Elle est basée sur l'analyse des représentations graphiques des mouvements obtenus avec WiiGraph, un logiciel interactif qui utilise les deux télécommandes Nintendo Wii pour afficher graphiquement la position des utilisateurs. Les interactions potentielles avec WiiGraph permettent aux étudiants de vivre des relations mathématiques spatio-temporelles à travers des tâches qui les engagent dans des défis kinesthésiques pour l'apprentissage où ils peuvent collaborer ou être compétition. Nous analysons comment la composante kinesthésique affecte l'activité mathématique ainsi que le sujet impliqué dans cette activité provoquant de nouvelles subjectivités mathématiques. Pour mener cette analyse, nous choisissons une vision participationniste de l'apprentissage avec les instruments, qui joint l'idée de «jouer les instruments mathématiques» proposée par Némirovsky et al. (2013) et celle de «l'action distribuée» par Rotman (2008) et de Freitas et Sinclair (2014), pour lesquels l'apprenant se réinscrit toujours dans des mathématiques qui sont modifiées par sa rencontre avec la technologie utilisée.

Introduction

In their work about playing mathematical instruments, Nemirovsky and colleagues (2013) assume a non dialectic approach to mathematical tool use that questions any dualism between perceptual and conceptual, "between bodily, tool-mediated expression and mental structures or schemes" (p. 376), which would entail an acquisitionist rather than a participationist view of learning. While, in an acquisitionist perspective, learning is conceptualised in terms of mechanisms that students are expected to acquire, pursuing a view of the second kind entails recognising the students' ways of talking, moving and feeling as communicating and learning within the classroom. Nemirovsky et al. propose that mathematical thinking and learning inheres in transformations in lived bodily experience. Lived experience has to be intended as a temporal flow of perceptuo-motor activity, that is, perceptuo-motor activity "is always permeated by expectations, recollections, fantasies, moods, and so on" (p. 378). These aspects are part of the ways in which students speak, do, and feel as well as of their changes over time. The authors are interested in analysing this flow in the context of using what they call a mathematical instrument (which will be discussed in the next section). In the study, they point out that dynamic geometry environments (DGEs) and motion detectors are both families of mathematical instruments that have received much attention from research. For example, they argue, dragging in a DGE can support important integration of motoric and perceptual aspects, being the first given by dragging geometric elements in a diagram by hand and the second by visual feedback on the whole diagram. Motion detectors instead involve similar aspects in how they allow students to explore the modelling of their own body movement by means of real-time graphs.
Thinking about tool use in teaching and learning mathematics, we should be concerned not only with the students' ways of communicating, but also on how the mathematics taught and learned changes in interacting with the tool. As Rotman (2003) claims, "the effect of the computer on mathematics (pure and applied) will be correspondingly far-reaching and radical; that the computer will ultimately reconfigure the mathematical matrix from which it sprang and will do this not only by affecting changes in content and method over a wide mathematical terrain, but more importantly by altering the practice and overall nature, and perhaps the very conception we have of mathematics." (p. 1675). In his more recent book Becoming besides ourselves, Rotman (2008) adds that mathematics is permanently altered by its encounters with new technologies, thus the future of mathematics will see a reduction of an alphanumeric hegemony and, especially, a move toward visual and dynamic mathematical expressions. Drawing on the image of the post-alphabetical world of Rotman, Sinclair (2014) argues that this will imply a new kind of sensory politics at play, since digital technologies have been "steadfast in their acts of dissensus" in relation to the dominant regime of sense-making. The dynamic geometry triangle, for example, carved out "a new dimension for time, thus changing what was taken to be common sense about how you saw, drew and talked about triangles." (p. 175).

What is intriguing in this discourse is the kind of mathematical subjects that students are becoming in learning mathematics with technology, or, briefly speaking, the fact that "in these environments driven by the hand or body, the human is constantly reinscribing herself into the idealized, abstract mathematics." (Sinclair, 2014, p. 168). It is through this 're-inscription' that the students' encounters with the mathematics in terms of their lived-in experience become sites of agency in the classroom, and that agency is constituted across the learning situation, being no longer attributed to the individual learner as the only centre of mathematical activity, but to her engagement with the material surrounding. In this perspective, we pursue a participationist vision of learning with tool aligning with this re-inscription of the self or, following Rotman, with the increasingly break down self, who becomes plural, distributed and besides herself, in a word: posthuman. We do this in the specific context of graphing motion, which we find challenging for the particularly important position taken on by motion: concerning the way that mathematics struggles to reckon with time, in terms of the learners' kinaesthetic engagement, and for its being so constitutive of self.

Agency and mathematical instrument

The vision we adopt in this paper troubles the traditional conceptualist idea that materials and tools have confined properties of their own. Instead, it has an interactionist nature for which materials and tools are not inert but are constantly interacting with each other and with the human body. In so being, it is far from the Cartesian reading of the human body, according to which the human body is exceptional in its freedom and will, and the human mind animates the body while other bodies lack any such agency. We follow here De Freitas and Sinclair (2013, 2014), who propose an alternative theoretical view of the body, drawing on materialist theories that consider freedom and agency as "dispersed across human and non-human agents" (de Freitas & Sinclair, 2014, p. 39), without centring "human will or intention in the orchestrating of experience" or conceiving "the human body as the principal administrator of its own participation" (ibid., p. 19). Instead, they unbind the body from its skin for shedding light on the ontologies of both body and mathematics, and breaking "with binaries that set organic against inorganic, and animate against inanimate, so that matter might be re-animated more generally and seen in terms of potentiality and emergent generative power" (de Freitas & Sinclair, 2013, pp. 459-460). In so doing, they rethink the boundaries of the body in the mathematics classroom, so that boundaries are constantly re-created and assemblages emerge as the body and as the unit of analysis in the learning experience of students.

This distances us from discourses of identity, causality and determinism. It also opens room for a discourse of subjectivity that, rather than locating knowing in the individual body, attempts to adequately address the collective social body. In line with posthumanist theories of subjectivity, subjects "are constituted as assemblages of dispersed social networks" (de Freitas & Sinclair, 2014,
p. 33), and the human body has to be conceived in terms of distributed networks where the material and the social are fused. Attention is shifted to distributing agency across a network of interactions, the properties of which are constantly changing (Rotman, 2008). This is an even more fascinating position when referred to interactions with tools. For example, Rotman talks about "becoming beside ourselves" to capture the new acentred sense of subjectivity that is emerging this century, in part, claim de Freitas and Sinclair (2014), "because of new digital technologies that herald and hail a network 'I' which thinks of itself as permeated by other collectives and assemblages" (p. 36). This vision problematizes the idea that any one part of the assemblage is the source of action, intention or will. Plural agency entails the formation of new assemblages and new folds upon the working surface, where digital tools are like everything else in being materialities that do not have determinate boundaries. Instead, they operate within the relations of the ever-changing assemblage participating in the network 'I'. Thus, distributing agency allows theorizing the role of tools in learning mathematics in ways that the traditional cognitivist could not. Rotman would outline how mathematical activity co-involves the concepts, the learner and the material world, including tools, and that this co-involvement means that "mathematical activity does not just produce more mathematics (or more learning), but also produces a new person in a new material world" (de Freitas & Sinclair, 2014, p. 109). In this sense, subjects constantly reinscribe themselves into the mathematics.

Focusing on learning mathematics with tool, we propose to join the vision of distributed agency with the notion of mathematical instrument introduced by Nemirovsky et al. (2013). Nemirovsky and his colleagues propose an alternative perspective for tool use, which talks about mathematical knowing as constituted by (not dialectically related to) embodied tool use, and attempts to avoid a deterministic view in line with the Cartesian reading of the human body in interaction with tools. In the particular context of a science museum exhibit, these researchers study emergent fluency with a mathematical instrument as a way to access certain kinds of mathematics. In so doing, they "locate tool fluency as well as mathematical thinking and learning in the process of perceptuomotor integration when learners engage with others and physical artifacts" (p. 378). In this view, a mathematical instrument is defined “as a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics.” (p. 376). The term instrument is used, instead of tool for example, because it intentionally connotes the culture of music, where one cannot speak of “a violinist’s expertise as something divorced from the quick movements of her fingers over the strings and the trained dance of her eyes across a musical score” (p. 377). In a similar way, one cannot talk about mathematical expertise divorcing it from the “skillful motoric and perceptual engagement with the tools of the discipline.” (p. 377). So, the metaphor of playing mathematical instruments is used to conceptualize mathematical instruments analogously to musical instruments. Nemirovsky and colleagues claim: “[f]luent use of a mathematical instrument allows for culturally recognizable creation in mathematical domains, just as musical instruments enable practitioners to produce distinct kinds of music that members of musical communities acknowledge.” (p. 373). Then, emergent tool fluency is described in terms of perceptuomotor integration, a phenomenon that occurs when the perceptual and motoric aspects of using a mathematical instrument are intertwined. The relevant point is that, for these researchers, perceptuomotor integration is constitutive of mathematics learning processes and is common to using a wide variety of tools, not only those considered in their study.

**WiiGraph as a resource in the classroom**

In the context of graphing motion, the paper presents an activity in which we used the Nintendo Wii as a resource for the learning of mathematics. The choice is not casual, since the Wii offers with its devices new game experiences to its users, with lots of possibilities in terms of kinaesthetic and proprioceptive engagement. The main devices are the Wii Remotes (Wiimotes), the controllers for playing games with the Wii, and the Balance Board, a platform through which it is possible to play
Interactions with WiiGraph

WiiGraph works with two Wiimotes connected via Bluetooth to the computer where the software is running and via infrared technology to the sensor bar of the Wii. The software allows displaying the location of the two controllers over time on a single Cartesian graph area, while users move holding them in front of the sensor bar, in a large space for interaction devoted to embodied explorations. A display area is needed to project the computer screen so that the interactions can be shared within the classroom context. In order for a graph session to start, each Wiimote has to be directed steady at the sensor bar, in which case a coloured diffuse circle appears on the graph area. With the circle visible for both controls, WiiGraph produces two real time coloured graph lines corresponding to the two controls' movements (the graphs of the two functions $a(t)$ and $b(t)$, being $a$ and $b$ the distances of the controllers from the sensor). Several graph types, composite operations, targets and challenges can be set. The two that are the main focus of this paper are the "Make your own Maze!" target and the $a+b$ operation that can be activated in the case of the Line Graph type. The first option allows for the creation of a target maze, with a certain complexity (given by number of inflection points, thickness, tension and layout of the line), which becomes the challenge target for the users. Figure 1a shows an example of a session of this type, also giving the final score of the two "players". As a resource for learning, this first option is captivating to design tasks that make pairs of students to be rivals, each player against the other, challenging each other to get as close as possible to given graphs. The second option permits to have a third coloured graph on the screen, which results from the sum of $a$ and $b$ over time, that is the graph of the function $(a+b)(t)$. As a
resource for learning, this option is intriguing if we think of tasks that require couples of players to obtain specific lines as sum graphs (this can be realised through plain tasks or using a "Make your own Maze!" target for the sum). In these second kinds of tasks, the students need to be collaborative to reach their goal. Obviously, the same can be replicated with a different operation among the four that are available in the software.

WiiGraph also offers another interesting modality for designing collaborative tasks, that is, the Versus Graph type. This type plots an ordered pair of the distances of each user over time (with the production of the ordered pair that remains implicit). Practically, in this case, graphs are obtained by pairing the two users' functions of position versus time, one for the horizontal axis and the other one for the vertical axis. So, the spatial graph corresponding to two parametric functions (where the parameter is time) is constructed. As a resource in learning mathematics, this option may favour tasks that require the students to create shapes or plane figures, like for example rectangles, rhombuses, and circles (see a trial for the circle in Fig. 1b), thus entailing the need for an interactive coordination between collaborating users. The case of the circle is of very concern whether we think of approaching it as limit of regular polygons with a progressively increasing number of sides, and in relation to the study of the sinusoidal functions as characteristic components of circular motion.

We also believe that the modalities of being collaborative/being competitive deserve lots of attention from the pedagogical point of view. They offer great potential for new learning spaces, where, beyond the mobile and the kinaesthetic, the influence/interference between students who are "playing" the tools together are a compelling resource in the teaching situation, which might engage learners in completely new manners and at differently deep levels, use their strengths and curiosities to fuel sophisticated mathematics, as well as stimulate discussions about the values of collaboration and competition.

![Image](image.png)

Figure 1. (a) "Make your own Maze!" and (b) Versus Graph sessions; (c) Students playing

**Method**

Our study has involved a 9th grade group of 30 students and their regular teacher in mathematical investigations through the use of WiiGraph. The study lasted a total of 9 two-hour weekly meetings, which were preceded with an individual questionnaire about believes on function and graph. The activities were carried out in a lab room used as a laboratory space for mathematics lessons. We set the room projecting the computer screen on an IWB, with the students sat around the interaction space, apart from the two holding the controllers. This way, all of them could watch the on-line acquisition of WiiGraph and the real-time creation of the graphical lines. All the experiences and tasks were designed in collaboration with the class teacher along the entire course of the study. In addition, during the meetings the students worked in various manners. They took part in some individual tasks, group works and collective discussions led by the researchers (the two authors), who were always present in the classroom. All these phases were also filmed through the use of one or two mobile cameras. Data for the analyses comes from the movies and from the written productions of the students. In the next section, we present examples from the classroom and we briefly discuss them using the theoretical commitments that we have elaborated above.
Discussion of classroom examples

Episode 1. Being rivals or Make your own Maze!

The first example concerns a task with a competitive nature. The task asked the students, divided into groups of three, to choose one among them for a challenge. The challenge consists of competing with a class-mate from another group to match a target graph, using the "Make your own Maze!" modality of WiiGraph. This modality furnishes as a target a graphical line with a given shape and thickness, which can be set up through the options. Different target graphs were given to different couples of competing groups. For example, Figure 1a shows the light blue tick graph that was the target for the first two players: Emanuele and Oliver. We can see that this line had two humps. The other two lines in the Figure were obtained through the movement of the two students, and captures the functions $a(t)$ and $b(t)$.

Before moving in front of the sensor, each player received suggestions from his group-mates about the suitable movement to perform. Then, the students moved (Fig. 1c) and Emanuele won. He got a score of 28 out of 35 (blue line) against 22 out of 35 (pink line), obtained by Oliver.

In their being rivals, Emanuele and Oliver are affected by their moving close to each other, as well as by the feedback that the software gives them in terms of the real-time lines appearing on the screen. This is apparent in how the two students move during the challenge, when their tension towards winning is clearly manifested by their posture and voice engagement (for example, their laughing). At the end, Oliver is unsatisfied about his result, which he associates to having made a "mistake", as he affirms when the researcher asks the students if they have reflections about their experience:

Oliver: I made a mistake on the second [hump] (Points towards the hump on the graph, from his chair) to go up, because I had to move faster. I didn't realise that

Researcher: Where, do you say? (Invites him to better explain at the IWB)

Oliver: Here I have really gone out (Without touching the board but being quite close in front, mimes with his open right hand the increasing pink line piece, which is out of the target graph: Figg. 2a and 2b. The gesture occurs in a plastic way with the head and body softly tended towards the IWB so as to go along with the slope of the line), because if I would have moved faster (His right index finger runs along the corresponding piece of the target graph exactly staying within its thickness: Fig. 2c. Turns to look at the researcher, laughing) I would had been more in the graph (Repeats the previous gesture moving his finger faster than before)

Figure 2. (a) Oliver's gesture and posture for the line's slope; (b) Following the target line

The brief episode shows how Oliver bridges the 'distance' between his "mistaken" pink line and the corresponding target graph in the graph area. His words are relevant in the discourse, especially in
their being entangled with other components, like his ways of moving and feeling. At first, Oliver uses the verb "to go out" to speak about the pink line piece that goes out of the target thickness, as witnessed by his miming gesture of the line. He is comparing the two graphs but, at the same time, the subject "I" associated with the motion verb fuses the line and the motion experience (see the use of the past tense and of "really"). In doing so, it is constituted as a plural "I", which is distributed between the space and time of the motion challenge, the space-time graphs and the feeling of being mistaken, accentuated by Oliver's plastic and soft bodily movement so close to the IWB although without touching it (Figg. 2a and 2b). In a second moment, Oliver justifies ("because") his having "gone out" through the logical "if" that suddenly brings into discourse a new dimension: that is, a new virtual pink line related to a new "faster" movement. This time, Oliver first runs his finger along the suitable piece of graph, making present the virtual line within the thickness of the target graph, while he explicitly speaks of his motion experience through the use of the verb "to move". Then, the changed speed with which he repeats the running gesture actualises the higher speed of the new imaginary movement, while also actualising again the new pink line and its slope. The new line means to be "more in the graph". On the one hand, this last verbal expression seems to underline the imaginative presence of the student "inside" the graph, merging the line and the motion experience again, in contrast with the present situation (past in movement) in which Oliver is outside of the graph. On the other hand, it also seems to actualise the mobility of the pink line in the situation: the line assumes different slopes depending on the movement.

Interpretation of Episode 1

We can interpret the episode saying that Oliver reinscribes himself into the temporospatial nature of his challenge with Emanuele. He goes back to the spatial and temporal dimensions of the motion experience, by thinking of the new speed ("faster"). But he does this in the present spatio-temporal dimension of the graph ("on the second", "out", "in the graph"), by actualizing in word and gesture the virtual line, which only comes to exist in relation to the current line. Perceptual and motoric aspects of the challenge are both recovered through this actualization. Oliver is 'moving' back and forth, between the graph space and the interaction space, making the slope of the line and the speed of the motion experience change together, one depending on the other. The assemblage of meaning develops through his ways of doing, gesturing, talking and feeling (like his last laugh), which also mobilize the mathematics at play. In his being part of this changing assemblage Oliver is learning with tool, starting to gain some fluency with WiiGraph used in the Make your own Maze! modality.

Another aspect that we see as relevant in the episode concerns the fact that the lived-in challenge with Emanuele is present in Oliver's tension towards correcting himself and finding a way of getting a better graph, which would imply a better score. The particular experience of being rivals, allowed by the software in this case, is not neutral with respect to the ways in which Oliver communicates with the class. For example, his use of the adverb "really" seems to underline the force and affect of what exactly occurred and to justify his losing of the challenge. Briefly speaking, in this moment, it is likely that Oliver is not simply thinking of him as being mistaken because he was not able to be "more in the graph", just in terms of a line that goes out from the target thickness. Rather, for him it is much more a question of not being able to be "more in the graph" with respect to his rival.

Episode 2. From competing to collaborating

The other couples of competing groups were given the same challenge, with new target graphs, and the result of each challenge was discussed collectively. After this phase, the competing groups were asked to join in order to face a written task together. The task was essentially made of two requests. The first request was formulated as follows:

"You have tried to move, competing with each other, in order to be as faithful as possible to a given graph on the screen. You have also obtained a measure of your accuracy (precision with
respect to the complexity of the graph). Write down your reactions and sensations just after the challenge (distinguishing the voices of the two groups)."

A second request asked the students to give advice to an imaginary friend, who has to compete (but never used WiiGraph), and to explain the strategies to reach a score as good as possible.

Emanuele's group and Oliver's group produced a protocol in which they drew their target graph, and two answers for the first request, which are very similar between each other (see Fig. 3). Oliver and his group-mates wrote: "Our sensations are: that of changing speed in order to create a different slope to create different kinds of "humps". The second difficulty lies in finding the starting point such that it is possible to have a "secure" point from which the graph is to be arisen." (Fig. 3, left). Emanuele and his group-mates, instead, wrote: "Our difficulties were little. I did not have so many difficulties apart from the change of speed in the curves. Our main difficulty, which was making us to lose points, has been finding the correct starting point" (Fig. 3, right).

