Algebraic Thinking- More to Do with Why, Than X and Y

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Abstract

Algebraic thinking is a crucial and fundamental element of mathematical thinking and reasoning. It initially involves recognising patterns and general mathematical relationships among numbers, objects and geometric shapes. Using historical evidence, this paper will highlight how the ability to think algebraically might support a deeper and more useful knowledge, not only of algebra, but the thinking required to successfully use mathematics. It will also provide a framework for educators of primary and middle years' students to develop the necessary thinking strategies required to understand algebra.

Introduction

Mathematics is often seen as the gate-keeper of the mathematically intensive vocations. For nearly thirty years this metaphorical gatekeeper has worked very effectively with governments' world wide identifying a steady decline in the participation rates of students undertaking advanced mathematics courses at a secondary school level. For example, only 12% of Australian students enrol in advanced mathematics courses with only one-third of these students being young women. The declining participation rates and limited engagement with mathematics is slowly impinging on the availability of competent individuals pursuing careers in the mathematical rich vocations offered at a tertiary level (Norton & Windsor, 2008). Critically, this non-participation is negatively impacting on the employment opportunities available to people. More alarmingly is the fact that a limited understanding of mathematics may directly hinder a person's effectiveness to participate in our modern society where information, discussions and rhetoric are immersed and in some cases shrouded by mathematics. As Booker, Bond, Sparrow & Swan (2009, p7) state, individuals who lack an ability to think mathematically will be disadvantaged and at the mercy of other peoples interpretation and manipulation of numbers.
Algebra is the crucial link between the predominantly arithmetical approach of the primary school curriculum and secondary mathematics subjects such as calculus, quadratics and trigonometry. However, in the recently published Foundations for Success (US Department of Education, 2008, p 18) it was noted that the sharp falloff in mathematics achievement begins as students reach middle school where, for many students, they are introduced to algebra for the first time. Arcavi (2008) states that algebra, in many ways, intimidates students and affects their attitudes towards mathematics. These conceptual and attitudinal impediments have long been seen as reasons why student struggle with some advanced mathematical concepts at a school secondary level. The question that needs to be addressed is how do educators ensure that all students have the opportunity to successfully participate in algebra? If this issue is addressed in the primary and middle school context then it may influence students to participate in the mathematically rich subjects undertaken at secondary school.
Simply bringing the subject of algebra to the earlier grades does little to address the underlying problems of student misunderstandings (Kriegler, 2006). Importantly, educators need to consider the thinking required for understanding algebra. It is widely acknowledged that to understand number, students initially use additive thinking structures before transitioning to multiplicative structures. Surely, to understand algebra students need to develop the thinking required to identify, understand and communicate generality which is the essence of algebra. To develop this thinking- often referred to as algebraic thinking- Kaput (2008) suggests an increasingly longitudinal view of algebra; that is, a view of algebra not as an isolated course or two, but rather as a strand of thinking and problem solving, beginning in primary school and extending through students' mathematical education. By connecting and seeking out the generalities inherent in number, geometry and measurement, algebraic thinking and algebra can become the unifying strand of primary and middle school curriculums.

Algebraic Thinking

Algebraic thinking promotes a particular way of interpreting the world. It employs and develops a variety of cognitive strategies necessary to understand increasingly complex mathematical concepts and builds upon students' formal and informal mathematical knowledge. Essentially students are using, communicating and making sense of the generalities and relationships inherent in mathematics, rather than just the identification of a single numeric answer or objective fact. Chazan (1996) implores that educators appreciate the algebraic thinking already done by students, their parents, and other members of the community, even though it is not necessarily expressed in x's and y's. Developing students' ability to think algebraically is a precursor not only for participation in the subject of algebra, but also importantly to be able to think broadly about problem situations. Algebraic thinking provides an extra dimension to an individual's understanding and use of mathematics because they seek out and understand the generalities, as well as the specifics of a problem.

Algebraic thinking can emerge from the number, geometry and measurement activities primary school students engage with daily at school. By illustrating ideas and using concrete materials, models, diagrams, tables and patterns of objects students can 'see' the relationships between the concepts. Students who think algebraically are aware of the inherent links and interconnectedness of mathematics and this thinking can be developed in all students. They understand that mathematics is a system of interpretation where the concrete and the abstract are interwoven. This would suggest that using concrete materials is fundamental because so many of the ideas of algebra are not intrinsically obvious. As Booker et al (2004, p.14) suggest students need to be assisted to develop algebraic thinking using structured materials, materials through which the underlying ideas are understood and appreciated. Furthermore, Lins & Kaput (2004) suggest that students who are engaged in algebraic thinking attempt acts of generalisation and seek to communicate those of generalities. Their thinking involves, usually as a separate endeavour, reasoning based on the forms of syntactically structured generalisations, informed by syntactically and semantically guided actions. By linking the concrete and the abstract, educators parallel the historical development of algebra.

History and Algebraic Thinking

Reflecting on and analysing the historical development of algebra can provide an awareness of the ways mathematical thinking and understanding has developed. A rich tapestry of information is available to link the epistemological and the historical. According to Ernest (2006), analysing history from a deep epistemological perspective for psychological purposes moves mathematics away from the traditionalist view of mathematics to a more humanistic position. Devising a pedagogical approach that is in sympathy with its historical development takes into consideration all the elements of knowledge creation and appreciates and values all human activities associated with mathematics. Using the history of mathematics can benefit and influence the way educators teach and importantly develop a greater sensitivity concerning how students learn algebra.