Figure 3. Emanuele's group (right) and Oliver's group's (left) difficulties to match the target graph

As a second point, the answer to the request about advices for the imaginary friend, and strategies to adopt, claims:

"The relevant thing is staying inside the blue thickness and holding the controller towards the sensor, and remembering that the closer you are to the sensor, the closer you are to zero on the graph. A thing to look at is that the steeper the piece is, the bigger speed will have to be, in order to remain inside the thickness. One has to move in a direction perpendicular with respect to the sensor." (Fig. 4)

Figure 4. The answer to the second request of the written task

After dealing with the second request, the group could use WiiGraph to check its suggestions. The six students came in front of the sensor to try a new challenge, in line with their strategies. The striking thing is that, at this point, Emanuele and Oliver changed the way of playing (and being) in
the challenge, by performing their movements without competing but collaborating with each other. They decided together the starting point and the changes in speed, which were the difficulties that both expressed in the first part of the written task. This was so unexpected that the researcher wondered whether the two students were not supposed to be in a competition. Oliver responded with a resolute "No, no, in a collaboration", immediately before starting. Thus, the students moved in coordination with and next to each other, so much that they proceeded, silent and focused, with the same pace and their facing arms touching each other, keeping the controllers very close (Fig. 5).

![Figure 5. Emanuele and Oliver collaborating with each other](image)

**Interpretation of Episode 2**

We can interpret this second episode pointing out the way that fluency with the tool is progressively growing for the students, even in a phase in which they are solving a written task. Perceptual and motoric features of the previous competing experience come evidently out in the answers given by the students. Starting from the initial request, the two groups share the critical aspects entailed in matching a target graph, that is, the starting point of movement and the speed at which to move. The question of speed is significant concerning perceptuo-motor integration in the use of tools. While, for Emanuele and his group-mates, different speeds mean different slopes, which, in turn, mean "different kinds of "humps"", Oliver and his group talk about change of "speed in the curves". In both cases, the idea of speed comes with the graphical shapes assembling the visual and kinaesthetic experiences involved in the creation of a graph with WiiGraph.

This fluency is present even in the answer to the second request, where the steepness of the curve is captured in terms of speed, and being closer to zero is being closer to the sensor. But fluency is becoming functional, on the one side, to the original competitive task, for which one has the goal of remaining "inside the thickness" as much as possible (or, in other words, of winning the game). On the other side, it is clear that the new task is making way for the students' becoming collaborative, through focus on advices to be given to somebody else (the imaginary friend), who never took part in similar experiences with the tools, and on consequent strategies of motion. The task is becoming challenging for the students in a new manner: they want to play the game and win. The nature of the task implies a shift in the students' interaction with the software, from their being rivals to their becoming players against the computer, thus collaborative with each other. This is shown in a powerful way through the need for checking the correctness of the written suggestions, for which attention is no longer on the two players who are competing but on their capacity of saying a third person *how to play*. The assembling of meaning here grows through the ways in which the students talk about accuracy in the situation (through difficulties, suggestions, strategies), and move, again animating the curves and the mathematics at play. In being part of the changing assemblage, the six students are learning and gaining more and more fluency with the particular use of the software.

**Episode 3. Being collaborative or $a+b$ in Line**

The third episode is about a task born with collaborative nature. The students first encountered the Line modality for the sum $a+b$, moving in front of the sensor without knowing the chosen options. Through the kinaesthetic experience, they discovered that a third real-time line was appearing on
the screen, and was produced by adding the values of \(a\) and \(b\) over time (see Fig. 6). After this, the researcher led a collective discussion about how to obtain through the new modality a horizontal line, and about whether there exists a unique way or not. Then, in the following week, the students joined their groups and faced a written task on the sum. In particular, the students had to imagine to be in a situation in which a class-mate is absent and were asked how they would describe the activity of the previous week, and how they would explain the functioning of \(a+b\).

![Figure 6. Line modality for the sum \(a+b\)](image)

Alessandro and Luisa worked with Massimiliano, who was really absent the week before, when the students encountered the sum for the first time. This episode focuses on an initial brief moment of group work, in which Alessandro and Luisa explain to Massimiliano the meaning of the sum. The three students are sitting around a table, each on one side. The short dialogue develops as follows:

**Luisa:** Then, there are two people, that their graph, that is, each of them performs a movement *(With the pen in her right hand mimes some humps in the air in front, looking at Massimiliano)*, which is on the graph *(Gazes and points with the pen to the graph area of WiiGraph)* and, that is, the graph *(Mimes again the humps in the air with her right hand, the pen in the left hand)* is the sum of these movements of two people *(Looks at Alessandro, smiling)*, and so

**Alessandro:** It is as if there was, then, that is, it is as if, say, there were three people, that is, there are two people *(Turns and looks at the interaction space where the people should be)*, who perform two movements *(Mimes the two people moving with his two open right hand fingers little moving back and forth in the air, gazing to the interaction space: Fig. 7a)*, and it is as, that is. If they stay, one at 1 [feet] and one at 2 [feet] *(Looks back at the interaction space)*, it is as if there really was a third person, who moves *(Turns towards the researcher, mimes a quick movement in front of him, with his right hand moving a little forward in front of his torso: Fig. 7b)* at 3 [feet] *(Turns again towards the interaction space)*. It is a sum, that is, that is get as \(a\) plus \(b\) equal to \(c\) *(Looks at Massimiliano)*

**Massimiliano:** Ah, yeah *(Nods)*

**Luisa:** As if there was a \(c\) *(Looks at Massimiliano)*

**Alessandro:** Right

**Luisa:** That is, there the movement is that of \(c\) *(Repeats the previous hump gesture in the air with her right hand: Fig. 7c)*
Interpretation of Episode 3

We can interpret this episode looking at the two ways in which Luisa and Alessandro explain to Massimiliano how the graph of \( a+b \) works. Luisa mainly talks of what happens on the graph area, thinking of the movements of two people as being "on the graph". Reference to the two people is interesting in terms of her fluency with the tool, since her lived-in perceptual experience of the two graphs of \( a \) and \( b \) is embodied with the motoric aspects of the experience for which two people have to move at the same time ("each of them"). The two people are the starting point. The sum depends on them, since "the graph" is thought of in terms of "these movements of two people". But Luisa is not precise in her explanation as she witnesses by gazing and smiling to Alessandro so as to looking for help. Alessandro modifies her original idea introducing a "third person", and adding much more precision to the explanation. In doing so, he specifically talks about an imaginary situation ("as if", repeated many times), which would involve "three people" and would take place in the interaction space (as shown by the insistently act of turning towards it). There, the third person would "really" move, and move together (in collaboration with) the two people who are performing the "two movements" that still are the starting point. Agency is plural and distributed among all the actors in the situation, which are, beyond the students, the imaginary moving people, the tools, the graphs, the sum of functions. The relation between the two original people and this third person (that marks the need for collaboration) is clearly expressed by means of the numerical examples of 1, 2 and 3 feet. Then, it is transformed into the algebraic expression \( a+b=c \), which implies that a variable of position ("As if there was a \( c \)") could be also attached to the movement of the added person ("there the movement is that of \( c \)"). The beautiful example shows that the lived-in experience of the students with WiiGraph is crucial in the assembling of meaning for the sum function. The ways of moving, gesturing, gazing, talking and feeling of Luisa and Alessandro reveal their perceptuo-motor engagement and fluency with the software. In playing the instrument, the new graph of the sum is mobilized and learned, even adding a form of imaginary collaboration among movers that will presuppose the presence of a new mover. All these aspects can be also found in the written explanation that the group produced. In fact, the students wrote: "The two people move in front of the sensor in the same way in which they moved the other times but, on the graph a third movement is represented, which is the sum of the first two". They also added that "the first thing is that it is necessary to collaborate", and the introduction of \( c \) is fundamental in this respect as shown in Fig. 8.

Figure 7. Luisa and Alessandro explaining \( a+b \)

Figure 8. Written protocol for \( a+b \)
Conclusive remarks

In this paper, we have pursued a non-dualist participationist vision of learning mathematics with tool. This vision moves from drawing on Rotman's idea that the self is always breaking down and reinscribing, becoming plural, distributed and posthuman, through her engagement with the material surrounding, from which we cannot separate learning. We have considered the particular context of graphing motion, by the aid of a software application called WiiGraph, which allows students to reason on a graphical approach to function using in the same time two controllers of the Nintendo Wii. The aspect of playing (with) the controllers (in the usual experience, devices to play games) was fused with the aspect of playing WiiGraph and the Wiimotes as mathematical instruments in the classroom. Students were never alone in playing with the software. The game experiences available in WiiGraph occasioned lots of possibilities, with respect to the learners' kinaesthetic and proprioceptive engagement, as well as in terms of competitive/collaborative engagement between learners. From the didactic point of view, they became a matter of interest and motivation towards the mathematics. From the pedagogical point of view, they gave rise to issues about task design, becoming a very resource for learning, implying new unexpected and unexplored teaching and learning spaces. The discussion of some classroom episodes has shown that the students gain fluency with the devices and with the software, and develop mathematical understandings, through their lived-in experience in which perceptual and motoric aspects are continuously entangled. All the encounters of the students with the mathematics are made of recollections, expectations, fantasies and moods related to their embodied and kinaesthetic interactions with the tools. The only site of agency is not the student body of the students. Instead, agency is distributed and constituted across the learning situation. The body that is always becoming is that of the students and of their relations with all the human and non-human agents that are involved in the situation. In this perspective, we can talk about learning in terms of this moving assemblage. The mathematics itself is transformed and changing in the assembling of meaning, along various dimensions, like the digital and the material.

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Vector subspaces generated by vectors of $\mathbb{R}^n$:
Role of technology

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Abstract: In this work we report how the use of technology (e.g., Geogebra) promotes –in students– learning the concept of vector subspace generated by vectors of $\mathbb{R}^n$, $n \geq 3$. Two cognitive theories support this research: shifts of attention (Mason, 2008) and representations (Duval, 1999, 2003, 2006). Eleven college students (between 19 and 26 years old) from an instituto tecnológico located in Mexico City participated in the research. Data was collected by doing group interviews (groups of two or three members); students worked on previously designed Activities using Geogebra and pencil-and-paper. Our results show that implementing the use of technology and pencil-and-paper, the students correctly determined vector subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, and partially of $\mathbb{R}^n$, $n \geq 3$.

Résumé: Dans ce papier nous reportons comment l'utilisation de la technologie (e.g., Geogebra) favorise –chez les étudiants– l'apprentissage du concept celui de Sous-espace vectoriel engendré par des vecteurs de $\mathbb{R}^n$, $n \geq 3$. Deux théories de type cognitive appuient cette recherche: celle appelé comme Change d'attention (Mason, 2008) et celle des Représentations (Duval, 1999, 2003, 2006). Dans cette étude ont participé 11 étudiants (d'environ 19 à 26 années), du niveaux supérieur d'un Institute Technologique de la ville Mexico, au Mexique. La collecte de données a été faite par des entrevues en équipes de deux ou trois étudiants. Dans les entrevues, les étudiants ont réalisé des Activités, désignées préalablement, en utilisant Geogebra et ainsi que papier/Crayon. Nos résultats nous suggèrent que, avec l'usage de la technologie et ainsi que de papier/crayon, les étudiants ont pu déterminer correctement les sous-espaces vectoriels de $\mathbb{R}^2$ et de $\mathbb{R}^3$, et aussi partialement de $\mathbb{R}^n$, $n \geq 3$.

Background and research question

Research in mathematics education –about teaching and learning of linear algebra concepts– report learning difficulties of this area in conventional teaching environment (pencil-and-paper). With respect to this problematic, several authors such as Dorier, Robert, Robinet and Rogalski (2000, 2011), Sierpinska (2000), among others, have detected that such difficulties are related with the obstacle of formalism, which is closely linked with the learning of concepts of vector space or related, e.g., set generated by a vector of $\mathbb{R}^n$, linear combination of vectors of $\mathbb{R}^n$, set generated by vectors of $\mathbb{R}^n$, $n \geq 2$, etc.

With the purpose of overcoming the obstacle of formalism several researchers (e.g., Gol & Sinclair, 2010; Stewart & Thomas, 2010, among others) have implemented the use of technology to reduce the learning difficulties of linear algebra concepts. Research reports can be found where they use CAS (Computer Algebra Systems; e.g., Matlab, Mathcad, Maple, Derive, etc.) and DGS (Dynamic Geometry Software; e.g., Sketchpad; Cabri Geometry II –farther on was Cabri–, and more recently Geogebra, etc.).

About the use of CAS –with respect to the teaching of linear algebra concepts– Harel (1997) reports the outcomes from using Matlab as a didactic medium, in the study of basis of space columns of echelon form matrices reduced line by line using CAS. According to Harel, the results obtained by using that CAS were satisfactory, which induces to believe that CAS potentially enhance the learning of abstract linear algebra concepts. Nonetheless, in the research of Pruncut (2008), when using CAS Maple, as a didactic medium, the outcomes did not turned as expected with respect of

1 The Activity or Activities (upper case) in this paper, refers to the activities that the students did with the purpose of leading them to understand linear algebra concepts.
the students thinking about linear algebra concepts. Related research show that the use of CAS as didactic mediums does not enhance the learning of the abstract concepts.

Gol (2012), being aware of the works of Pruncut, search for alternative software to use so that students would develop geometric intuitions that would allow them the learning of the eigenvalues and eigenvectors concepts. The author acknowledged in the research that such goal could not be achieved by using CAS due to the static limitations of this kind of software while working with mathematical objects, and decided to go for the sketchpad (DGS) as an alternative so that the students would understand in a reflective way such concepts.

From the research of Gol as well as others where the DGS was use as didactic mediums it can be inferred that the DGS software play an important role in the learning of concepts, in particular of linear algebra. For example, Sierpinska, Dreyfus and Hillel (1999) showed that with the use of Cabri, the students learned mathematic concepts of linear algebra; it was emphasized that it could not been accomplished in a pencil-and-paper environment (p. 217). Furthermore, Sierpinska (2000) used such software with the students with the purpose of avoiding the obstacle of formalism. In that research, the author states that students were capable of expressing themselves in ways such as: “the vectors were the same or not”, “parallel vectors”, “vectors that have the same direction”, etc. (p. 210), while the students that worked in conventional environment of pencil-and-paper were not capable of “visualize” vector properties like the ones mentioned above.

Recent research (e.g., Soto & Romero, 2011; Uicab & Oktaç, 2006, among others) show that the use of dynamic software, with the emphasis of overcoming the obstacle of formalism in the learning of concepts related with vector spaces, leads to satisfactory but not definitive results. Researchers that have implemented the use of DGS as a mediator in the learning of abstract linear algebra concepts have reported how it is achieve –by the students– the learning of the concept of linear transformation of vectors (Soto & Romero, 2011; Uicab & Oktaç, 2006); eigenvalues and eigenvectors (Gol & Sinclair, 2010); linear dependency of vectors (Aranda & Callejo, 2010); linear combination of vectors, linear dependency or independency of vectors and linear transformations (Andreoli, Beltrametti, & Rodriguez, 2009); vector space basis, set generated by vectors and linear dependency and independency of vectors (Stewart & Thomas, 2010). Furthermore, Aydin (2014) studied the main tendencies of the role of technology with respect to the learning of abstract concepts in linear algebra.

The results of these researchers put in evidence that the learning of concepts of linear algebra is promoted by the use of technology, despite, there still are some difficulties to fully achieve it. In this paper, it is aimed to respond the question: How does the use of the DGS (e.g., Geogebra) influences in the learning of the concept of vector subspace of $\mathbb{R}^n$, $n>3$?

**Conceptual framework**

This research is based on the cognitive theories of *shifts of attention* (Mason, 2008) and *representations* (Duval, 1999, 2003, 2006). The theory of *shifts of attention* is based on three basic concepts: *attention*, by the means that the observation –by the student– takes place, which allows to *sustain, discern, relate, perceive, and reason* the topic being studied and without it, would not be possible to make sense what the student learned; *being aware of…*, consist in verifying if the student makes sense of what he/she aims to know, and precise the knowledge/senses of what the student already knows, and the *attitude* as a means of willingness of the student to want to learn.

The theory of *representations* (Duval, 1999, 2003, 2006) is based in two concepts: *semiosis*, as apprehension or production of semiotic representations (as a means that the individual uses to externally expose his/her mental representations –images about an object– achieved by the use of signs: natural language, algebraic formulas, geometric figures, among other), and *noesis*, cognitive doings such as the learning of an object.
The data was analyzed considering the areas where the two theories coincide, which are: (1) object association, that considers that fact that the learning of concepts (in mathematics) can be shown in several representations; (2) cognitive position, the change of awareness from implicit to explicit (Mason, 2008) that occurs when the student can see (Duval, 2003) and willingly manages the topics being studied as a whole, and (3) the student makes sense of what she or he aims to know, it is recognized when students master the concepts attention, being aware of..., and attitude (Mason, 2008), and can support—with relative ease—the transition from one type of semiotic representation to another (Duval, 2003) of the mathematic topic being studied.

Methodology

Participants. In this research, eleven college students participated (between 19 and 26 years old) from an instituto tecnológico located in Mexico City. The students took a linear algebra course where technology was not used to teach the class. They were chosen by their teacher according to their willingness (attitude) to collaborate and previous academic performance in their linear algebra course (pencil-and-paper course environment). This research was conducted by assembling four groups of two or three members. Data was collected by doing group interviews, done directly by one of the authors of this paper, on predesigned activities with the aim to recollect evidence of the work of the students. None of the students had previous experience using Geogebra to solve linear algebra activities, hence some training was provided before starting to gather data.

Design and implementation of the tools for the gathering of data. As part of the employed methodology in this research the following was designed: a Pre-test, three Activities, and a Post-test. The students approach individuality the Pre-test, and the Post-test, but not the three Activities which were conducted in the groups. Previous to the training for the teaching of the software commands related to current discussed subject, the students took a Pre-test with the aim to know the software mastering level of the students with respect to vector subspace generated by vectors of \( \mathbb{R}^n \), \( n > 3 \).

After taking the Pre-test, the students commenced the Activities; then after finishing the Activities the Post-test was conducted. The thematic content of the three Activities are: Activity I: Set generated by a vector of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), Activity II: Linear combinations of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), and activity III: Set generated by vectors of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). The data collection took place in a classroom and each collecting section lasted approximately three hours.

The activities. The Activities were designed by the authors of this paper and implemented by one of them, which will be called \( I_n \). An interview was conducted alongside the Activities and was videotaped. During the interview, the analysis and thoughtful thinking about the claims of the students were favored by via in depth discussions aimed at learning concepts. It was permitted to the students to ask questions with respect to understating the questions being asked. The previous work of the students was discussed and analyzed at the beginning and at the end of each section.

Data analysis and discussion of results

Due to length limitations, only the data collected by a single group (Team 4) composed by two students (\( E_1 \) and \( E_2 \)) would be presented in this paper.

Activity I: Set generated by a vector of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). The initial discussion in this Activity was about the concept of scaling a vector. The students did not have difficulties in accurately and efficiently interpreting the new vector generated by scaling a vector; for convenience, in this paper, the new vector will be called the scalar product vector. From a list of vectors of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) that students had sketched in pencil-and-paper, and in Geogebra, \( I_n \) asked: How do you define a vector from the list of vectors, is it a scalar product vector? \( E_1 \) Answered:

1. \( E_1 \): Because any scalar that multiplies a vector is [resulting vector] contained in the same straight line [where the initial vector is located].
In the expression $-2 \left( \frac{1}{2} \right) = \left( -\frac{2}{2} \right)$, the scalar $-2$ that multiplies $\left( \frac{1}{2} \right)$ is “any” scalar?

No, it is just some specific scalar [right after, E2 added to the answer of his team mate].

Because there is only a scalar [multiplying the vector], that if it multiplies the vector would result in a [single] scalar product vector (cognitive position, Mason, 2008; Duval, 2003).

Following, the students write in their own words how they understand the relation of the scalar product vector (see Figure 1).

With the aim so that students make sense of the amount of the scalar product vectors, Inv requested to the students to trace in Geogebra the scalar product vectors (see Figure 2a), and Inv asked:

Are all scalar product vectors [vectors generated by scaling the initial vector $u$] of $u$ shown on the computer screen? E1 answers:

Yeah, though there is a limit marked [pointing to the computer screen] we know the straight line goes to infinity.

You are stating that there is an infinite number of scalar product vectors $u$, but on the screen all those scalar product vectors of $u = \left( \frac{1}{2} \right)$ can be seem? (See Figure 2b)

Well not, but we assume [that the scalar $c$ shapes up the product scalar vectors of $u = \left( \frac{1}{2} \right)$] and that the vector $cu$ goes to the origin location to the minus infinity and to the plus infinity.

What geometric location describes point B [of Figure 2a]? Is it an entire straight line?

It is not a segment of a straight line?

Because there is an infinite number of scalars that generate a straight line [then makes a correction]; no, because there is an infinity number of vectors [scalar product vectors of $u$] that generate a straight line [the student makes sense of he/she wants to know Mason, 2008;
Duval, 2003].

In Figure 3, the students explain the responses to the line 8.

The work of the students in Geogebra allowed them to visualize (cognitive Position, Mason, 2008; Duval, 2003) the concept of scaling a vector in $\mathbb{R}^2$ (see figures 4a) and 4b).

With the work done by the students up to the Task Io) it allow I$_{nv}$ to ask how do you define the concept of a set generated by a vector. The silence of the students reflected the trouble they had in answering the question. Hence, I$_{nv}$ asked they question in a different way: Think about the work you (the students) have done up to the Task Io) then answer, if there is a relation between “the get generated by a vector $u$” and “the set of product vectors (vectors generated by the scaling of the initial vector $u$)?”

12 E$_1$: We could say that the set generated [by a vector $u$] is the straight line that contains the initial vector along with the product scaled vectors making a straight line [following E$_2$ added to the answer of his classmate.]

13 E$_2$: Both of the set of vectors are the same [strongly stating]: “any set generated by a vector is a set of the scalar product vectors of the initial vector”, and “any set of the scalar product vectors of the initial vectors is a set generated by an initial vector”, [therefore] the set of vectors are the same [the student E$_2$ makes sense of what he wants to know; Mason, 2008; Duval, 2003].

14 I$_{nv}$: According to what you have seen in the computer screen with respect to the Activities done up to this point. How do you define a set generated by a vector of $\mathbb{R}^2$ and $\mathbb{R}^3$?

15 E$_2$: According to what we have seen in the files […] we could observed that a vector along with its scalar product vectors generate a straight line that contains them in $\mathbb{R}^2$ and $\mathbb{R}^3$.

---

2 The Task or Tasks (upper case) in this paper, refers to the specific work that the students did for the activities. For example, “Task Ia)” refers to task “a” of Activity I.
which is the same as the set generated by a vector: that is, the scalar product vectors of a vector would now be named a set generated by a vector: that is, the scalar product vectors would now be called a set generated by a vector in \( \mathbb{R}^2 \) or \( \mathbb{R}^5 \), which generate a straight line. \( [E_1 \text{ accepts what } E_2 \text{ said}, \text{cognitive position}, \text{Mason, 2008; Duval, 2003}.] \)

Activity I end it by requesting the students to determine the set generated by two vectors of the vector spaces of \( \mathbb{R}^2 \), \( \mathbb{R}^4 \), \( \mathbb{R}^5 \) and \( \mathbb{R}^n \). They gave their answers (Column D from Task Iy) from Figure 5) in terms of the number of zero rows from the echelon formed matrix from these vectors (see Column C from Figure 5), since this matrix representation would allow them to determine whether the vectors of Column A are or are not scalar product vectors (see Column F). In Figure 5, can be observed how the students accurately responded for Row 5 (position 5D), but in rows 3, 4 and 6, did not know what to answer (positions 3D, 4D, and 6D). The aim of \( I_n \), was that the students to accurately answer, for the Activity III, Activities such as the ones planned in positions 3D, 4D, and 6D from Task Iy) (see Figure 5); for this, it was necessary to study for Activity II the concept of linear combination of vectors, Activity that is discussed below.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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<tbody>
<tr>
<td>1</td>
<td>Set of vectors</td>
<td>Matrix associated with the vectors</td>
<td>Echelon form matrix</td>
<td>Set generated by the vectors of Column A</td>
<td>Does it generate a straight line?</td>
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<tr>
<td>2</td>
<td>[ \begin{pmatrix} 1 \ -2 \ -3 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 &amp; 2 \ 0 &amp; 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ -2 \end{pmatrix} ]</td>
<td>Yes</td>
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<tr>
<td>3</td>
<td>[ \begin{pmatrix} 3 \ -2 \ -3 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 2 &amp; 3 \ 0 &amp; 2 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ -2 \end{pmatrix} ]</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>[ \begin{pmatrix} 5,3,2,1,1 \ 3,6,5,2 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 5 &amp; 3 &amp; 2 &amp; 1 &amp; 1 \ 3 &amp; 6 &amp; 5 &amp; 2 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ -2 \end{pmatrix} ]</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>[ \begin{pmatrix} 2,1,1,1 \ 4,-6,2,0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 2 &amp; -3 &amp; 1 &amp; 0 \ 4 &amp; -6 &amp; 2 &amp; 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 1 \ -2 \end{pmatrix} ]</td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>

Activity II: Linear combination of vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Activity two starts by discussion when are two vectors equal to each other. From a list that the students had sketched in pencil-and-paper, and in Geogebra, \( I_n \) asked: When are two vectors equal to each other? \( E_1 \) and \( E_2 \) answered:

16 \( E_1 \): It can be when \( [\text{the vectors}] \) have the same magnitude and direction \( [\text{does not senses the meaning of a vector}] \).

17 \( E_2 \): I remembered that two vectors were \( [\text{are}] \) the same when component to component are the same \( [\text{does not mention if the vectors are in the same vector space}] \).

18 \( I_n \): What it says (in line 17) (talking to \( E_2 \)), Are you contradicting \( E_1 \)? What you are stating (in the line 16) contradicts (asking \( E_1 \) \( E_2 \))?

19 \( E_1 \): It does not contradict, because \( E_2 \) is talking about an initial and final component that is the
same as magnitude \( E_1 \) does not answered adequately, but tries to associates objects in two representations, Mason, 2008, Duval 2003. \( E_2 \) tries to complement the answer.

20: \( E_2 \): Because, if given a vector \([ \text{in } \mathbb{R}^2 \text{ or in } \mathbb{R}^3 ]\) that if displaced to another place in its own space \([\text{vector space}]\) it conserves its magnitude... \([E_2 \text{ pauses for a second but does not continues}].

\[
\begin{array}{c}
\text{Dos vectores son iguales cuando tienen la misma dirección, magnitud y sentido. Cuando compone a componente }
\text{son iguales y pertenecen al mismo espacio vectorial.}
\
(\frac{1}{2}) \text{ y } (\frac{3}{2}), (\frac{2}{3}) \text{ y } (\frac{2}{3})
\end{array}
\]

Two vectors are the same when they have the same direction and magnitude. When their components are the same and when they belong to the same vector space \( (\frac{1}{2}) \text{ and } (\frac{1}{3}), (\frac{2}{6}) \) and \( (\frac{2}{6}) \).

\text{Figure 6. Definition, algebraic and geometric, of the concept of vector equality.}

During a section of the interview (lines 16 to 20), the students were not able to adequately define the concept of vector equality; however, they improved their response when they wrote the answer (algebraic and geometric, \textit{object association}, Mason 2008; Duval, 2003, see Figure 6).

Following, \textit{I}nv conducted the discussion with the aim that students would learn the property of enclosed vector addition that are not scalar product vectors of \( \mathbb{R}^2 \). \textit{I}nv asked, given that the vectors \( (\frac{1}{-3}) \) and \( (\frac{3}{2}) \) are in \( \mathbb{R}^2 \), where is the sum of the scalar product \( \frac{1}{2} (\frac{1}{-3}) + (-1) (\frac{3}{2}) \)? \( E_1 \) responded the question:

21: \( E_1 \): In \( \mathbb{R}^2 \), because the vectors \( (\frac{1}{-3}) \) and \( (\frac{3}{2}) \) are located in \( \mathbb{R}^2 \), besides it doesn’t make sense that the sum of the scalar product vectors gives, for example, a vector of \( \mathbb{R}^2 \). \textit{[the students makes sense of what he wants to know; Mason, 2008; Duval, 2003.]}]

22 \( \textit{I}nv \): In how many ways can you choose the scalars \( c \) and \( d \) when summing the scalar product vectors \( e (\frac{1}{-3}) + d (\frac{3}{2}) \), and in which vector space is located?

23 \( E_2 \): In an infinite number of ways and is located in the \( \mathbb{R}^2 \) vector space.

24 \( \textit{I}nv \): Is there a vector that cannot be written as a sum of the scalar product vectors \( (\frac{1}{-3}) \) and \( (\frac{3}{2}) \)?
The students simultaneously answered “No”, to the questions asked by Inv in the line 24, but did not justified their answer. Following, Inv asked the student to calculate $c$ and $d$ from the equation $c \left( \frac{1}{3} \right) + d \left( \frac{3}{2} \right) = \left( \frac{a}{3} \right)$ by geometric means using Geogebra, if $\left( \frac{a}{3} \right) = \left( \frac{1}{3} \right)$ (see Figure 7a) and by an algebraic way using pencil-and-paper with the purpose that student would think about the unique existence of the values of $c$ and $d$, despite what the resulting vector would be (K in the Figure 7a), which they reason it during the Tasks IIq) and IIr), respectively, from the Figure 7b) and 7c).

With the aim that students would write the definition of linear combination of vectors, the following discussion section took place.

25 Inv: In what vector space is the result of the sum of the scalar product vectors $\left( \frac{3}{2}, \frac{-2}{2}, \frac{1}{3} \right)$?

26 E1: In $\mathbb{R}^2$ (and added), every time a sum of the scalar product vectors is done the result has to be $[s]$ in the same vector space that we are referring to [the same vector space that the scalar product vectors are in, object association, Mason, 2008; Duval, 2003].

27 Inv: With the experience you have gained when using Geogebra or the traditional pencil-and-paper way, what do you think is the outcome gained due to de adding of the scalar product vectors?

28 E2: That we could make adjustments [–In Geogebra– of each scalar product vector that is included in the addition] in such a way that it outcomes a resulting vector, but this new vector belongs to the same space from which was generated [same space as the individual vectors that were summed], it has to belong to the same space [object association, Mason,
29 \( E_1 \): The resulting vector belongs to the same space as the vectors that are being summed [\( E_2 \) interrupts].

30 \( E_2 \): Due to the nature of the two vectors in a vector space, the sum of them gives a vector that is in the same space [vector space].

The work in the previous section of the interview (lines 25 to 30) permitted the students to recollect sufficient supporting arguments to allowed them to accurately interpret that the sum of the scalar product vectors of any number of vectors, of a certain vector space, is another vector of the same vector space [the students makes sense of what he wants to know, Mason, 2008; Duval, 2003, see Figure 8].

Figure 8. Interpretation by the students of the result of summing the scalar product vectors.

Following, the \( I_{nv} \) directed the interview with the idea that students would accurately define the concept of linear combination of vectors, and asked, according to the Task IIdd) (see Figure 9), What is a linear combination of vectors? \( E_1 \) answered:

31 \( E_1 \): When the sum of the scalar product vectors result in a [new] vector of the same vector space.

32 \( I_{nv} \): What is the relation between a sum of the scalar product vectors and the concept of linear combination of those vectors?

33 \( E_2 \): The linear combination of vectors … (\( E_1 \) interrupts).

34 \( E_1 \): Is the sum of the scalar product vectors.

35 \( I_{nv} \): According to the experience gained by doing the Activities in Geogebra and by using pencil-and-paper, define what is a linear combination of vectors?

36 \( E_2 \): Is a resulting vector in that same space (\( E_1 \) interrupts).

37 \( E_1 \): Is the resulting vector of the sum of the scalar product vectors, which is located within the same space.

38 \( I_{nv} \) \( I_{nv} \) [Directs towards \( E_1 \) and asked] What \( E_2 \) said (line 36) contradicts what you mentioned (line 37)? (\( E_2 \) answers).

39 \( E_2 \): No, because in here it also says that the sum of the scalar product vectors is what we have been working on, which would outcome in a resulting vector […] but now it would not be called sum of the scalar product vectors, instead it would now be called linear combination of those vectors [being summed] [the student makes sense of what he/she wants to know, Mason, 2008; Duval, 2003].

After the above discussion took place (lines 31 to 39) and with the experience gained by working in an environment using Geogebra and pencil-and-paper, \( I_{nv} \) asked the students to define the concept of linear combination of vectors. The response of the students is showed in Figure 9.

Activity II ended with Task IIdd) shown in Figure 9. With the this Activity along with Activity I allowed the students to gain more skills to be able to address Activities that would direct them to the concept of set generated by vectors of different vector spaces which were discussed in Activity III.
**Activity III: Set generated by vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$.** From a list of vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ that the students wrote in Activity III, the students sketched in Geogebra the sum of the scalar product vectors from that list. Following $I_{nv}$ asked:

40 $I_{nv}$: What set of vectors generate two scalar product vectors $\mathbb{R}^2$?

41 $E_1$: The straight line that contains the vectors [cognitive position, Mason, 2008; Duval, 2003].

42 $I_{nv}$: What are the characteristics of an echelon form matrix composed of two vectors either column or row of $\mathbb{R}^2$ that are scalar product vectors?

43 $E_2$: The echelon form matrix has a row of zeros.

The purpose of the $I_{nv}$ in asking those question was so that the students would relate the characteristics of the echelon form matrix of a set of $n$ vectors of $\mathbb{R}^n$, $n \geq 2$ to the space generated by the set of vectors. Following, it was studied the vectors that are not scaled product vectors $\left(1\begin{array}{r}3\end{array}\right)$ and $\left(3\begin{array}{r}2\end{array}\right)$ aiming that the students would determine to which set of vectors generate these vectors. For these reason, it was asked to the student to work on a file in Geogebra where the sketched vectors where present (see Figure 7a). Then, according to what the students where observing on the computer screen, $I_{nv}$ asked:

44 $I_{nv}$: If the vector is located in the interior of the parallelogram (OBKD in Figure 7a), Can a linear combination of the vectors $\left(1\begin{array}{r}3\end{array}\right)$ and $\left(3\begin{array}{r}2\end{array}\right)$ that is that vector [vector cu and dv in Figure 7a] be found?

45 $E_2$: Can the scalar product vectors be adjusted so that the parallelogram would be the vector k, even if the vector is found in the limit of the parallelogram and even if is out of the parallelogram.

Figure 10a) shows the end result of adjusting a couple of times the scalar product vectors of $\left(1\begin{array}{r}3\end{array}\right)$ and $\left(3\begin{array}{r}2\end{array}\right)$ in such a way that if adjusting them an “infinite” number of times the union of the parallelogram would “cover” the plane $xy$; in Figure 10b) the students justify the reason why these vectors generate the plane $xy$ [cognitive position, Mason, 2008; Duval, 2003].
10a)

Figure 10. The students interpreted the set generated by the vectors \( \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \) and \( \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \).

By doing Task IIIq) shown in Figure 11 the students were able to determine (Column F, Task IIIq) the set generated by two vectors of \( \mathbb{R}^2 \) in terms of the echelon form matrix (Column D, Task IIIq). Under such conditions, it allowed the students to classify the vector subspaces of \( \mathbb{R}^2 \) (see Figure 12).

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<thead>
<tr>
<th>A</th>
<th>Vectors</th>
<th>B</th>
<th>Matrix</th>
<th>C</th>
<th>Echelon form matrix</th>
<th>D</th>
<th>Linear combination cu + dv</th>
<th>E</th>
<th>Set generated by the vectors “u” and “v”</th>
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<tr>
<td>2</td>
<td>“u”</td>
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Figure 11. Task IIIq), vector subsets of vectors of \( \mathbb{R}^2 \) defined as a function of the echelon form matrix.

10b)

Figure 12. List of vector spaces of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
From the results generated in Task IIIq) (see Figure 11) the students wrote a list of vector subspaces of $\mathbb{R}^2$ (see Figure 12a) and similar Activities of set of vectors of $\mathbb{R}^2$ leading the students to write a list of the subspaces of $\mathbb{R}^2$ (see Figure 12b). However when $I_n$ asked the students about the subspaces of $\mathbb{R}^4$, $E_2$, answer: $\mathbb{R}^4$ and the set that contains the zero vector (the trivial subspaces of $\mathbb{R}^4$). Despite the lack of graphic representations of vectors of $\mathbb{R}^n$, $i \geq 4$, the students associated (Mason, 2008; Duval, 2003) the set generated (subspace of $\mathbb{R}^4$) with the number of zero rows of the echelon form matrix initially composed of vectors either row or column of set of $n$ vectors of $\mathbb{R}^n$. This way, $I_n$ requested from the students to explained: Which are the characteristics of the echelon form matrix of four vectors of $\mathbb{R}^4$? And what are their corresponding vector subspaces of those vectors. They answered the following:

46. $E_1$ y $E_2$: [The echelon form matrix] does not have rows of zeros and it generates $\mathbb{R}^4$; it has a row of zeros, generates a tri-dimensional vector hyper-space [that passes through the origin]; it was two rows of zeros and it generates a hyper-plane [that passes through the origin]; if it has three rows of zeros and it generates a straight line [that passes through the origin] and if all its rows are zeros and generates a zero [zero vector].

Figure 13a) shows how the students determined the subspaces generated by four vectors of $\mathbb{R}^4$ starting from the number of zero rows of the echelon form matrix of these vectors, while in Figure 13b) it is requested to the students to determine a set of vectors that would generate a tri-dimensional hyper-space of $\mathbb{R}^n$.

![Figure 13] Subspace generated by four vectors of $\mathbb{R}^4$ and a set of vectors of $\mathbb{R}^n$ that generates a subspace of this vector space.

**Conclusions**

The available data collected from the Activities completed by the Team 4 along with the data analysis allowed us to infer a partial answer to the question: How does the use of the DGS (e.g., Geogebras) influences in the learning of the concept of vector subspace of $\mathbb{R}^n$, $i \geq 4$? The data analysis indicates that when students implemented the use of technology, they were able to improved learning [to relearn in a deeper thinking way] the concept of vector subspace generated by vectors of $\mathbb{R}^n$. The improvement was reasonable for $i = 2$ and $i = 3$, but if $i \geq 4$ then the improvement in their learning was only partially achieved.

By implementing the use of the echelon form matrices allowed the students to determine the vector subspaces generated by vectors of $\mathbb{R}^n$, $i \geq 4$. Via this way, we could observed that technology partially promotes the learning of the concept of subspaces generated by vectors $\mathbb{R}^n$ and that the pencil-and-paper environment, extends the understanding of the concepts of subspaces of $\mathbb{R}^2$ and of $\mathbb{R}^n$. However, there still are difficulties in the understanding of this concept when students try to extrapolate the concept for sets of vectors of $\mathbb{R}^n$, $i \geq 4$. 

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REFERENCES


Expanding contexts for teaching Upper Secondary school mathematics

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Abstract: This paper describes how the reading of a literary work—The Sand Reckoner, concerning the work and life of Archimedes—created the opportunity of an expanding learning space. Tools and resources of different knowledge domains (funds of knowledge) came together to transform traditional classroom practices where students were actively involved in (re)negotiating their own learning processes as well as their conceptions of mathematical discourse. The research analyzes a year-long, interdisciplinary, didactical intervention based on a teaching experiment methodology; 10th grade students in a public school in Athens brought together different funds of knowledge and Discourses that coalesced to both destabilize and expand the boundaries of official school Mathematics Discourse.

Résumé: Le présent article décrit comment la lecture d’ une œuvre littéraire -Le Reckoner du Sable, concernant les travaux et la vie d’Archimède- a créé la possibilité d’ un espace d'apprentissage en expansion. Des outils et des ressources des différentes domaines de connaissance (fonds de connaissance) se sont réunis pour transformer les pratiques pédagogiques traditionnelles où les étudiants ont été activement impliqués à la (re)négociation de leur propre processus d’apprentissage ainsi que leurs conceptions de Discours mathématique. La recherche analyse une intervention didactique interdisciplinaire annuelle, basée sur la méthodologie de l’ expérience pédagogique. Les élèves d’une classe de seconde, d’ un Lycée public d’Athènes, ont rassemblés des différents fonds de connaissance et des Discours dont la synthèse a déstabilisé et repoussé les limites de la Discourse Mathématique de l'école officielle.