The history of mathematics informs us that the development of algebra and consequently the ability for individuals to interpret, think, and communicate algebraically, progresses through three distinct yet overlapping stages of development. Researchers (Katz 2007; Bashmakov & Smirnova 2000) define these three stages as the rhetorical stage, the syncopated stage, and the symbolic stage. This chain of development, first identified by G.H.F. Nesselmann in *Die Algebrader Griechen (The Algebra of Griechen, 1842 cited Puig & Rojano, 2004),* attempts to summarise how algebraic thinking strategies developed over a 4000 year period. From his work the commonly adhered to definitions for each of the three stages are; the rhetorical stage, where the calculations are expressed completely and in detail utilising everyday written and spoken vernacular; the syncopated stage, where frequently occurring concepts and operations are replaced by consistent abbreviations instead of the complete words; and finally the symbolic stage whereby all possible forms and operations are represented in a symbol based system.

History would suggest that to understand and solve problems of an algebraic nature, individuals operate and manoeuvre their thinking continually between the rhetorical, syncopated or symbolic stages. For example, the Lucus' Tower of Hanoi puzzle, whereby disks are moved from one rod to another in the least number of moves without a larger disk being placed on a smaller disk, can be examined algebraically. At the rhetorical and syncopated stages, students describe and identify the relationship between the minimum number of moves and the number of disks. The description may be summarised using a table, diagrams or simply a model to develop the generalities and identify the least number of moves, for any disk configuration. At the symbolic stage to fully understand the relationship between the disks and the number of moves, students will make links with the relationships identified at rhetorical and syncopated stages of thinking.

An Example of Algebraic Thinking within a Primary School Context

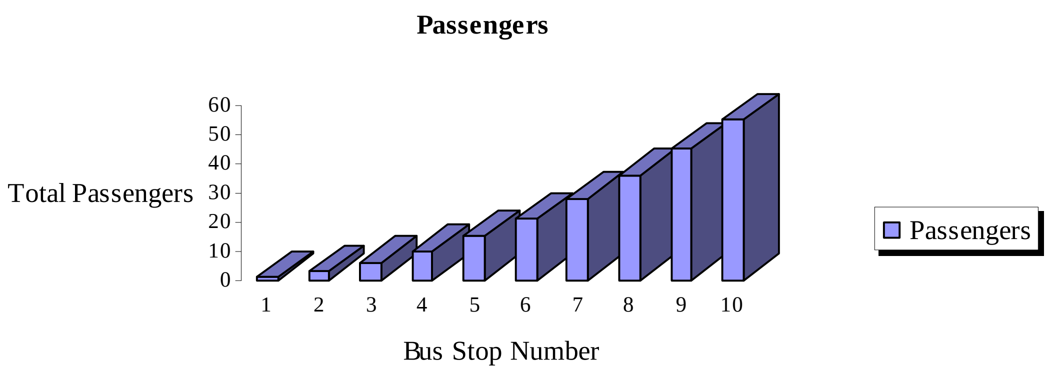
To build-on and extend a class of year seven students' numeration and computation understanding and to develop their algebraic thinking skills, a variety of different problems were presented to them. The class were required to work in small co-operative groups, whereby they would verbally present to their peers and teacher their understanding of the patterns and generalisations they identified within the problems. All of the groups were able to write a short explanation, however some went beyond these explanations and explored alternative representations of their thinking. For example, Dale a 12 year old boy, who had an excellent understanding of numeration and computation concepts, observed that the number of passengers boarding the bus was the same as the bus stop number. Because no passengers were 'hopping off' the bus, Dale identified the total number of passenger as the sum of all the bus stops. He represented the 'Bus Stop Problem' firstly, by identifying and summarising his thinking using a table, he then proceeded to graph the information. When asked why he constructed the graph he simply stated that the graph made it easier for him to see the pattern. In conjunction with his peers he formulated a description of the pattern.

Problem- The Bus Stop

One day the bus conductor noticed that passengers were boarding the bus in the following way. At the first bus stop, 1 passenger got on, 2 got on at the second stop, 3 at the third stop and so on. The capacity of the bus was 72. What was the number pattern that the conductor noticed?

Dale's Responses

Bus Stop	1	2	3	4	5	6	7	8	9	10
Passengers Getting On	1	2	3	4	5	6	7	8	9	10
Total Passengers	1	3	6	10	15	21	28	36	45	55



Rhetorical Description: The number of passengers getting on is the same as the bus stop number. The total number of people on the bus is the sum of the passengers already on the bus and the next bus stop. To work out the total passengers on the bus take the bus stop number people are getting on at, add one to this then multiply it by the bus stop number and half this total

Conclusion

Like many fundamental mathematical concepts, algebraic thinking is best learnt by communicating and linking real objects and materials with the symbols of mathematics. Importantly, by extending algebraic thinking beyond algebra's purely symbolic realms, educators may give all students the opportunity to learn how to generalise, justify and reason using algebraic methods. Crucially students must use materials to "see", describe and reason about generality. Secondly develop the necessary understandings to summarise those generalities by using graphs, tables or diagrams. Finally, using the representational systems of mathematics and algebra they communicate those generalities succinctly and with understanding. Educators can ensure that the catch cry of "algebra for all" is a legitimate goal.

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