Introduction

Moje and his colleagues (2004, p. 41) argue that the active integration of multiple funds of knowledge and Discourse is important to support youth in learning how to navigate the texts and literate practices that are necessary for ‘survival’ in secondary schools. In what comes to be called a ‘hybrid’ or ‘third’ space, these seemingly oppositional categories can actually work together to generate new knowledge, new Discourses, and new forms of literacy (Moje, Ciechanowski, Kramer, Ellis, Carrillo & Collazo, 2004). A number of studies have examined the integration of the literacy practices and texts of different domains of knowledge, funds of knowing, and Discourse in Science learning in school (Barton & Tan, 2009; Barton & Tan, 2008; Basu & Barton, 2007; Moje 2004; Moje 2001); others have examined ways of constructing a ‘third space’ for improving mathematics teaching/learning (Razfar, 2012; Flessner, 2009; Thornton, 2006).

In our paper we claim that introducing the reading of literary works, performing arts and other similar practices in teaching Geometry has the potential to create a hybrid (third) space, with new tools and new Discourse—a blend of standard and non-standard mathematics Discourse—where a richer repertoire of students’ participation possibilities is enabled.

The study

Our aim in this article is to discuss how a group of adolescent students engaged in the cross-curriculum project, “In the footsteps of Archimedes,” through the reading of the literary work, “The Sand Reckoner,” which concerned the work and life of Archimedes. Students were encouraged to communicate mathematics through performing arts (using mind and body) and through a variety of practices connected to reading literature. These classroom experiences challenged students’ stereotypic images of what constitutes Mathematics knowledge and Mathematics learning (as
outcome as well as procedure) through the realization of the relative connections among the disciplines.

In this project we explored the following main research questions:

How can the reading of a literary work in the Mathematics classroom create a (hybrid) third space in which students can renegotiate the dichotomy of the Discourses of Science and Humanities? How might students’ experience of the expanded mathematical space transform their conceptualizations of mathematics and motivate their participation?

Data selection

The use of ethnographic research techniques (i.e. participant observation and interviewing) helped us to gather empirical evidence concerning students’ experiences; the majority of students’ activities were also videotaped and analysed. Semi-structured interviews aimed to explore how students themselves perceived and processed their experience of participating in the project through mind and body. Our data collection was completed with a questionnaire given to the students at the end of the project implementation.

The Project in practice

The project was carried out for one whole school year in a State Lyceum in an inner city school in Athens, Greece, with one class of 10th grade students (12 girls and 12 boys), the participation of 2 teachers, and the teacher librarian. Throughout the project, 28 different thematic teaching interventions and activities were carried out, most of which lasted 2-4 teaching periods. Nearly every week, teaching sessions with different themes were held according to the references that were in the chapter of the book that the students regularly used and that they had read prior to the class activity. In this way, the teaching of different mathematical topics was been presented as the elaboration of meanings constructed during the reading of the book. A mixing of different tools coming from different contexts were used, challenging dichotomies such as body-spirit, formal-informal learning, listening-doing, etc.

The whole investigation was integrated at the end of the school year in a performance addressed to students’ parents—an application of their learning further motivated their participation.

In the next paragraph some of the teaching issues concerning Mathematics are presented, followed by the practices and tools that were exploited. In almost all sessions, different worksheets were used.

- Measurement in Antiquity: The value of our decimal system. The system of naming large numbers from the work, ‘The Sand Reckoner,’ by Archimedes: revision of the properties of powers. (Dramatisation)
- The Unsolved Geometric Problems of Antiquity (an introduction to cubic routes). Neusis construction: Proposition 8 from the Book of ‘Lemmas’ by Archimedes (demonstration of instruments for the solution of the geometrical problems, reading relevant extracts from the “Parrot’s Theorem” by Denis Guedj.
- Resolving mathematical problems with the use of physics principles. Finding the area of a parabolic section and the volume of a sphere by Archimedes (documentary film about Archimedes’ Palimpsest)
- Patterns in mathematics (activity “Balancing mobiles” from the book “Mathematics from History, The Greeks” of M. Brading)
- The relationship of Mathematics and Physics with music (Power Point presentation, ‘Rhythm and numbers’, music by cello)
- Graphs of functions, the slope and the rate of change of a function (“Fortune line”: a graph to show the fluctuations in the intensity of feelings of one of the novels’ heroes)
- Mathematics in our life: ‘Maths in society’, ‘Women mathematicians’
‘Maths and Nature’, ‘Maths and War’, ‘The wonderful world of fractals’
(Thematic exhibition, ‘Radio broadcasts’)
• The moral and political responsibility of the modern scientist.
(‘role-playing roundtable discussion’)
• Applied and pure Mathematics (expressive reading of extracts from the second ‘Socratic dialogue’ by the Hungarian mathematician A. Renyi)

**Discussing the results**

**Students’ experiences**

Both from the responses we got in the semi-structured interviews with the pupils, and from the observation during the whole process, it seemed that the pupils were motivated within these expanded contexts, and that they became creative and cooperative through their engagement in new teaching practices. Students referred positively to the whole project and highlighted that the whole process was beneficial not only at a cognitive but at an affective level as well. The following quotes are indicative:

• ‘The activities widened my knowledge’,
• ‘It made me learn how to do a research’,
• ‘It gave us the opportunity of expressing our creativity and the special skills each student has’,
• ‘The round table discussion motivated me a lot. ‘It was fun’
• ‘The spirit of cooperation was strengthened’

Because our sample was small for quantitative analysis, we are attempting to present some data just to enhance our qualitative, anecdotal evidence: Eighteen from the twenty-three students would like to repeat similar activities in the following year, and thirteen consider these activities could be a regular part of the teaching process. Pupils considered that they gained from the whole program: the skill of cooperating 69,5%, knowledge 54%, abilities 43,5%, self-confidence 50%, critical ability 37,5%, creativity 37,5%. They also found teachers cooperation effective (14) and interesting (17).

To the question, ‘which activities did you like more and why?’ twenty of the twenty-three students chose the ones with Drama-in-Education techniques. The students defined these activities as being original, interesting, and different. Their main reason for identifying these activities was the possibility of working together (7 answers) while at the same time expressing their creativity. Moreover, students were surprised to have the chance to use both mind and body in doing mathematics.

**The experiential character of learning:** Student interviews revealed that Drama-in-Education techniques were the tool that mediated mathematical knowledge by offering the experiential dimension, motivating students to become active learners who owned their own learning (November, 2012). ‘You think the same way as the hero of the book...’ ‘I was anticipating the experiential activities with greater interest than the other ones’.

**The role of creating the third space:** Moje et al. (2004) speak about hybrid space where different funds of knowledge and Discourses coalesce to destabilize and expand the boundaries of official school Discourse. In our classroom intervention this view was affirmed, as demonstrated in the following student quotes:

‘I was impressed the way all disciplines were integrated’,
‘A whole world opened up that is waiting to be explored’,
‘We managed to make connections between Mathematics and civilization and to learn that everything around us is connected to Mathematics’.

**The nature of mathematical knowledge:** In the responses we got in the interviews with the pupils, the impact on their perceptions of mathematics is emphasized rather than the cognitive level: ‘It
made me realize that Mathematics is a living entity’. ‘It is not only what we learn at school but something more beautiful and attainable’.

This further reveals the differences between student and teacher perspectives on the learning activity; whereas the teacher is focusing on the cognitive outcomes, the students are experiencing the third space in terms of aesthetics and emotions (Appelbaum & Scott Allen 2008).

In conclusion: The use of a literary book in school mathematics teaching, as well as the fusion of varied mediation tools and disciplinary discourses, may be understood in terms of the contextual, affective, and attitudinal approaches to a curriculum where mathematics may be humanized. These experiences also enable opportunities for incorporating unconventional and informal practices that generate a third space where the different Discourses establish a dialogue; Students in this third space have the opportunity to renegotiate both mathematics concepts and their own, personal perception of what constitutes mathematics knowledge. Most importantly, students in the third space can understand this perception as their own perception of mathematics, one they could carry with them into future studies.

REFERENCES


Objectification of the concept of variation about the quadrature problem

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Abstract: in this article we report the results obtained when implementing Activities related with the concept of variation about the quadrature problem with high-school students (grade 11) in Mexico City using paper-and-pencil and technological environments (e.g., GeoGebra). This is a qualitative research and is supported by the Theory of Objectification (Radford, 2006, 2008, 2014). The data collection was done by video-recording the students’ work while solving the Activities, worksheets and software generated files with GeoGebra. Our results show that the use of paper-and-pencil and technology promote the objectification of the concept of variation among students.

Résumé: dans ce papier nous reportons les résultats issus lors d’implémenter, avec des étudiants d'école secondaire 5 (degré 11) dans la ville de Mexico, au Mexique, des Activités liées avec le concept de variation sur le problème des quadratures, dans les environnements papier/crayon et ainsi que technologique (e.g., Geogebra). Cette recherche est de type qualitatif et elle est appuyée sur la Théorie de l'Objectivation (Radford, 2006, 2008, 2014). La collecte des données nous l'avons faite par vidéo-enregistrement lorsque les étudiants ont été en train de résoudre les Activités, feuilles de travail et ainsi que des fichiers générés par le software Geogebra. Nos résultats nous suggèrent que l'utilisation de papier/crayon et ainsi que celle de la technologie favorisent l'objectivation du concept de variation chez les étudiants.

Background and research problem

Educational mathematics research on the concept of variation (e.g., Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Vasco, 2006; among others) shows that the teaching of topics involving variation and change is commonly done in a traditional way through algorithms (using only paper-and-pencil) and lacks visual and geometric arguments. In this article we seek to answer the question: How does solving tasks [quadrature problem] designed in paper-and-pencil and technological environments influence the process of objectification of the concept of variation in the students?

Theoretical framework

This study is supported on the Theory of Objectification (Radford, 2006, 2008, 2014) founded on the concepts of labour, knowledge, knowing, and learning. This theory considers learning as a social process that goes back and forth between knowledge and the self. Through labour the subject transforms knowing into objects of conscience to give room to learning; however, such transformation (knowledge mediation) does not occur in an isolated way; it demands a labour together with the other, and during this labour not only is knowledge transformed but also the subject who acquires it, the one who learns (Radford, 2014, p. 138). One of the aims of the Theory of Objectification is to make evident how the subject learns the cultural knowledge through social interaction and semiotic means of objectification (Radford, 2003, 2005) as signs (written, verbal or gestural) and artifacts, fundamental sources of meaning production (Radford, 2006).

Indeed, the objects, the tools, the linguistic resources, and the signs which the subjects use intentionally in the processes of signification to carry out their actions and reach their goals constitute the so called semiotic means of objectification (Radford, 2003, p. 41). Among these, the gesture stands out, particularly the one made with the hands which may move in time and space with the purpose of conveying ideas. This goal is not always achieved with either written or spoken language.
Method

This is a qualitative research. The participants were 12 high-school students (ages 16 and 17) from different classrooms (grade 11) couring Analytic Geometry in a Mexico City school. The students were chosen by their teacher taking in account their good performance in mathematics. They were divided in six teams of two members each and were video-recorded while they solved the Activities. All the Activities were solved in paper-and-pencil and technological (GeoGebra) environments. The data collection was done through video-recording, work sheets and software generated files. Due to limitations of the workspace, in this article we only report the work of one team solving two of the Activities.

Data analysis and result discussion

The discourse made by the students during the development of the Activities implemented is a way of social interaction, a social practice; a reflection influenced by artifacts, either material or cognitive, such as gestures, language and objects, among others (Radford, 2006). It is a joint labour in which both the knowledge and the subject that acquires such knowledge are transformed. Therefore, in order to carry out the data collection, this article considers the relationship between the different ways in which the students clarify and convey ideas with language and artifacts while they discuss and reflect together (Radford, 2014), as well as the students’ written answers in each Activity.

Here, Activities A1 and A2 are described; we show the excerpts of the discussion and the reflection held by the students E1 and E2 from Team 1 when solving the activities (A1 in a paper-and-pencil environment, and A2 in a technological one with GeoGebra), and the data obtained from their work is analyzed.

Description of Activity A1 (paper-and-pencil environment)

In Figure 1, the straight lines $l_1$, $l_3$, and $l_5$ are parallel to the straight lines $l_2$, $l_4$, and $l_6$, respectively, while H is the midpoint of $EG$.

(a) Use paper-and-pencil to prove that the area of:

(i) $\triangle ABCDE$ and the area of $\diamond CDEF$.

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$\triangle ABCDE$ refers to any pentagon whose vertices are the points A, B, C, D, and E.

$\diamond CDEF$ refers to any quadrangle whose vertices are the points C, D, E, and F.
(ii) \(\triangle CDEF\) and the area of \(\triangle DEG\),

(iii) \(\triangle DEG\) and the area of \(\text{EHML}\) are equal.

(b) Provide convincing arguments to demonstrate that the areas of all these polygons are equal.

Development and analysis of Activity A1

(a) (i) Equality of the areas of \(\bullet \triangle ABCDE\) and \(\triangle CDEF\)

E1: Uhm, these \([\bullet \triangle ABCDE\ and \ \triangle CDEF; \ Figure \ 2a]\) have areas in common [points with his finger at \(\triangle BCDE; \ Figure \ 2b]\) and these [points with his finger at \(\triangle ABE\ and \ \triangle FBE]\) have the same height.

E2: Yes. [E1 immediately starts writing the proof shown in Figure 3a].

Figure 2. In (2b), E1 uses a gesture to explain to E2 why the areas of \(\bullet \triangle ABCDE\) and \(\triangle CDEF\) shown in (2a) are equal.

Figure 3. (3a) Written proof by E1 and E2 that the areas of \(\bullet \triangle ABCDE\) and \(\triangle CDEF\) are equal. (3b) English translation of answer given by E1 and E2 in section (a) (i) of A1.

(a) (ii) Equality of the areas of \(\triangle CDEF\) and \(\triangle DEG\)

E1: This area [points with his finger at \(\triangle CDE]\) is common to both [points with his finger at \(\triangle CDEF\ and \ \triangle DEG]\) and these two [points with his finger at \(l_2\ and \ l_4]\) are parallel. [See Figure 4a]

E2: Oh, yes. That’s right.

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\(\triangle DEG\) refers to any triangle whose vertices are the points D, E, and G.

\(\text{EHML}\) refers to any rectangle whose vertices are the points E, H, M y L.
E1: Then these two triangles [points with his finger at \(\triangle CEF\) and \(\triangle CEG\)] are equal.

E2: Yes, this point [he refers to the intersection point of the side \(\overline{CF}\) of \(\triangle CDEF\) and the side \(\overline{EG}\) of \(\triangle DEG\)]…

E1: Mmh…

Next, E2 refers to the common region between \(\triangle CDEF\) and \(\triangle DEG\). If \(P\) is the intersection point of the side \(\overline{CF}\) of \(\triangle CDEF\) and the side \(\overline{EG}\) of \(\triangle DEG\); then, the region common to both polygons is \(\triangle CDEP\).

E2: Yes well, when we cut the common [he refers to \(\triangle CDEP\)], well, not to this [he refers again to \(\triangle CDEP\)], but to this [he refers to \(\triangle CDE\) and points at it with his finger]. [See Figure 4b]

E1: Yes, this little triangle [points with his finger at \(\triangle CDE\); Figure 4b], now the others [he refers to \(\triangle CEF\) and \(\triangle CEG\)] have… Ok… [He means that the areas are equal].

![Figure 4](image_url)

**Figure 4.** E1 and E2 use gestures; in (4a) to identify the parallels \(l_4\) and \(l_3\) related with \(\triangle CDEF\) and \(\triangle DEG\), and in (4b) to identify the region common to these polygons.

Now, the students write down their answer on paper (Figure 5a).

![Figure 5](image_url)

**Figure 5.** (5a) Written proof by E1 y E2 that the areas of \(\triangle CDEF\) and \(\triangle DEG\) are equal. (5b) English translation of the answer given by E1 and E2 in section (a) (ii) of A1.

(a) (iii) Equality of the areas of \(\triangle DEG\) and \(\triangle EHML\)

E1: Now, for \(\triangle DEG\) and \(\triangle EHML\) (Figure 6a). This, as they told us, is the midpoint [he refers to \(H\), but at the same time points with his fingers to the points \(E\) and \(H\); Figure 6b] well yes, it’s done. [He means it is enough with that; it is proven.]

E2: Oh, yes, it is the same. Because this and this are parallel [points with his finger at \(l_6\) and \(l_5\). Right away, E1 starts writing down the proof shown in Figure 7a]…
Figure 6. E1 uses gestures to explain to E2 the equality of the areas of $\triangle DEG$ and $EHML$.

Figure 7. (7a) Written proof by E1 and E2 that the areas of $\triangle DEG$ and $EHML$ are equal. (7b) English translation of the answer given by E1 and E2 in section (a) (iii) of A1.

(b) Provide convincing arguments to prove that the areas of all these polygons are equal.

E2: Well, is one is equal to the other, and the other is equal to the other, well that’s it… *He means that each and every area involved is equal to the others*.

E1: Ok

Figure 8a shows the written answer the students provided for this section.

Figure 8. (8a) Written answer in section (b) of A1 provided by E1 and E2. (8b) English translation of the answer given by E1 and E2 in section (b) of A1.
During the development of this Activity, the students talk using colloquial language which is inaccurate in occasions while, in their written answer, they combine symbolic and colloquial language in an adequate manner. In the three sections of (a) in this Activity, E1 and E2 use a sign (gesture) to provide clarity to their explanation and justification as to why the areas of the polygons in question are equal, as well as to distinguish the other polygons involved in their speech. They discuss and reflect (social interaction) adding previous knowledge related with the area of a triangle and with the properties of parallel straight lines (as a result of their knowledge mediation), which they turn into action from the start (as shown in the dialogues and in Figures 3 to 7). The moment E1 and E2 point with their finger at different points, straight lines, and regions in Figure 1 to convey their ideas and justify their answers gives sense to the equality of the areas, first in (a) (i) and (a) (ii) with an addition (see Figures 3 and 5), and then in (a) (iii) by pointing out that the segment $EH$ is half of $EG$ (see Figure 7).

**Description of Activity A2 (technological environment)**

The students open the GeoGebra file in which Figure 1 of Activity A1 has been reproduced and they manipulate the dynamic construction. In this Activity the students are required the following:

(a) Explain in a clear manner how it is possible to find a square of equal area to this polygon [quadrature of the polygon] from any polygon given [regardless of the number of sides].

(b) Argument why the equality of the areas of the polygons is maintained every time one of their sides is eliminated, and why their perimeter varies.

(c)

**Development and analysis of Activity A2**

(a) Explain in a clear manner how it is possible to find a square of equal area to this polygon [quadrature of the polygon] from any polygon given [regardless of the number of sides].

E2: Of course, they don’t change their proportions *[he means the equality of the areas of the polygons while E1 manipulates the construction]*.

E1: No, because everything moves on the parallel lines *[see Figure 9]*. Mmh, how is it possible to find a square of equal area from a polygon?

E2: From any, right? *[He means any polygon]*.

E1: Well yes, It is true for any, that is regardless…

E2: As long as their proportionality is maintained *[he means the areas of the polygons]*.

E1: Ok, we may get a square from any polygon, but how do you explain that?
E2: That means, we could say that it may be found with the analytical process of equalizing the areas and finding the variables…

E1: Yes, but [...] is right it is true for any polygon [...] Oh, ok, then it is possible…  
[To construct a square of equal area to that of the given polygon while approximating the given rectangle to a square; Figure 9c]. It is the same, isn’t it? That as long as it does not go through any of its sides [as long as the sides of a polygon do not cut each other] the pull [of any vertex of \(\bullet ABCDE\)]...

E2: Yes, as long as it [any vertex of the polygon] does not extend infinitely and touch the parallels [of any vertex of the given polygon] its proportionality is maintained [he means the equality of the areas of all the polygons].  
[See Figures 9 and 10]
Figure 10. (10a) Intersection of the sides of the polygons. (10b) Some overlapped parallels. (10c) Polygons degenerated into straight lines.

E1: Their proportions are not lost unless their sides [of a same polygon] are crossed. [See Figures 9 and 10]

Next, the students start writing down their answer on paper (see Figure 11). E1 and E2 explore the construction; pull each and every of the vertices of the pentagon $ABCDE$ over all the work area; they observe at moments that some of the polygons are deformed until they are confused with either others (convex or otherwise) or a straight line or point while two parallels get as close as they want or if they separate or get close to the vertices of $ABCDE$ as much as they want (Figure 10). They say that if certain rules are not followed, it is possible that the proportions of the polygons are lost and that their areas are not equal; however, they do not validate such conjecture, which is not entirely true since disconnected polygons maintain their areas. Nevertheless, in general terms, the students use GeoGebra as a semiotic means of objectification since with it they think, reflect and act to provide a response to this section (see Figure 11).
(b) Provide argumentation as to why the equality of the areas of the polygons is maintained every time one of their sides is eliminated, and why their perimeters vary.

E1 moves the dynamic construction (see Figure 12) and continues the dialogue.

E1: Yes, these [areas] when certain properties are fulfilled...

E2: Because it doesn’t matter the sides it [the polygon given] has, the properties of such figures [he means the other polygons], despite their sides, will match the proportionality between them [he means the proportions between the sides of the polygons]. And their perimeters change because they are ultimately based on the number of sides. [See Figure 12b]
E1: It’s not all the properties [the ones maintained], it’s just that the area doesn’t change.

Next, the students start writing down their answer on paper (see Figure 13). They manipulate the dynamic construction and argument that the equality of the areas of all these polygons is maintained if certain properties of the polygons are fulfilled, like proportionality, regardless the variation in the number of sides. However, this explanation is ambiguous. Using the software, they clarify, explain, and justify that the perimeter of each polygon depends on the number of its sides, so the perimeter of each polygon varies and its area remains constant. These facts are observed in the dialogue between the students, in Figure 12, and in the written answer they provided (Figure 13); it is evident they resort to the software to (dynamically) strengthen what they had studied in Activity A1. Above all, they understand that the perimeter of every polygon varies even when the areas of all are equal.

Figure 13. (13a) Written answer provided by E1 and E2 in section (b) of A2. (13b) English translation of the answer provided by E1 and E2 in section (b) of A2.

Overall, in sections (a) and (b) of Activity A2, E1 y E2 discuss and reflect (social interaction) with the aid of software. They add both the knowledge used and those acquired in Activity A1 (as a result of the mediation of their knowing) which they set in play again while they manipulate the construction with the software (as a means of semiotic objectification) to give sense to the equality of areas in all the polygons involved, and to understand how to find a square of equal area to the given polygon (Figure 11a) and realize what varies and what remains constant in the given construction. In fact, the students use GeoGebra as a semiotic means of objectification since using it they think, reflect, and act to solve this Activity and provide an answer to what is asked in every section.

Conclusions

In the activities designed for this study, in the pencil-and-paper environment, the test is carried out from a known hypothesis through a series of logical and valid reasoning using a given figure (fixed) until reach a result that confirms it; likewise, there is the possibility of raising new hypotheses and make conjectures. GeoGebra allows us to reason, argue and create new hypotheses and conjectures differently than in the static environment formulated. The dynamic construction can be manipulated using the drag tool and slippers, to better understand the geometric behavior of the object of study and justify the statement given. However, you should not ignore the work with paper-and-pencil, because that accounts for the knowledge that students bring into play, and how they organize them in written replies when they meet each of the activities through the use of colloquial and formal language.

The gestures used in Activity A1 allowed the students to generate and clarify ideas to provide arguments for their proofs; while they elaborated conjectures with the software used in Activity A2, they understood why it is possible to find a square of equal area to that of a given polygon and they managed to objectify the concept of variation [through a process] when they say
that the perimeter of every polygon varies because it depends on the sides and that the only thing that remains unchanged is the area.

The design of the Activities (first, with paper-and-pencil, and then with technology) promotes the learning of the concept of variation inherent in the quadrature problem because the dynamic nature of the software (GeoGebra) allowed the students to validate or reject the conjectures made around a drawing on paper, to find properties that are not easy to detect in a paper-and-pencil environment, and to formulate new conjectures as when they say that the areas of the polygons are maintained as long as the sides of the figures do not cross or overlap each other and they do not grow infinitely or get reduced to a point. Our results suggest that the use of paper-and-pencil and technology promote the objectification of the concept of variation in students.

REFERENCES


Visual Strategy and Algebraic Expression: Two Sides of the Same Problem?

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Abstract: It has been advocated that the search for patterns and their organization in mathematical language is a central component of mathematical thinking. Hershkowitz, Arcavi and Bruckheimer (2001) investigated problem solving processes of a "visual-pattern-problem" which start with visual strategies for reorganizing the visual pattern and ended in a formal algebraic expression as a solution. In this study we had decided to use the same problem for investigating also a process in an opposite direction: starting from a given algebraic expression, analyzing it in a way that uncovers the visual strategies which are behind the algebraic components of the given expression.

Background

Many questions about the different roles of visualization in mathematics have been addressed in the last few decades of mathematics education research (Arcavi, 2003). Researchers have studied the ways in which children develop ways to describe visual patterns gradually grasping the basic concepts of algebra. Among the ways are the use of computer software that enables to perform visual manipulations in building visual patterns (Healy & Hoyles, 1999), and also software that enables the constructing of a model by visualizing sets of patterns gradually replaced by numbers and variables (Mavrikis, Noss, Hoyles, & Geraniou, 2013).

Hershkowitz et al. (2001) used a rich visual task (Figure 1) in order to invoke as many visual solution strategies as possible, that were categorized according to the visual-counting strategies used by the different solvers.

How many matches are needed to build the following nxn square?
Find as many strategies as you can

Figure 1: nxn square made of matches

This problem is representative of a whole class of “counting” situations in which the solutions' main
steps can be described as follows: observation, recording and understanding of regularity, finding and applying a “visual-counting” strategy, generalizing and capturing the generalization in a symbolic-algebraic form (see Figure 2).

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<td>1.</td>
<td>$4 + 3(n-1) + 2(n-1)(n-1)$</td>
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<td>$\frac{4n^2 + 4n}{2}$</td>
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<td>3.</td>
<td>$2 \cdot 2(1 + 2 + \cdots + n)$</td>
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Figure 2: Algebraic solution examples to the visual problem

The algebra expresses the visual-counting-strategies, and compresses them into expressions. This emphasizes that even though the problem is visual, without using algebra we would not have been able to write the solution in a compact way.

**Methodology**

Participants in the study were eight mathematics education graduate students from different programs in Israel. The participants took part in a group's activity that has two parts. In the first part participants were asked to individually find an algebraic expression that would represent the number of matches needed to build the nxn matches' square shown in Figure 1.

In the second part, participants were asked to go in the opposite direction; they were given six different algebraic expressions that represent solutions for the number of matches in the nxn square. The participants were then asked to choose an expression, to find a visual strategy for finding the number of matches in the nxn square, where the algebraic expression they choose is its solution. Then they had to explain to their peers the visual strategy they had found. Data sources included the individual written report, written by each student along his/her work's process, video documentation of the group activity, and field notes.

The analyses and interpretations of the students' reports were based to large extent on the findings of Hershkowitz et al. (2001). Next, we focused on categorizing the different characteristics of the visual strategies which are hidden behind the given algebraic expressions.

**Results**

Our initial analysis revealed three main characteristics in uncovering the visual pattern strategy which is the origin of the algebraic expression as a whole or its components. In the following we will illustrate these characteristics as they appeared.

**The dual role of numbers in the algebraic expressions - a quantity and a visual-construction-unit**

While trying to uncover the visual strategy which was the origin to algebraic expressions, we found out that the most popular component in the matches' square problem was "4". In some cases it was standing for a quantity. For example, when dealing with the expression: $4 \cdot \frac{n(n + 1)}{2}$, student C referred to the "4":

C: "Where does this four come from? I now need to decompose the, to decompose it to four parts. … The most natural way for me to get to four parts is to look at each triangle (One quarter of the square while divided to four parts by its diagonals).

In a different situation, student D was addressing the "4" as something else:

D: "… and I saw n four (nx4). OK?"
Visual counting strategies behind arithmetic operations

Arithmetic operations within the algebraic expression might serve as hints for the visual counting strategies upon which the expression is constructed. For example, the division by 2 operator in the expression: \( \frac{4n^2 + 4n}{2} \). This expression was the product of the visual counting strategy which was called by Hershkowitz et al. (2001), Shake and count. In this solution some of the visual constructs counted "shared" matches which are thus counted twice, so the solution process has to take into account fixing this double counting by division. When describing his way of constructing a visual solution suitable for this expression, student B referred to the operator:

"We have here four n squared, plus four n, divided by two. So first the division by two gives me a hint that something is counted here twice and at the end we divide".

Ambiguity in visual representations of algebraic expressions

One expected outcome was that each algebraic expression had several suggested visual solutions that could be described with it. This ambiguity manifested, for example, in the "3" included in the expression: \( 4+3(n-1) \cdot 2+2(n-1)(n-1) \). This expression was visualized commonly by what Hershkowitz et al. (2001) categorized as from one square on…As described visually in figure 3, and explained by student A:

A: The first square [upper left] has four sides, four matches, and here we have n minus one [showing left column], and here we have n minus one [showing top row], and each one of these, we add three. …

[background] Wait. I did not understand. Which three matches?
A: This has four matches [upper left square]. Here we have n minus one squares. n minus one sides, like, we multiply by three. One, and two, and three [counting sides of U pattern marked in red in figure] one-two-three, one-two-three
Instructor: All right.
A: And also these [showing the turned U figures that make up the top row]. So we have four, plus two times n minus one times three.

For this case, although the U and the ㄷ patterns are visually different, it took only one minor clarification for all of the students to accept that number "3" stands for them both, thus not explicitly giving any information about the visual solution apart from this part of the expression is constructed from three matches.
Expression form which is familiar as representing numerical counting strategy

Some expressions are familiar as the sum of numerical sequence. For example, the expression: $4 \cdot \frac{n(n+1)}{2}$. This expression was called by Hershkowitz et al. (2001) Starting with symbols. In this case, participants identified this expression as four times the sum of $1+2+3+\ldots+n$. This resulted in many instances of an arithmetic series that could be identified in the pattern in various ways. Two of which appear in Figure 4.

![Figure 4: Two instances of $1+2+3+\ldots+n$ (marked by dashed lines) in the $n \times n$ square of matches](image)

This exemplifies how visual reasoning can be guided, inspired and supported, a posteriori, by a symbolic expression known to be the solution. Therefore, the visualization process may not only involve the decomposition into units or the creation of auxiliary constructions, it may also be guided by a known symbolic result.

Concluding Remarks

Can a visual strategy be inspired by a symbolic expression? Our data show that yes, students might be familiar with the visual problem, and then may "see" the visual-counting strategy or even strategies in the certain algebraic expression given to them.

It is worth to note that this thinking direction is much more complicated than the opposite direction described at Hershkowitz et al. (2001) paper. At the previous paper the solver is going from the visual problem to look for appropriate visual solution's strategies, and at the end transform them into algebraic expressions. In this paper the solver has to go back and forth in tiny steps looking for hints that evolve from the components of the algebraic expressions.

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Teaching the derivative in the secondary school

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Abstract: In this paper I want to focus the attention on a specific mathematical content of the secondary curriculum: the derivative concept. Drawing on the work done in my PhD thesis, I outline two moments of the didactic transposition (Chevallard, 1985) of this notion in the secondary school. More precisely, I focus on the introduction of the derivative through the problem of the tangent to a generic function within two textbooks and in the case of a teacher in her grade 13 classroom. The aim is identifying how the local dimension intervene in the work on the function that has to be differentiated.

Introduction

In the secondary teaching, the derivative is one of the first and fundamental concepts of Calculus. It evokes and calls into question competences, notions and registers which are proper to the algebraic or the geometrical domain. In particular, it involves functions and their properties, limits and also geometrical objects such as the tangent line. At the same time, the derivative permits to solve several problems belonging to the Calculus domain, such as optimization problems, zeros approximation methods, primitives of functions, and many others. Therefore, the introduction of the derivative notion represents a crucial node for students, and also for teachers.

In Italy, this moment occurs at the last year of upper secondary school (grade 13). As I could verify working with some teachers during my PhD, the derivative notion is considered as a cornerstone in the mathematics curriculum of the last year of secondary school, with relevant applications to physics. Their experience shows teachers that learning the rules to differentiate a function is quite simple and automatic in terms of computation. Nonetheless, conceptualizing the derivative as a mathematical object, and in particular as a function, for then reemploying it as a tool (Douady, 1986), may not reveal so immediate. One of the aspects that make this process difficult is fostering, on the teachers’ side, and grasping, on the students’ side, a local dimension in the work on the function that has to be differentiated.

With a particular interest in the teaching practices with the derivative concept in secondary school, I articulated my study around the following research question: how does the local dimension intervene in the development of derivative-related practices? In particular, I focused on the role given to the local work on functions in the intended curriculum (mathematics to be taught) and in the implemented curriculum (taught mathematics) when such practices are introduced.

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7 The expressions intended and implemented curriculum were introduced by the Second International Mathematics Study (SIMS) in the 70s-80s, along with that of attained curriculum, which consists in “the mathematics that the student has learned and the attitudes that the student has acquired as a result of being taught the curriculum in school” (Mullis & Martin, 2007, p.11).
Theoretical framework

This study is grounded on the Anthropological Theory of the Didactic (ATD), which has been elaborated and disseminated by Chevallard during the last thirty years (Chevallard, 1985, 1992, 1999; Bosch & Gascón, 2006). My focus is on the didactic transposition of the derivative concept in the secondary school context. The didactic transposition is a process that “starts far away from school, in the choice of the bodies of knowledge that have to be transmitted. Then follows a clearly creative type of work – not a mere ‘transference’, adaptation or simplification –, namely a process of de-construction and rebuilding of the different elements of the knowledge, with the aim of making it ‘teachable’ while keeping its power and functional character.” (Bosch & Gascón, 2006, p.53).

Within such a frame, I coordinate three theoretical lenses coming from three different theoretical approaches. In order to describe the intended and implemented teaching practices, I refer to the notion of praxeology, which is central in the ATD. A praxeology consists of a type of task, a technique to solve it, the justification that such a technique is efficient and the theoretical arguments that support this justification. For instance, among the praxeologies related to the derivative concept, it is particularly relevant the one that allows determining the equation of the tangent line to a generic function at a point. The peculiarity is that this praxeology has been already practised with conics in grade 9-11, with techniques and justifications whose validity is not extendable to a generic curve. From the teachers’ point of view, reworking this praxeology permits to introduce the derivative as a fundamental tool to solve the type of task in the generic case. Thus, we distinguish two planes:

- the mathematical praxeology that has to be constructed around the type of task of determining the equation of the tangent (T\textsubscript{tangent} in the following);
- the didactic praxeology that consists in the teacher’s organisation and management of the development of the mathematical praxeology, through different didactic moments.

I am referring here to the model of the didactic moments elaborated by Chevallard (1999) distinguishing different (but not ordered) steps in the construction and the practice of a praxeology that determine the teacher’s didactic praxeology. More precisely, I am interested in

- the moment of the first (significant) meeting with the task;
- the technical moment, where a technique is developed or at least an embryonic form of it;
- the technological-theoretical moment, where the justifications for the techniques are formulated and grounded on a specific theory.

Working on T\textsubscript{tangent} entails introducing a local regard on the function that has to be differentiated. A key question is: how this local dimension is introduced?

In order to detect if and how a local work is done on the function, I use the lens of the perspectives (Rogalski, 2008; Maschietto 2002; Vandebrouck, 2011), that are different ways to regard a function while working on it. We can recognise that a certain perspective is adopted on a given function f if certain properties of f are exalted. One can be interested in a pointwise property of f that is valid at a specific point (e.g., f(2) = 4, x=3 is a zero of f). In this case, the enhanced perspective on the function is pointwise. Moreover, one can consider the function as a whole object or some global properties of it that are valid in a given interval (e.g., f is even, f is increasing in [0,1]). In this case, the enhanced perspective on the function is global. In addition, and this is the case of the differentiability property, one can concentrate on a local property of f that is valid in a neighbourhood of a given point (e.g., f is discontinuous in x=2, f has a maximum point in x=1). In this case, the enhanced perspective on the function is local, in the sense that it highlights property of the function that are valid on a family of neighbourhoods that contain the given point. It is not enough to know what value the function takes at that point, and it is not necessary to choose a particular interval: a local property is valid for an infinity of open intervals containing the point.
In order to recognise the perspective adopted on a function, it is certainly important to know what has been said or written about the function. Nevertheless, if we apply this lens for analysing teachers or students working on functional objects, we realise that it is the combination with other semiotic resources, different from the oral or written speech, that actually inform us of the adopted perspective. Therefore, I consider the semiotic bundle (Arzarello, 2006) as a third lens for analysing the semiotic resources activated by the teacher and the students during the work with functions, and their mutual relationships. The semiotic bundle is defined as a bundle of semiotic sets (speech, gestures, sketches, drawings, symbols, …), their internal relationships, and the coordination between two or more resources that are simultaneously active. The great variety of semiotic resources that can be activated by the teacher or the students while working on \( T_{\text{tangent}} \) can exploit different registers of representation (algebraic, symbolic, graphical, etc.) on functions and reveal or hide a particular perspective on them. When different semiotic resources (e.g., speech and gestures) converge to underline the same perspective on a function, this unity may enhance such a perspective and foster its activation. However, it is also possible that two or more different semiotic resources simultaneously active highlight different perspectives on a function.

Through the presented theoretical framework, I approach my research question in the following terms: how the derivative-related praxeologies are constructed and how the local perspective intervene in this process?

**Methodology**

In this paper, as I said above, I focus on the transposition of the derivative in the intended curriculum and in the implemented one. I concentrate especially on scientific high schools, where we can suppose to find a more intensive local work on functions. As for the intended curriculum, I consider in particular the praxeology for determining the equation of the tangent in two of the most widespread textbooks in Piedmont (Turin region): Bergamini et al. (2013) and Sasso (2012). As for the implemented curriculum, I propose the case of a teacher dealing with the introduction of the derivative in her grade 13 classroom. It is one of the three case studies that I observed and analysed for my PhD thesis. The teacher used one of the analysed textbooks. I interviewed her before entering the classroom about her usual practices and plans for the lesson. I chose to present this particular case because something in the lesson led her to modify her usual practices. The interesting unexpected outcome is that her didactic transposition of the derivative did not turn out to be a transposition of the textbook one.

**Textbooks analysis**

As far as \( T_{\text{tangent}} \) is concerned, in the analysed grade 13 textbooks, we can recognise the didactic transposition of one of the “scholarly” definitions of differentiable function (see for example Bramanti et al. 2000, p.171)

\[
\text{DEF. Let } f: (a,b) \rightarrow R. \text{ We say that } f \text{ has derivative at } x_0 \text{ in } (a,b) \text{ if } \\
\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \exists \text{ and is finite. This limit is called first derivative of } f \text{ at } x_0 \text{ and it is denoted with } f'(x_0). \\
\text{The straight line whose equation is } y = f(x_0) + f'(x_0)(x-x_0) \text{ is called tangent line to the graph of } f \text{ at the point } (x_0, f(x_0)).
\]

Indeed, at the beginning of the chapter on the derivative within both Bergamini et al. (2013) and Sasso (2012), the following phases are developed.

First of all, the problem of the tangent to a generic function is introduced and the tangent line is seen as the limit of a dynamic secant line or a sequence of secant lines (see Fig. 1).
Fig. 1: The tangent as the limit of a sequence of secants (Sasso, 2012, p. 259).

Thus, firstly, the gradient of the secant line is found and then the limit as \( h \) goes to 0 is applied in order to obtain the gradient of the tangent line:

\[
m_{\text{secant}}: \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = m_{\text{tangent}}: \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}
\]

Finally, the derivative is defined as this limit if it exists and is finite.

Basing the analysis on the perspectives, we can notice that in a first phase pointwise and global perspectives on the function are activated with the work on the secant line. Then, the limit symbol is introduced and the local perspective suddenly intervenes.

Through the lens of the semiotic bundle (here we have words+graph+symbols), we can detect a potential, but rather implicit, activation of the local perspective. The local perspective on functions is potentially activable by a student who disposes of this material. Nevertheless, it is difficult without any mediation to correctly establish the relationships among the words that dynamically describe the static graph on the page (e.g., “Q is approaching P”, “as Q gets closer and closer to P”) and the introduced symbol: \( \lim_{h \to 0} \).

**Case studies analysis**

Two of the observed teachers transpose the textbook transposition in their classroom. As it happens on the textbooks, they resort pointwise and global perspectives on the graph of a generic function, by using the secant line as an intermediary. Then, the introduction of a local dimension is delegated to the limit symbol, which is justified through terms of movement, such as “Q is approaching P” or “as Q gets closer and closer to P”.

I propose the analysis of a different didactic transposition: the case of V. In the preliminary interview, to the question “How do you introduce the derivative notion?”, V. answers that she usually starts with the tangent line definition. Indeed, in classroom she immediately poses the problem of defining the tangent to a generic function at a given point. In particular she asks to students: “Which properties must a tangent line have?”. An open discussion arises and the students, as V. expected, recall all the operational definitions and techniques they used with conics. Within the model of the didactic moments, we can recognise that this first phase of the lesson is devoted by the teacher to a technological-theoretical discussion around the problem of the tangent in the case of a generic function. The main concerns indeed consist in defining the mathematical object they are working with and explaining why all the conics-related techniques are no longer successful. V.’s intention, as she declares before the lesson, is disarming the students of all their previously used techniques and introducing the derivative as a tool to solve the generic type of task \( T_{\text{tangent}} \). Nevertheless, the discussion in classroom produces an unexpected but correct definition of tangent. Let us analyse the key moments of the discussion through the lenses of the perspectives and of the semiotic resources.
The first definition proposed by a student (S1) is: the tangent intersects the curve at a single point.

1 S1: [the tangent] must intersect [the function] at a single point.

2 S2: […] But, if it is so, not all the points has a tangent line … I’m imagining a sloped function (tilting his hand) then maybe the tangent line in that case could intersect the function in another point, right?

3 V: […] So, are you thinking of something like this? (She sketches the curve in Fig. 2)

4 S2: Yes, there is the tangent line but it touches other points of the function.

5 V: For example, if I search for the tangent line here? (She points at the maximum point on the curve, see Fig. 2) How do I imagine it?

The first definition (line 1) enhances a pointwise perspective on the function, exalted by the pointwise speech indicators “at a single point”. Leaning on S2’s global remark (line 2), V. proposes a graphical non-example (line 3, Fig. 2). Her global sketch of a whole section of the graph on an interval contextualises her pointwise pointing gesture on it (“here” in line 5, Fig. 2), in order to foster the students to look at the whole graph of the function, in a global perspective.

Another student (S3) proposes to localise this definition by adding “in a suitable interval”, but the definition remains pointwise.

6 S3: To avoid what S2 said, we can take a suitable interval (moving his two indexes up and down together, as in Fig. 3) where the tangent line satisfies our conditions.

7 V: So, we limit the zone.

8 S3: At that point, if I want a tangent line to a point in that interval, I can do it without any other intersection of the line in that interval.

9 V: […] So, we take a point, wherever we want, this one (x₀, y₀), we limit to a suitable neighbourhood (she sketches the situation at the white board, see Fig. 4) and what do we require there?

10 S4: There, that the line intersects [the function] only at that point.

11 S5: It is not enough.

12 V: It is not enough, why? […] It could do so (she draws the situation in Fig. 5).

With his words (line 6), S3 makes a limiting gesture with his fingers (Fig. 3) that the teacher reproduces at the whiteboard as two vertical lines around the point (Fig. 4). Although the student’s gesture (Fig. 3), strengthen by the teacher’s sign on the graph (Fig. 4), has already the intention of enlarging the pointwise perspective of the first definition, the speech indicators at this stage are global for the student (“a suitable interval”, line 6, “in that interval”, line 8) and local only for the teacher (“a suitable neighbourhood”, line 9). The proposed definition (line 10) amended by “there” that means “in that interval” falls again in the pointwise perspective. The local handholds are too weak to make the students enlarge their perspective. However, S5 recognises that this property is not enough (line 11) and, leaning on his intervention, V. makes a graphical counterexample that exalts the pointwise character of the given definition (Fig. 5).

Fig. 2: V.’s non-example to the first definition. Fig. 3: S3’s gesture for “a suitable interval”.

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The third proposed definition is pointwise: the tangent is the perpendicular to the radius of the circle, to which one of the students locally adds “the circle that best approximates the curve”. The teacher then recognises that it can be a correct approach, but technically too difficult for them.

Thus, another student (S6) further proposes: the tangent must all lie in the same region of plane.

13 S6: It must all lie in the same half-plane, except for the point.
14 V: What do you mean?
15 S6: A function detects two half-planes.
16 V: Yes. They aren’t half-planes, but regions of plane.
17 S6: Ok. And the straight line must always lie in the same region of plane.
18 V: Yes. Always?
19 S6: In the interval (measuring a short distance with his hands, as in Fig. 6)
20 V: Locally. All we are saying is only local (she sketches two vertical lines on the white board) [...] Ok, S6. And if I draw a function like this (she sketches the curve in Fig. 7) and I ask you to find the tangent in this point (indicating the inflection point). Is there the tangent in that point or not?

S6 formulates the global definition, speaking about half-planes, then corrected in regions of plane by V. (lines 13-17). Prompted by the teacher (line 18), he adds “in the interval” (line 19) making the same limiting gesture as before (see Fig. 3 and Fig. 6). V. again converts this gesture in two vertical lines on the whiteboard, but she strengthens the local feature of this shared sign, accompanying her sketch with the words (“Locally. All we are saying is only local”, line 20). Then, she proposes a local graphical counterexample where the tangency point is an inflection point (Fig. 7). This graph not only fosters the students to reject the claimed property (line 24), but also evokes the case of $y = \sin x$ and $y = x$ (line 21-22). In the previous months, V. has made the students work on remarkable limits like $\sin x$ over $x$ going to 1 as $x$ goes to 0, by speaking in terms of asymptotic equivalence and supporting graphically this property.
Thus, finally, a student proposes a correct local definition: the tangent is the straight line that best approximates the given curve in the neighbourhood of the point, and S6 writes an equality at the whiteboard (Fig. 8). We can interpret it as a previous technique (linked to the asymptotic equivalence property and the remarkable limits), re-employed at the level of justification in relation with the given definition.

Fig. 8: Equality proposed by S6 to express that \( f(x) \) and \( mx+q \) are asymptotically equivalent.

V., confronted to this unexpected further development by the students, says: “It is an approach that I have never tried before. Let’s try together now”. The definition and the equality given by the students lead to the target technique, but through a technology that the teacher has not pre-prepared. V. works on S6’s justification and a graphical-symbolical reformulation of the type of task allows her to find the right technology from which the gradient of the tangent \( m \) can be deduced. In particular she applies a vertical translation to the \( x \)-axis of the vector \((0, f(x_0))\) which permits her to compare the infinitesimals \( CB \) and \( AB \) as \( x \) goes to \( x_0 \) (see Fig. 9). She accompanies this action on the graph by saying:

25 V: Why it [the equality proposed by S6, Fig. 8] doesn’t give me the idea of asymptotic estimate? Because the asymptotic estimate is valid for infinitesimal quantities, which go to 0. Thus, here first of all I need an indeterminate form 0 over 0, the two quantities must go to zero, and then I compare the speed with which they go to zero.

By expressing \( CB \) and \( AB \) in symbols, she can finally write the equality in Fig. 10.

Fig. 9: V.’s graphical reformulation of \( T_{\text{tangent}} \).

Fig. 10: The right technology to find \( m \).

With the lens of the praxeology, V.’s local words, graph activity and symbols (Fig. 10) can be interpreted as the justification for the technique for finding \( m \). Such a justification is supported, at the theoretical level, by the local asymptotic equivalence property. Within the model of the didactic
moments, this second phase can be interpreted as the strict interrelation between the technological-theoretical moment and the moment of elaboration of a technique for determining the gradient $m$ of the tangent and then for finding its equation. Starting from the theoretical definition of tangent and from the asymptotical equivalence property, the technology is formulated using the graph (Fig. 9), the speech (“the asymptotic estimate is valid for infinitesimal quantities, which go to 0”, “an indeterminate form 0 over 0, the two quantities must go to zero”, line 25) and the symbols (Fig. 10) in order to find the target technique for $T_{tangent}$:

$$tg: \ y - f(x_0) = m(x - x_0) \ \text{where} \ m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The complete mathematical praxeology for $T_{tangent}$ constructed by V. and her students is summed up in Table 1.

<table>
<thead>
<tr>
<th>$T_{tangent}$</th>
<th>Determining the equation of the tangent to a function $f$ at the point $x_0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Technique</td>
<td>$tg: \ y - f(x_0) = m(x - x_0) \ \text{where} \ m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$</td>
</tr>
<tr>
<td>Technology</td>
<td>Among all the straight lines passing through $(x_0, f(x_0))$, the tangent is the one that best approximates the function. The infinitesimal quantity $f(x) - f(x_0)$ is asymptotically equivalent to the infinitesimal quantity $m(x-x_0)$ (see graph in Fig. 9). The condition $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x-x_0)} = 1$ is then satisfied. Since $mm$ is a constant we can bring it out of the limit sign, obtaining: $1 \ \lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x-x_0)} = 1 \iff m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.</td>
</tr>
<tr>
<td>Theory</td>
<td>- Definition of tangent as the straight line that best approximates the curve in the neighbourhood of a point.</td>
</tr>
<tr>
<td></td>
<td>- Property of being asymptotically equivalent.</td>
</tr>
<tr>
<td></td>
<td>- Analytic equation of a straight line.</td>
</tr>
<tr>
<td></td>
<td>- Limit theory.</td>
</tr>
</tbody>
</table>

Table 1. Mathematical praxeology for $T_{tangent}$.

**Discussion and conclusion**

Analysing this classroom experience allows us to retrace the didactic moments of the development of a mathematical praxeology for determining the equation of the tangent. It comes out a didactic transposition of the derivative that is not transposed from the textbook transposition. The local perspective on the function that has to be differentiated is present in the speech (e.g., “the asymptotic estimate is valid for infinitesimal quantities, which go to 0”, line 25). It is marked by the students’ gesture (see Fig. 3 and 6) and by the teacher’s sign at the whiteboard (see Fig. 4). This is a sort of *semiotic game* (Arzarello & Paola, 2007) where

“the teacher uses one of the shared resources (gestures) to enter in a consonant communicative attitude with his students and another one (speech) to push them towards the scientific meaning of what they are considering” (Arzarello & Paola, 2007, p.23).

In the case of V., the relationship between the semiotic resources is more complex. Indeed, the teacher exploits one of the shared gestures, but without repeating it. In recalling it, she changes the semiotic resource, converting the gesture into the written sign “| |” and accompanying it with a meaningful mathematical speech, which prompts the students from a global perspective on “the interval” to a local perspective on “the neighbourhood”. This semiotic game is an important strategy of the teacher for enhancing the local perspective. She starts from the way the students
refer to an interval without specifying if their perspective is global or local on the function, and constructs on it the reasoning within a local neighbourhood.

A turning point for the work in the classroom is represented by the teacher’s local counterexample of the tangent at the inflection point (Fig. 7). Through the graphical sign, V. unconsciously evokes in the students’ mind the case of \( y = \sin x \) which is asymptotically equivalent to \( y = x \) as \( x \) goes to 0. The recalling of this property and of the related praxeology leads the students to propose a completely local definition of the tangent enriching the theoretical base. In addition, one of the student proposes a hint of technology supported by such a theoretical base, introducing a possible symbolic formalisation. It is from this moment on that the teacher starts manipulating symbols. The local perspective, which has been gradually developed and strengthen by the definition of tangent, is thus transferred to symbols \( \lim \) and \( x \to x_0 \) that are proposed by the students and borrowed by the local praxeology of the remarkable limits. The local dimension on the generic function \( f \) is conveyed by the reasoning in a neighbourhood, which is inherent in the idea of best linear approximation.

In the technological part of this praxeology, the justifying speech is centred on the definition of the tangent as the best linear approximation of the function in a neighbourhood of the point. There is no allusion to pointwise and global aspects of the function referring to a secant becoming tangent or to global increments that has to be reduced. The local perspective on the function permeates each part of the praxeology.

In conclusion, this didactic transposition of the derivative notion could represent a challenging but also powerful alternative to the traditional scheme, whose procedure appear sometimes obscure and artificial for students.

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Using of the Cartesian plane and gestures as resources in teaching practice

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Abstract: in this article we report how a secondary school physics teacher in Mexico City used the Cartesian plane and gestures as resources to promote the understanding on the concept of reference frame among his students (between ages 16 and 18). The teacher’s use of resources arose from a question posed to him by one of his students while solving a problem linked to the concept of acceleration. This is a qualitative research supported by the theory of use of resources as a means of reflection during teaching practice in the classroom. This theory is linked to the documentational genesis (Gueudet & Trouche, 2009; 2012), and was used in the analysis of our data. The data collection was done by video-recording of the class when the physics teacher worked with his students. Our results suggest that even this physics teacher has difficulties understanding the concepts related with the movement of objects.

Résumé: dans ce papier nous reportons comment un enseignant de physique d'une école secondaire dans la ville Mexico, au Mexique a utilisé le plan cartésien et ainsi que des gestes comme ressources pour favoriser la compréhension de leur étudiants sur le concept de système de référence. L'utilisation de ressources par l'enseignant ont émergé à partir d'une question posée à lui par un de leur étudiants pendant la résolution d'un problème lequel est lié au concept d'accélération. Cette recherche est de type qualitatif, et elle est appuyée sur la théorie autour l'utilisation de ressources comme moyen de réflexion pendant la pratique de l'enseignant en salle de classe. Cette théorie est liée à celle de la genèse documentaire (Gueudet & Trouche, 2009, 2012), et elle a été utilisée dans l'analyse de nos données. La collecte de données a été faite par vidéo-enregistrement de la classe lorsque l'enseignant de physique travaillait avec leur étudiants en salle de classe. Nos résultats nous suggèrent que même cet enseignant de physique a des difficultés pour comprendre les concepts liés au mouvement des objets.

Introduction and research problem

Several works on mathematics practice agree in the need of paying close attention to the use of resources; in understanding what they are and the way in which they work as extensions of the teacher in school practice (Adler, 2000). This author points out that the resources used in teaching practice need to become a focus of attention (p. 221). Besides, she notes that the resources are not restricted to material objects. She classifies them in human (teachers, parents, and teacher’s knowledge, among others), material (textbooks, calculators, and mathematical objects like the Cartesian plane, among others), and socio-cultural (language). Guzmán and Kieran (2013) state that the way in which resources support teachers or not in their efforts to solve problems in class, clearly has an impact in the students’ experience in problem solving. For his part, Radford (2012) uses the term artifact to point out the relevance of researching their use and understand their influence in the teaching and learning processes.

Starting from the importance in the analysis of the teachers’ use of resources in the classroom when solving problems, in this article we seek to answer the question: How are the non-physical resources [representations and gestures] used by the teacher so that his students give meaning to the reference frames?

Conceptual framework

Gueudet and Trouche (2009) propose a theoretical approach similar to the one proposed by Adler (2000) concerning resource conceptualization. For instance, they do not consider resources merely as those coming from material objects, but as all those that take part in the understanding and
solving of a problem. In their proposal, these authors make the difference between resources and documents. So, documents are developed through which they call “documentational genesis.” Documentational work is the core of the teachers’ activity and of their professional development.

In the documentational genesis, the documents are created from a process in which the teachers build schemes of utilization of the resources for situations within a variety of contexts. The process is exemplified in the equation: document = resources + schemes of utilization (Gueudet & Trouche, 2009, p. 205). The schemes cover particular rules of action and are structured through the uses and the operational invariants during the activity. The uses correspond to the observable part of the scheme, which happens during the teacher’s action. In contrast, the operational invariants correspond to the cognitive structure that guides the teacher’s action. Then, the schemes are only observable through the actions the subject carries out when working with the resources (Gueudet & Trouche, 2009).

These authors use the term resource to emphasize the variety of artifacts that they consider and, at the same time, that an artifact (physical or psychological) is a cultural and social medium provided by human activity (e.g., computers and language); produced with specific purposes (e.g., problem solving, Gueudet & Trouche, 2009). During the subject’s activity with the use of artifacts, there are two processes: instrumentalization and instrumentation; the first one occurs when the subject takes over the artifact and determines the way it is used. Instrumentation is the influence of the artifact over the subject’s actions [activity] (Gueudet & Trouche, 2009).

So, in documentational genesis, one of the objectives is to conceptualize the teacher’s activity goal-oriented, considered as social activity. Such consideration of the activity leads to paying attention to the social contexts of the different groups (Gueudet & Trouche, 2012) in which it is present. This way of conceptualizing the activity is linked to the interest of the authors in mediation and mediating artifacts. Hence, the theoretical proposal by Gueudet and Trouche (2009) is related to other research whose approach is semiotic mediation (Mariotti & Maracci, 2012; Radford, 2008; Arzarello, 2006, among others). Mariotti and Maracci (2012) consider an approximation over the teacher’s role and describe how the use of artifacts when doing activities may be increased. They also analyze how the use of resources is related to their function as a tool of semiotic mediation. Radford (2008) takes the mediated characteristic of the thought in Vygotsky’s sense to refer to the role of the artifacts during social practices. Vygotsky (quoted in Kozulin, 2000) considers that the tools and the psychological signs, like language, are used to control one’s activity and to influence on others’ activities; so, we think with and through cultural artifacts (Radford, 2008).

In one of his research works, Radford (2009) proposes that mathematical thought is not only mediated by written symbols but also by actions, gestures and other types of signals. Then, thought is produced as well through a sophisticated semiotic coordination of voice, body, gestures, symbols, and tools (Radford, 2009). In the same approach, Arzarello (2006); Arzarello, Paola Robutti and Sabena (2009) consider gestures as part of the resources activated in the classroom: speech, registers records, objects, etc. Hence, gestures are a resource (semiotic tools) used by both the students and the teachers in teaching and learning mathematics.

**Method**

This is a qualitative study. The research was carried out in a physics laboratory of a high school in Mexico City (grade 12) and the participants were: the physics teacher, who practices since over 30 years, and 11 students (ages 16 to 18). The students (grouped in teams of 3 and 4 members) carried out an experiment regarding a ball falling down an inclined plane to get distance-time Cartesian graphs and to interpret them in terms of the physics concept of reference frame. All the participants were video-recorded during the (five) sessions; they were given worksheets to guide the activities. After the five sessions, the videos of every session were analyzed besides gathering and analyzing the worksheets; no interviews with the participants were conducted. In order to fulfill the purpose of
this research, we show excerpts of the moment (last session) when the teacher talks with Peter (pseudonymous). In this dialogue, they talk about the concept of reference frame, which arises from a question by another student (E1) related to the concept of acceleration. Data analysis is focused on the use of resources (gestures and concept of reference frame) by the teacher to answer to the student’s question.

**Analysis of the resources used by the teacher**

We present some excerpts of the dialogue between the teacher [Prof.] and Peter [P.] after another student (E1) in Peter’s team asked the teacher a question related to negative acceleration. We show moments during which the teacher uses the resources of the concept of reference frame and the use of gestures in his statements.

**E1:** Teacher, what would be an example of negative acceleration?

**P.:** It is that deceleration is to stop accelerating, that is, that a mobile stops little by little.

**E1:** What would be an example of negative acceleration? [Addressing the teacher].

**Prof.:** It is only called deceleration until the moment you reach zero.

**P.:** Oh! Ok.

**Prof.:** Yes? You are decelerating until you reach zero. Which is related to negative acceleration. It’s only that negative acceleration doesn’t stop, it does not stop there. It goes on, right? For example, when you throw an object, it decelerates.

**E1:** And it will have a negative acceleration if you pass zero.

**Prof.:** It always has a negative acceleration; it always has a negative acceleration, it’s just that deceleration, you finish when it stops [the object]. And here it keeps on acting so that now [he is interrupted by E1 who says: “it is negative”]. Then, the body goes up, but our acceleration, negative. Then, it stops [when it reaches its highest point], but because the acceleration continues, it goes down [he means that the body changes direction with respect to the initial one].

At first, a student (E1) asks the teacher a question related to what negative acceleration is. Here, the teacher retakes what Peter said: “It is that deceleration is to stop accelerating”; Peter tried to relate the negative acceleration [concept] with a known physical phenomenon; however, he [the teacher] is unclear in his explanation on the existing relation between deceleration and negative acceleration; regardless, he says: “Which is related to.” The relation that the teacher establishes is as follows: when the teacher says “You are decelerating until you reach zero”, he means the moment in which the mobile stops. When he says “It’s only that negative acceleration doesn’t stop, it does not stop there. It goes on…” he refers to negative acceleration (concept) obtained from a reference frame. That is, the teacher does not clarify how a concept such as negative acceleration may be used to analyze movement (deceleration). In his speech, the teacher implicitly argues that, from a reference frame, the direction of the object goes to where the acceleration has been determined to have negative sign. However, accelerations with positive sign (positive accelerations) may occur and the mobile would still decelerate (slow down), depending on the reference frame chosen by the observer; hence, the importance of choosing an adequate reference frame to analyze a certain movement of objects. In a textbook (Giancoli, 2006) used as support during the development of the courses in this school level, the author expresses the care that must be taken for not conceiving that deceleration necessarily means acceleration is negative. “When an object slows down, sometimes it is said that it is decelerating. But we must be careful: deceleration does not necessarily mean that the acceleration is negative.” (p. 25). In reality, it means that the magnitude of the velocity decreases. Later, he provides an example and says: “when you throw an object, it decelerates”. Here, the teacher does not clarify whether he is making reference to a vertical throw. We infer it is
about that movement when he says: “Then, the body goes up […] Then, it stops, but because the acceleration continues, it goes down.” The teacher’s example corresponds now to a movement that has a behavior in two directions (when it goes up and when it goes down), but in which the orientation of the reference frame that shows the negative acceleration down is constant. He also fails to link the deceleration with the changes in velocity of the object. From that moment, the teacher’s objective is that the students understand how the sign (positive or negative) of physics quantities (e.g., acceleration) are established; to do so, he will need to use the concept of reference frame. Until this point, we do not observe the teacher using any resource besides language. The dialogue with the students continues, and the teacher retakes the experiment done by the students on an inclined plane:

**Prof.:** In the inclined plane [Referring to the experiment done by the students], where did you consider the positive goes to? [P. asks: “come again?”], where did you consider the positive goes to in the inclined plane? [The students get nervous] Yes, you considered positive and negative, right? [P. answers: “yes”] Where was the positive going to?

**P:** Uhm… positive, well, what I managed to understand was that the positive was taken from, well from the part, how do you say? From the uppermost length, well the centimeters that were from the distance to the floor to the hundred and twenty centimeters that were on the practice. And from that point to the floor, well to where the rail [referring to the inclined plane] ended was, uhm, the plane.

The teacher uses the physics concept of reference frame as a resource (of the mathematical object type) to interpret the physics phenomenon. However, he uses it in an unclear way. The reference frame involves a direction (orientation) and a mathematical origin (which may coincide or not with the phenomenological origin). Here, the teacher refers only to the direction when he asks “where […] the positive goes to?”, but Peter seems to refer to the (mathematical) origin of the reference frame, which he relates to the phenomenological origin, that is “from the part”. This lack of understanding between the teacher and Peter may be due to the spontaneous way in which the teacher introduces the concept of reference frame. What is more, the moment when the teacher asks: “Where did you consider the positive goes to?”, the other team members shy away from the conversation. Besides, the teacher provides no reason as to why he asks; at no moment he mentions “reference frame” either even when his objective is to use said resource so that Peter understands how the sign of acceleration is established [considered]. Attention should be paid when Peter says “what I managed to understand” since he refers to the way in which the experiment was conducted; while the teacher intends to carry out a conceptual analysis which guides the rest of the discussion with Peter. The discussion continues:

**Prof.:** Where did you consider the positive goes to? [After Peter remains thinking, the teacher continues] The orientations, positive, negative, are arbitrary, right? And if you take the positive goes up, that does not mean that the ball goes up. And if you take the positive goes down, that does not mean that the ball goes up. And if you take the positive goes diagonally, that does not mean that the ball goes up. The ball has its behavior and that’s it [P. says: “yes”], right? The other is the interpretation [P. says: “Oh, ok.”] Right? Where do you take the positive goes to?

**P:** Well, uhm, in the moment when the ball goes down.

Within the resource of reference frame, the teacher focuses his attention on the observer (when he says: “they are arbitrary”), who chooses the conventions to measure [interpret] the physics phenomenon (inclined plane), but he is not explicit about the student being the observer in the

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8 In the experiment, the students video-recorded the movement of an object (tennis ball) on an inclined plane. Later, they used the video and with the aid of software they obtained tables of values for the experiment; later, they sketched the graphs (with Excel) that represented the movement of the ball.
experiment. Thus, we see the use of the resource by the teacher (Gueudet & Trouche, 2009). For his part, now Peter focuses his attention on the phenomenon and starts linking it to the data collection. Nevertheless, he does not perceive the relation between the experiment (physics phenomenon) and the reference frame. It is here when the teacher uses another cultural-semiotic resource (gestures); i.e., a resource used intentionally by the teacher to achieve an objective (Gueudet & Trouche, 2009) and that at the same time, is used in a social process of meaning creation through the teacher’s actions in a specific time and space (Arzarello, 2009; Radford, 2008) observed in the following excerpt.

Prof.: Where do you consider the positive goes to? Not the moment [when], but where to? [P. whispers to E1: “Help me!” see Figure 1] I let this go [He takes an object (mousepad) and placed it at a distance above the table; Figure 2a] what is going to happen to it? [P. says. It’s going to fall down] [Prof. Lets the object go; see Figure 2b] Where do you consider the positive goes to? [P. thinks] Then, I help you. I consider the positive goes down. Then, this body [referring to the object he is picking up at the same time and then lets fall down], did its distance increased or decreased? [P. does not answer].

From the excerpt above, we observe the social dimension of the moment when something more than a negotiation of meanings takes place. (Radford, 2008) concerning the resource the teacher is using. Peter feels nervous and uncertain when he says: “Help me!” (see Figure 1). After what Peter answered what he considered a reference frame when he said “what I managed to understand” (in the second excerpt), we found a moment when he could not answer to the teacher’s question. Therefore, the interaction between Peter and the teacher plays a key role to mutually recognize the institutional meanings addressed. This social interaction affecting the individuals is decisive in Peter’s learning process (Radford, 2008) and also in the way the teacher conveys knowledge.

Later on, the resource of gesture is used by the teacher to represent the physics phenomenon and he expects for Peter to understand the concept of reference frame; the cognitive part that guides the teacher’s action (Gueudet & Trouche, 2009). In the teacher’s gestures, there is only the representation of the physics phenomenon (Figures 2a and 2b), and with language he refers to the concept of reference frame when he says: “I consider the positive goes down.” However, Peter seems to only observe the gesture without its meaning; what he observes is the phenomenon of the object falling down, the visible part of the resource (Adler, 2000). When the teacher says: “I consider the positive goes down”, there is no element of either orientation or origin of the reference frame in his gesture. That is why when asking whether the distance increases or decreases, the teacher does not realize he needs an origin from which the distance of the object increases. In the following excerpt the teacher modifies his resource by reflecting.
Prof.: What is the distance? [He picks up the object again; Figure 2a] [P. says: Uhm… at X centimeters] Zero, ok. Zero. What is the distance? [He lowers the object a little with respect to the highest point at which he had held it; see Figure 3a] Zero was here [he points with his finger at where the object was at the beginning; see Figure 3b].

P.: Yes, uhm… we could say minus one or something like that.

Prof.: No, I cannot assign minus one. […] Positive goes down [He places the object back where he had placed it at the beginning and lets it fall down], I say, arbitrarily. What is the distance? [He places de object below the initial point (zero for the teacher); see Figure 3c].

Reflecting upon the resource he is using, the teacher (Gueudet & Trouche, 2009) realizes that he needs an origin from which the distance can be measured, so he adds a new gesture (Figure 3b) that represents the origin of the reference frame and uses it in two moments; first, when he says “Zero, ok. Zero”; then, when he states “Zero was here.” We infer the teacher’s reflection on the movement of the object because of the development of the speech he used with the students when he tries to explain the meaning of the sign of acceleration. He had not previously used the gesture to represent the origin of the reference frame, but as the teacher’s speech and action evolved, a new gesture (resource) appeared. After he lowers the object and asks: “What is the distance?” (see Figure 3a) the gesture appears to make reference to the origin of the reference frame (see Figure 3b). It is worth pointing out that the first moment also indicates that the origin of the reference frame (mathematical origin) coincides with the phenomenological origin (start of the movement), but the teacher neither says nor makes it explicit. The dialogue continues:

P.: It goes down. [Prof. asks again: “what is the distance?” making reference to Figure 3c] Oh!, I don’t know.

Prof.: Well, calculate it more or less [Someone from another team says: what’s the distance between his finger and what he’s holding?].

P.: […] It’s that I got nervous, let’s see, let’s say some ten centimeters. [Prof. lowers the object (he moves it closer to the table); see Figure 4a] Like twenty-five. [Prof. lowers the object even more (he moves it closer to the table); see Figure 4b] Like thirty-five, forty; let’s say forty. [Prof. lifts the object and places it above his finger (origin); see Figure 4c] Like at ten.

Prof.: Minus ten. [At the same time, P. says: “oh, good point, minus ten”].
Peter does not understand why the teacher makes the gesture (Figure 3b) or what it references. He still refers to the movement that follows the object (physics phenomenon) when he says: “It goes down.” It is only when another member of the team asks explicitly about the distance between the teacher’s finger and the object that Peter manages to understand and keeps on answering as the teacher moves the object away from his finger (Figure 4b). Once the teacher determines the reference frame with its origin, he makes a new gesture (Figure 4c) so that Peter notices unsuccessfully the role orientation plays, represented here by the negative value [of the acceleration] that the object would have when placed above the origin of the reference frame. It is now that Peter focuses again only on the distance between the finger and the object (see Figure 4c) without relating it to the concept of reference frame. The excerpt of the dialogue that follows shows the closing of the discussion between the teacher and the students:

**Prof.:** Yes? Why?

**P.:** Because your considering that the positive goes down.

**Prof.:** And does that mean that this [Referring to the object] goes up? [P. says: “no”]. No, then. Did the distance increase or decrease?

**P.:** No, none of them.

**Prof.:** How not? Let’s see [Raises the object to the starting point (origin) and lets it fall down] It is not in the same place [P. says: “Good point”]. Does the distance increase or decrease? [He raises the object again and lets it fall].

**P.:** It decreases.

**Prof.:** It increases! Didn’t we say that positive goes down?

**P.:** Yes, it’s true.

Peter answers correctly after the teacher used the convention (positive goes down). However, when the teacher asks him: “Did the distance increase or decrease?” Peter gets confused again. This [student’s] confusion is caused by the teacher’s question because it is ambiguous; the teacher is not clear as to what increases. The teacher expects an answer regarding the origin of his reference frame (finger). But if we take into account how he varies the distance with respect to the table, which is what the object comes closer to, then the distance decreases; in this sense, Peter’s answer is right. The argument the teacher is not clear about says: “It increases! Didn’t we say that positive goes down?” The orientation of the reference frame is independent from the distance that the object keeps with respect to the origin.

**Conclusions**

The gestures used as semiotic resources by the teacher complement another resource (reference frame) despite the fact that the physics concept of reference frame is a topic of study in itself. Here, the teacher used said concept as a resource so that the students understood the negative sign that the physics amount of acceleration may have. From Gueudet and Trouche’s (2009) theoretical perspective, the concept of reference frame was observed as a resource from its uses and the continuous process of its development. It was observed that the resources are not isolated, but are
part of a set of resources that the teacher uses with a purpose, and that he modifies from his reflection (Gueudet & Trouche, 2009). The predominant resources in the teacher (besides spoken language) were the concept of reference frame and gestures. The gestures contributed to create and to learn ideas (Arzarello, 2006). So, the way in which mathematical cognition is mediated as well by actions, gestures and other types of signs is shown; and that leads to pay attention to gestures as a source to observe the process of concept formation (Radford, 2009).

It was observed that in the teacher’s gesture (Figure 3b) both the mathematical object (origin of the reference frame) represented with the index finger and the physics phenomenon are involved; in such a way that he places the object in movement within a reference frame by using a gesture. We must underline that regardless the use of resources generates new knowledge, the student’s understanding of its physics meaning [of the concept of negative acceleration] turned out to be complicated and incomplete due to the semantic load of those resources. The results obtained in this research, part of which are reported in this article, allow us to state it is adequate to continue the research about the use of resources in teaching practice. Such research will help us to better understand the practice when teaching school physics concepts in classroom environments.

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The deltoid as envelope of line in high school:  
A constructive approach in the classroom

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Abstract: This paper presents a design of educational module (EM) aimed on the one hand at retrieving to the teaching of the traditional syllabus some mathematical terms connected to geometrical constructions, and on the other to introduce, by virtue of the new issues posed by dynamic geometry software (DGS), different teaching perspectives from the traditional ones so that students can be encouraged to experience the true meaning of mathematical discovery. The focus is based on the tracing of the deltoid curve - triangular curve - derived by an envelope of lines. The peculiarity of EM lies in the methodological approach which privileges a practical strategy based on the combination of the usual tools like ruler and compasses with their modern versions offered by a DGS like GeoGebra. In the concluding paragraph we will report a coherent analysis of the method and the results obtained from the experimentation in class.

Résumé: Cet article présente une conception de module éducatif (ME), qui vise d'une part à la récupération à l'enseignement du programme d'enseignement traditionnel certains termes mathématiques liés à des constructions géométriques, et de l'autre d'introduire, en vertu des nouvelles questions posées par la géométrie dynamique logiciel (DGS), différentes perspectives d'enseignement de celles traditionnelles afin que les étudiants puissent être encouragés à découvrir le vrai sens de la découverte mathématique. La mise au point est basée sur le tracé de la courbe deltoïde - courbe triangulaire - dérivée par une enveloppe de lignes. La particularité de ME réside dans l'approche méthodologique qui privilégie une stratégie pratique basée sur la combinaison des outils habituels comme règle et compas avec leurs versions modernes offerts par un DGS comme GeoGebra. Dans le paragraphe de conclusion, nous ferons rapport une analyse cohérente de la méthode et les résultats obtenus à partir de l'expérimentation en classe.

The beauty of shape is not, as people normally believe, that of living beings and of the paintings which represent them, but the rectilinear and circular beauty of figures, plane and solid, that can be obtained with compasses, ruler and square ruler. Because these are beautiful not, like the former, in a relative way, but in themselves and by their very nature.  
(Plato, Φίληβος, 51c)

Theoretical Framework

The use of dynamic geometry software (DGS) continually opens up new teaching perspectives in the teaching-learning of geometry because it enhances the constructive aspect without detracting from deductive accuracy, from clarity of hypotheses and related consequences pertaining to the discipline (Hannafin et al, 2001; Laborde, 2001; Arzarello et al, 2002; Hohenwarter et al, 2008; Leikin et al, 2013). Thanks to DGS the graphic-constructive phase, both prior to the acquisition of some concepts and geometrical properties, and subsequently as verification and/or further study, is
not only enjoyable, but also greatly helps teaching, as it offers both visualization and exemplification and/or exploration. In short the surveyor’s traditional tools (ruler, square ruler, compasses), retrieved and simulated by DGS, on the one hand facilitate geometrical intuition, while on the other they raise and stimulate interest and the learner’s imagination enabling speculation, which is sometimes immediately verifiable, thanks to the immediate computer feedback (Jones et al, 2000; Hollebrands, 2007; Hohenwarter et al, 2007; Ruthven et al, 2007; Baccaglini-Frank, & Mariotti, 2010). In this work we put forward an educational module (EM) with inter-disciplinary applications, which highlights the intrinsic relationship between geometry and drawing and has a dual objective: on the one hand to retrieve and reintroduce some mathematical terms connected to geometrical constructions in the teaching of the traditional syllabus, and on the other to show, by virtue of the new issues posed by dynamic geometry software (DGS), different teaching perspectives from the traditional ones so that students can be encouraged to experience the true meaning of mathematical discovery. In specific terms, the focus is based on the tracing of the deltoid curve - triangular curve - derived by an envelope of lines. The envelope technique before the invention of DGS represented a real challenge for teaching, not only for reasons of timing in class, but also because it is difficult to manage, especially as far as visualization and graphic representation are concerned. Indeed, the definition of a curve, intended as a place of points which satisfy a given property, usually implies the geometric construction of one of its points with the classical tools like ruler and compasses (R and C) which today are effectively substituted by the corresponding tools offered by DGS. The connection between a drawing and a geometric object in everyday teaching practice is nearly always established through a process of approximation. This is based on the idea that with subsequent, better attempts the drawing can eventually achieve something close to the ideal figure. Geometric constructions made with traditional tools (R and C) also fit this framework and are opposed to free-hand constructions in purely empirical terms of precision. In Italian higher secondary schools students come across constructions made with tools – when this happens – as part of the Drawing and Art History subject, something which in most cases reinforces the practical aspect of constructions and their separation from a geometric context. The use of tools is then seen in practical rather than theoretical terms (Mariotti, 1995). However, in this way a fundamental aspect is ignored and remains unknown to students: each tool contains some knowledge, which is useful for the solution of a particular class of problems. In this sense a geometric construction appears like a geometrical problem (Mariotti, 1996) whose solution can be worked out within a given theoretical framework. Indeed the essential didactic value of the Euclidean frame has always been the perception of its nature of a comprehensive frame which begins with the ‘simple and evident’ and progresses to the complex and ‘non evident’. Geometric construction, suitably contextualized in the teaching practice, helps the students to begin just this complex path which starts with the simple and evident (R and C) and moves on to the complex and ‘non evident’ in a tangible, critical and rigorous way. The integrated tools offered by a DGS represent a valid aid along the way as they progress in the same way from what is predefined to what is made by the user. Therefore we can say that the ED favours a constructive approach based on a combination of traditional tools (R and C) and their modern versions offered by a DGS like GeoGebra. When integrating different tools in the classroom different dimensions have to be taken into account: the relation between the use of tool and learning, the role of the teacher in technology-rich mathematics education, and the characteristics of technological tools (Barzel et al., 2005).

Construction as graphic representation of a curve, as an envelope of lines, takes place then through a series of ‘manipulative experiences’ finalized at balancing the conceptual elements with the drawing of the figure (Fischbein, 1993) of the geometric object. What is special about EM is its effective combination of the tools offered by a DGS like GeoGebra and the traditional ones (R and C) in so far as the latter instruments enable the learners to understand the logic underlying a DGS on the one hand, and on the other to appreciate the amazing abstraction process involved that since the classical age has made geometry not just a collection of
empirical experiences but a rigorous theory as shown by Euclid in the ‘Elements’.

**Design activities in classroom**

A plane line can be considered as generated by the 'continuous' movement of a point or a straight line: in the first case, it is the place of all the positions of the moving point; in the second, it is the envelope of the positions of the mobile straight line.9

(Luigi Cremona, definitions related to line lane)

Design activities in the classroom have a dual objective: raising students' curiosity and interest for geometry on the one hand, and on the other retrieving and consolidating geometric concepts and methods, already known to students in broad terms but which they have not fully mastered yet. That is why we chose a practical approach based on ‘manipulative experience’ implemented through problem posing and problem solving in order to gain insight into Euclidean geometry. In particular, we set up two workshop activities which entailed the construction of a deltoid curve first using traditional tools (R and C), and then virtually by using the GeoGebra spreadsheet.

**Activity One**

The class is divided into groups, and each group gets the required material and tools (graph paper, ruler, compasses, protractor, pencils and rubbers); then the teacher sets the following task for the geometric construction:

- Draw a circumference C with centre O and radius \( r = 2 \text{ cm} \);
- Trace a diameter naming the two extremes A and B;
- Starting from the B point subdivide the circumference into \( n \) (\( n \) divider of 360) arches of equal width \( \alpha \), let \( B_i (i=1, \ldots, n) \) be the second extremes of the arches (Example \( \alpha = 10^\circ \));
- Starting from the A point subdivide the circumference into \( n \) arches of equal width \( 2\alpha \), let \( A_i (i=1, \ldots, n) \) be the second extremes of the arches;
- Trace the straight line passing through points \( A_i \) and \( B_i \) \((i=1, \ldots, n)\).

Materials and task interact motivating the students to come to terms with the new concept; the sheet of graph paper acts as a background which enhances the drawing and at the same time supports the students while they work with the required accuracy. In other words, making a construction with R and C to draw a deltoid curve means starting from other geometrical objects using only real tools; this requires not only good manual skills at drawing but also a first level of abstraction. The direct manual work is obviously centred on the role of the traditional tools in the construction of Euclidean geometry as a hypothetical-deductive science highlighting the process of idealization through which these real tools are turned into abstract ones, characterized by the properties expressed by the postulates. Once all the groups have finished the task the teacher (T) asks the students (S) some stimulus-questions. The examples below are to be intended as basic guidelines. Here is an excerpt from the protocol:

T: the geometric construction obtained shows a new figure, let’s try to understand better .......

S: it’s an unusual figure .... Because from the drawing of straight lines we get some border curved lines;

S: the construction obtained “generates a closed curve with three points”;

T: have you come across any ‘closed curves with three points’ before?

S: I saw a similar construction in a museum where two “pictures” had been made with taught wires which replaced the drawn lines;

S: it makes me think of an equilateral triangle with the sides curved inward;

T: Which properties do you think verify the drawn straight lines?

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9 Plücker, 1839

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S: all the straight lines are tangent to the curve;
T: Are these properties casual or do they occur regularly and if so in which conditions?
S: so it is a curve obtained from all these tangent lines....
T: what happens if the measure of the radius varies from the circumference, does the figure remain the same or does it change?

The methodology of guided discovery encourages the students to formulate the first conjectures and to explore further; after that the teacher introduces the concept of an envelope of curves, a term which contains simple but fascinating concepts. The set of traced straight lines forms a curve called *envelope* which can be assimilated to an arch of circumference; the ordered composition of these arches generates a curve with different shapes, open or closed, which is quite impressive.

The conversation triggered and guided by the teacher is very important because it avoids the construction of formal games and it educates to reasoning before formulating conjectures and hypotheses, stimulating creativity, intuition and the imagination. As a result, the majority of students continue to investigate, to tally up points, monitor, reflect and experiment trying to go deeply into the mathematics knowledge system; a small minority, instead, has reservations about the need to repeat the process of construction with different radius values, as it would be repetitive and therefore boring. To re-establish class motivation, the teacher can then suggest the repetition of the process of construction in the computer lab using the GeoGebra spreadsheet. To recap, the starting point of the activity is a declarative and static task of a problem posing nature which asks the students to carry out an imperative and functional piece of traditional work (tracing of the curve with R and C) finalized to a representation which has a lot of learning potential. Such a problem-posing activity in fact goes beyond the traditional logic of the repetitive execution of a drawing (same mechanical operations of subdivision) because at cognitive level it triggers the ability to understand and interpret knowledge (hermeneutics), to investigate (ability to discover and produce knowledge) and heuristic (ability to invent and create new knowledge).

**Activity Two**

In this second activity the students, again divided into groups, are asked to repeat the previous process of construction, but this time they should do so using the GeoGebra spreadsheet; therefore the “predefined objects” available on the tool bar will be used: point, medium point, circumference, rotation.

The steps of the solving algorithm are as follows:

1. Draw two points, A and B in the Euclidean plane;
2. Determine the medium point (C) of A and B points;
3. Trace the circumference with centre C passing by point B;
4. Construct point B', rotated of B by 10° angle with respect to C;
5. Construct point B_n', rotated di B by an angle n \cdot 10^\circ with respect to C (n = 1, ..., 35) (Figure 1);
6. Trace the envelope lines (Figure 2).
Figure 1. Subdivision of the circumference into arches with same width.

Figure 2. Output of the envelope of the deltoid curve.

The repetition of the construction algorithm helps the learner to understand and use the geometrical shapes in question; it provides the answer inherent to the question of the invariable: the drawing of the curve does not change for different values of the circumference radius. At this point the teacher tells the students that the triangular curve obtained is called deltoid due to its shape being similar to the Greek letter ‘delta’, and continues by saying that the discovery of the curve cannot be ascribed to a particular person due to its relation with another curve named cycloid (rolling curves) studied by Galileo and Mersenne as far back as 1599, and later conceived by the Danish Roemer in 1674 while studying the best shape for a gearwheel. The first to actually seriously consider the deltoid curve was Euler in 1745 in relation to an optical problem, while later in 1856 Stainer studied the curve in such depth that it was nicknamed ‘Steiner’s ipocycloid’. In any case a more extended study of the curve within the context of its historical background could be the object of future lessons.

With regard to the present, the teacher begins a problem solving activity by asking the students to compare the geometric constructions made, noting analogies and possible differences. A lively discussion takes place among the groups. The teacher keeps to herself. At this point all the students agree that both constructions do not show any differences regarding the geometrical objects used; but they notice that “on the computer we can move points without having to start all over again”. To sum up, from the discussion it becomes apparent that the repetition process should be streamlined; consequently the students are required to study a strategy for the resolution of the problem. Improving the previous process of construction becomes very important from a teaching point of view: the tools used previously are no longer sufficient. This is when the GeoGebra slider tool can represent a valid aid. A numerical slider visible on the spreadsheet is thus created, finalized to the tracing of the straight lines so as to reduce the time...
needed for the execution of the previous algorithm. This is a very tricky phase for the students because they have to take quite a big jump into abstraction: the difficulty lays not so much in the creation of the spider, but in identifying the object or objects to apply it to.

The numeric slider requires the allocation of a variable specifying the numeric interval (min and max) and the increase; the obstacle is represented by the identification of the geometrical object (there can be more than one!) which will be modified by the slider.

The basic geometrical objects of the construction are points A and B on which the slider will work; therefore the following points need to be defined:

- A' (rotated by A by an angle $n \cdot 10^i$ clockwise);
- B' (rotated by B by an angle $n \cdot 10^i$ anticlockwise);

from which follows the tracing of the straight line passing by points A' and B'.

The creation of the slider is fundamental because it represents the tool designated to further improve the construction process of the deltoid curve as an envelope of lines.

Below you can find the steps of the algorithm:

1. Draw two points, A and B in the Euclidean plane;
2. Determine the medium point (C) of A and B points;
3. Trace the circumference with centre C passing by point B;
4. Define a slider $n$ (a whole number from 1 to 36);
   - Construct a point A', rotated of A of an angle $-20 \cdot n^i$;
   - Construct a point B', rotated of B of an angle $10 \cdot n^i$;
   - Trace the straight line passing by points A' and B' (Figure 3).

The envelope of the deltoid curve can be obtained moving the slider along (Figure 4).

![Figure 3. Partial output of the envelope of the deltoid curve.](image-url)
This second activity carried out in the computer lab is very educational because it:

- implies the real comprehension of the algorithm of construction, through the simplification of the geometrical objects and their complexity;
- enables the learner to answer the question about the invariant of construction;
- adds value to the students’ learning, because they get the opportunity to improve their skills and to become familiar with the formal and rigorous language of mathematics without being compelled to do so by a teacher.

To sum up, the second activity acts as support because it values intuition and at the same time enables the learners to make generalizations - impossible to attain with a static image - which lead to manage intuitive discovery within a rational framework, and ultimately to accept not only traditional deductive but also inductive processes of learning.

**Concluding Remarks**

The EM gives an opportunity for ‘direct experience’ at different levels with mathematical facts: students have really had the chance to work on geometrical objects constructively, exploring properties, formulating and testing conjectures, also thanks to the tools offered by GeoGebra software. Approaching geometric constructions once more and becoming familiar with the meaning of some terms, like the envelope, is extremely stimulating both for the students and the teacher.

Using once again phrases like ‘repeating the construction’, ‘the point subject to such a condition’, etc. means to reflect on the progress made by teaching technologies in the last few decades. In fact, the bond only to these instruments (R and C) is a kind of definition of ‘geometric procedure’ for the problem’s resolution and, in some way, the idea of geometric construction anticipates the modern concept of algorithm that will be defined precisely many years later.

The geometric construction part from some data, which are the geometrical objects departure, and through a finite number of steps constituted by the elementary operations allows realizing a new object: the curve deltoid. The finiteness of the number of steps is an important aspect from the computational point of view. The choice to represent the deltoid curve with the envelope technique through the use of tools which combine tradition and modernity brings into the classroom alternative methods and procedures to the usual teaching approach, and introduces the students to the culture of Mathematics.

The methodology has been assisted by the practice of self-reflection enabling a coherent analysis of the characteristics of the two laboratory activities that, below, are shown in Figure 5.
This methodology values the very essence of geometric construction of a curve by starting from other given objects; at the same time the students have a chance to experiment new ways of working using the knowledge acquired, testing hypotheses along the way and reasoning both inductively and deductively while also broadening their cultural horizons.

Manipulative ‘experience’ has motivated the learners while at the same time the teacher becomes a facilitator, collaborator and guide. The close link between the tools used has transformed the operative phase into an active research procedure which in turn has led to the formulation of new concepts and the mastering of the procedure.

However, the EM is in no way limiting. The creative teacher can use the design as a springboard for new teaching initiatives which are instructive and engaging.

Geometry can be meaningful only if it expresses its relations with the space of experience [...] it is one of the best opportunities to mathematicise reality (Freudenthal, Mathematics as an Educational Task).

REFERENCES


**Sitography**

URL: mathematica.sns.it/media/volumi/460/CREMONA_curve_piane.pdf
Writing as a Metacognitive Tool in Geometry Problem Solving

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Abstract: This work reports a teaching experiment, which explores the use of writing as a metacognitive tool in high school geometry problem solving. We develop a qualitative research study, to explore how explicit writing directives can help students to understand, organize and monitor the steps involved in the different phases of a cycle of activities for Geometry problem solving in the third year of secondary school.

Résumé: Ce travail rapporte une expérience d’enseignement sur l’utilisation de l’écriture comme outil métacognitif dans la solution de problèmes de géométrie dans l’enseignement secondaire. Nous avons développé une étude de recherche qualitative, afin d’explorer comment les consignes d’écriture peuvent aider les élèves, de manière explicite, à comprendre, organiser et contrôler les étapes impliquées par les différentes phases d’un cycle d’activités pour la résolution d’un problème de Géométrie, pendant la troisième année du cycle secondaire.

Background and research problem

From 2006 to 2013, the Ministry of Public Education applied nationwide a guided multiple-choice evaluation in language, mathematics, and other subjects of the national curriculum in each school cycle of basic education. This test known ENLACE for its acronym in Spanish, helped to diagnose basic education in México, but an undesired side effect of this type of evaluation was that many teachers and pupils, in their efforts to improve the marks of their schools, overemphasized strategies to succeed in multiple choice tests, in detriment of the skills for dealing with open ended mathematical questions. In our work, with the purpose of improving students' problem-solving skills and foster reflective activity when working with open ended questions, we use explicit writing directives to help them expressing their understanding of geometry problems, and to organize, monitor and justify the steps for their solution.

Theoretical framework

In his classic book How to solve it, Polya (1957) provides an outline in four steps of the mathematical problem-solving process:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back and review

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10 Ndt. Équivaut à la classe de quatrième en France
Polya gives advice to educators on the four steps, and exemplifies them through a careful analysis of non-trivial mathematical problems accessible to high school or beginning undergraduate students. Additionally, he compiles a dictionary of basic heuristics like working backwards, exploring limit cases, making diagrams, solving simpler related problems, etc., in order to help the problem-solver to progress in difficult cases. Schoenfeld (1985a) delves further into Polya’s ideas, studying the cognitive underpinnings and practical difficulties of mathematical problem solving, taking into account mastery of mathematical knowledge or resources, heuristics, executive or control issues and beliefs about mathematics and mathematical activity. In his own empirical research, Schoenfeld (1985b) encountered the major role that executive or control issues, have in the problem-solving processes. Control is concerned with the ways individuals use information at their disposal to take major decisions about what to do in a given problem. Control actions have global consequences in the evolution of solutions, as they determine which paths are taken or abandoned, and how resources are used. Schoenfeld studied the interplay of declarative knowledge, self-regulation and belief or intuition in these control processes, and described them as metacognitive phenomena (Schoenfeld 1987, 1992).

Metacognition and its entailments for learning and teaching have become important themes for mathematics education research in the last three decades (Schoenfeld 1985a, 1985b; Harman 1998; Collins et al. 2005). Metacognitive skills are needed in many school subjects, but according to Veenman (2012), they are honed mainly through four kinds of activities: reading text, problem-solving, discovery learning and writing. Skillful reading and writing have great impact on problem solving activities. Hyde & Hyde (1991); Hyde (2006), have pointed out the importance, for students in basic education involved in mathematical problem solving, of explicitly trying to describe and represent mathematical concepts, questions, assumptions and solutions. In this way they identify and clarify previous knowledge engaged in the problem solving processes, and can better organize, monitor and reflect on their work, thus strengthening their thought. The philosophy is that language, mathematics and thought – in its cognitive and metacognitive dimensions–, are better fostered together.

A substantial part of Hyde’s work comes from a deep analysis of the vast body of research on reading comprehension. Hyde (2006) identifies some of the most successful strategies in this area, and tries to understand how they work in order to contextualize and apply them in mathematical subjects, and in particular in mathematical problem solving. Hyde connects these contextualized strategies to the four phases of Polya’s problem solving approach, and formulates a Braid Model of Problem Solving, in which a complex repertoire of questions intends to sort out many of the practical difficulties for implementing Polya’s four phases. In classroom use, these questions may be used as needed by the teacher combining them as well with different forms of collective participation. Our teaching experiment described below, draws heavily on Hyde’s ideas, but emphasizes writing rather than reading comprehension, and uses a very simple orientation scheme because we want to give to the student accessible tools he can use without teacher’s assistance.

**Methodology**

In our teaching experiment, a five step plan for the problem solving process is followed:
The students are guided through these steps by the explicit directive of writing the answers to simple questions formulated in colloquial Mexican Spanish, whose rough English equivalent are: What are the givens? What do I need to find? Which mathematical idea am I using? Which drawings or diagrams can help me to find a solution? What steps will I follow to find a solution? How may I justify my solution? Are there other ways to find an answer?

A set of non-trivial geometrical problems was compiled from different sources, and reduced to 12 problems, after filtering and clarifying the statements working through them with the help of high school students and teacher trainees from Mexico City.

In the teaching experiment, the problems were worked out in 20 sessions of 45 minutes each, with a group of 10 highly motivated third year students from Secondary School 99 in Mexico City, working to prepare their admission exams for College. In the beginning four sessions work was done collectively. First and second session were devoted to the compilation of a glossary in which student made explicit and came to an agreement about the meaning of important concepts of high school geometry. Third and fourth sessions were used to explain the writing directives, and to work out and discuss collectively model examples of their application. The rest of the sessions the students answered individually worksheets for the 12 problems, having the guiding questions written on the blackboard as reminder. Data sources for analysis are the students’ written productions, and the field notes of the teacher-researcher.

**Preliminary findings.**

A detailed analysis of students’ productions is not yet complete, but we have noted that the writing directives for the 1st and 3rd steps of the problem solving plan were useful for all students. In most problems, 7 of the 10 students usually separately display givens, what is looked for, and one or several figures with the relations needed to solve the problem, while 3 of them regularly point out all these elements within a single figure. None of the 10 students made use of manipulatives to clarify some aspect of the problems. All students made an effort to give a clear, extended and complete justification of their answers, although 3 of them usually managed only to make an orderly account of the steps followed, with scarce justification or none at all. There is as well great variation in the quality of justifications in different problems in most students.

In the following, we will examine samples of student’s work in one of the 12 problems, whose statement is shown in Figure 1.

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**Problem.**

In the figure, ABCJ & EFGH are equal squares. JD = DF & DE = 3EF. The perimeter of the figure is 1818 cm. What do the sides of rectangle DEIJ measure? **Note:** The figure is out of scale.

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**Figure 1. Problem statement**
Figure 2 shows the answer given by student who correctly solves the problem using the simple intuitive strategy of decomposing the given figure on equal squares to recover the dimensions of rectangle DEIJ. He writes a brief, precise description of the procedure used, although makes some minor mistakes, i.e.: He writes boxes instead of box sides, but is referring to segments, as he unequivocally counts 18 in the perimeter. He also writes DJ=DI=404 instead of DJ=EI=404. It is worth noticing that he makes first an attempt to crosshatch the original figure in the worksheet, but corrects and then crosshatches his own drawing “complying with the given information” as he comments inside parenthesis in his writing.

We have seen many students using a similar crosshatching strategy with this problem. For instance Figure 3 shows work on the same problem by a student not participating in the teaching experiment. He directly crosshatches the given figure, without noticing the remark that the figure is out of scale and so, being unable to use the given conditions JD=DF and DE=3EF. He wrongly assumes that the length of the sides of the squares is given by the quotient between the perimeter and the number of squares, and proceeds to calculate the perimeter of DEIJ from that incorrect dimension.
From our point of view, identifying what is given and what is looked for in the problem formulation, and beginning to work explicitly writing these elements, makes a big difference for the students. Writing what is given provides an initial orientation which remains on sight, and functions as a control element that helps to correct mistakes and take into account relevant relationships and conditions.

Some students directly make use of the initial orientation provided by a complete, clear, and in orderly writing of what is given and what is looked for. Figure 4 shows the work of a student who draws the rectangle whose dimensions are required above the written statement of what is looked for, and uses the original figure as a reference whose sides he tags with the proper dimensions obtained by straightforward calculations using the given relationships. He grasps clearly what is given, and his justification globally describes the operations made, apparently considering them self-explaining. This brevity contrasts with the work of other students, like the example shown in Figure 5, who gives a detailed reconstruction of his train of thought, and writes in detail each relationship used, and each operation made. This student makes an effort as well to write in an orderly narrative sequence, clear visual disposition of mathematical expressions, and is as well one of the few students using punctuation marks.

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**Figure 3.** Worksheet of a student who did not take part in the workshop

**Figure 4.** Sample worksheet with student’s answer

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First since I know that a small piece value is three times the other I counted all the sides and then divided the perimeter of the whole figure by the sides of the figure and I got the measurement of the segment and then I just add up to know the sides.
Well, first I know that the perimeter of the figure is 1818 cm and the figure has 8 sides, the base and the height are the same IA=DF, JD=IE. I said to myself that DE=3EF, but also that the figure is not in scale. So then I saw that on the side DF=4FE is the same as the side of IA=4JA, just like DC=3AB is the same as IH=3FG, so I added all of the little parts that I had and it added up to 18. 4 at FD, 4 at IA, 3 at DC, 3 at IH, 1 at CB, 1 at AB, 1 at FG, 1 at GH. And I divided 1818 ÷ 18 and got 101, and that’s what FE measured, so I just multiplied 101 times 3 to obtain the base of ED and IJ, which was 303 cm and I multiplied 101 times 4 to get the height of the rectangle, and I got 404 cm.

Figure 5. Sample worksheet with student’s answer

Besides working more directly with relationships some students mobilized other resources like using algebraic symbols. Figure 6 shows the work of one of such students, who also felt comfortable in mentioning the concepts used while giving a detailed account of the steps in his work.
Figure 6. Sample worksheet with student’s answer

It is worth noticing that in his writing this student gives clues of developing self-regulation skills. To verify the correctness of the calculation of the unknown side $x$ of the small square in the figure, substitution an addition of all parts should be made, “but you already know is all right”, as a sound procedure has been followed. As the teaching experiment unfolded, students relied more and more on their own revision of written procedures rather than asking their classmates or teacher if solutions were right.

All 10 students finally managed to get correct answers for the 12 problems, but some of them only after revising several failed attempts. The writing directives were generally useful, but of course do not fully account for the success of the whole group of participants. An important factor was the sheer effort and quantity of work they invested in the workshop. They were studying for their admission exams for higher education and thus highly motivated. They were as well better prepared than average high school students not going to higher education. This may as well explain why none of the students feel the need to use concrete material to clarify the problems.
Conclusions

In the teaching experiment, the writing directives used to guide the problem solving process were generally useful for students. The 1st and 3rd steps of the problem solving plan helped students to clarify the problem statement. The simple action of identifying the data and relationships given in the problem formulation, as well as what is looked for, and begin to work with these explicitly written elements on sight, provides an initial orientation that helps students to take into account relevant relationships and conditions, avoid mistakes, and thus mobilize their knowledge in a proper direction. Particularly important as well, was the effort to produce clear and explicit justifications of the steps to get the solution. That effort helped to check their answers and to become aware of sources of error and facilitated the rectification of failed attempts of solution. These processes foster the development of self-regulation skills which manifested in the teaching experiment as students progressively relied more and more on their own revision of written procedures to assess the correction of their solutions.

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