Math lessons for the thinking classrooms

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Abstract

Teaching mathematics means teaching learners to think – wrote Polya in How to Solve It? 1957.

This paper intends to offer mathematics teachers suggestions for incorporating reading, writing, and speaking practices in the teaching of mathematics.

Through explicit examples and explanations we intend to share ways of engaging students in deep learning of mathematics, especially using and producing written and oral texts. More specifically, we plan to broaden and deepen teachers' understanding of strategies for guiding students' thinking so that they grasp mathematical concepts and processes, and also bridge the divide between mathematical processes, and written and oral communication.

This paper presents a core math lessons which provides numerous opportunities for the students to get actively engaged in the lesson and think about the new concepts, algorithms and ways of solving problems/ exercises.

The lesson was designed for the 7th graders (13 year-olds). It was chosen to illustrate teaching by using reading and writing for understanding math processes. The teacher's reflections after the lesson and some samples of the students' work and feedback are included in the paper.

The material in this paper is based on the author's own extensive teaching experience; and her work in the Reading and Writing for Critical Thinking project in Romania.

Introduction

Unfortunately, lately, in Romania, math has been less and less about teaching learners to think, and more about teaching learners enough math to pass a national examination. So, teachers "show" students algorithms and recipes for solving exercises and problems in the final exams. Many students learn the drill given by the teacher, and solve problems never asking "why am I doing this or that?"; although they understand almost nothing, they can get a good grade in the final examination. In most cases, *thinking* is not a process involved in this scenario.

According to G. Polya (How to Solve It?, 1957), teaching students to think does not necessarily mean sharing a lot of information with the students, but rather finding ways to develop the students' abilities to use information.

Through this lesson we will demonstrate concrete ways in which using a variety of written and oral texts can increase student participation and support thinking and learning in the mathematics classroom.

Informed by a cognitivist perspective, the lessons is organized around a three-part structure – A-B-C - of planning and instruction that consists of: **Anticipation**, in which the teacher uses reading, writing, and discussion strategies to activates students' background knowledge; **Building Knowledge**, in which the teacher guides the students' inquiry and helps them construct an understanding of the new content; and **Consolidation**, where students are led to summarize, apply, interpret, critique, and innovate relying upon the understandings they have constructed, assess their own learning, and ask new questions.(Crawford *et al*, 2005)

The lesson: The Square of a binomial

In this lesson we will demonstrate several methods of using writing to facilitate the understanding of mathematical processes. In mathematics classes, students are often required to prove theorems and properties and to solve problems and exercises. A large part of the content learned by students in mathematics classes consists of methods of proving and solving.

Students must know how to approach a problem or proof, how to reason, and what to do in order to come to the proof's conclusion or the problem's solution.
Comprehending the text of the problem or theorem is essential if students are to be capable of constructing the solution or proof. Throughout my years in the classroom, my students have often told me that they did not understand a problem or theorem when, in fact, it was its text that they did not understand, because they had not read it carefully. Students are very rarely asked to read the proof of a theorem or the solution to a problem. Reading mathematical texts – problems, theorems, proofs of theorems, and solutions to problems – and understanding them provide a basis for mathematical learning (Mower, P., 2003). On the other hand, the cognitive processes involved in the writing process require students to think before they use algorithms or solve problems. Often, students solve mathematics problems mechanically, working through the steps of the solution without understanding their purpose. If, however, they explain these steps in writing, they are encouraged to think beyond the algorithm and to see the logic behind the succession of steps of the solution. The writing activities exemplified in the following lesson require students to understand why a specific algorithm leads to the solution of the exercise or problem, encourage them to reflect upon the process, solicit the analysis of mathematical processes, and allow students to explore the logic of mathematics.
This is the fourth lesson of the *Algebraic Computation* unit, seventh grade and the first lesson of the *Strategies for Shortening Computation* sub-unit. In this unit, in previous lessons, students learned and practiced calculations with real numbers represented by letters (addition, subtraction, multiplication, division, and integer exponentiation).

Preliminary concerns	
What is the topic or question? What question and information should students investigate during this lesson?	Strategies for shortening computation – square of a binomial Where does it come from? How do we use it?
Why does it matter? Why is this knowledge worth having? What opportunities for thinking and communicating does this lesson afford?	The square of a binomial is often used in mathematics – in order to shorten the algebraic computation – when calculating algebraic expressions, solving quadratic equations. The lesson, through both oral and written communication activities, will allow students to analyse mathematical processes, to prove that there is more than one way to demonstrate a certain formula, to think about ways mathematicians work.
What are the objectives? What knowledge should the students gain? What should they be able to do with that knowledge? What strategies for thinking, investigating and communicating will they learn?	Content objectives: Students will be able to: ▪ Explain the meaning of *the square of a binomial*; ▪ Demonstrate the formula of *the square of a binomial*; ▪ Use the *square of a binomial* formula in two types of simple exercises; ▪ Understand the algorithm of using the formula of *the square of a binomial* in simple exercises. Process objectives: Students will be able to: ▪ Reflect on and learn the process of using the formula of *the square of a binomial* in simple exercises; ▪ Communicate using the vocabulary of algebra; ▪ Transfer mathematical content into their own words; ▪ Practice with basic proof reading and proof writing; ▪ Consider various approaches of a proof writing.

The lesson we will present uses the A-B-C structure. In this lesson we will use the following strategies: directed listening thinking activity (DL-TA), concept definition, proof reading, proof writing, and writing solution steps.

Anticipation: In this stage of learning, the students are prepared for what they will acquire from the lesson. Here, anticipation is accomplished through a **directed listening thinking activity** (**DL-TA**).

Activity 1 - Task: please answer the question: *What do we know about how mathematicians work?* – by thinking about the way you work with math. Please draw a line down the center of a sheet of paper so that you have two columns. In the first column write down some of the ways you work when you are learning math, and in the second write down how mathematicians work by thinking whether your ways of working would be used by a mathematician.

Students work in pairs and a few pairs are invited to present their answers in front of the class.

Activity 2: The teacher tells the students a joke about mathematicians. She starts the joke, but at one point she stops and asks students to tell her what the punch line is.

Mathematician Joke

A mathematician and a physicist are talking over a cup of coffee in the hallway of their university. Suddenly, for no apparent reason, the coffee machine bursts into flames. The physicist grabs the fire extinguisher from the wall and puts out the fire.

The following week, the two scientists are once again having a cup of coffee in the hallway, next to their new coffee maker. Suddenly, this coffee maker also bursts into flames.

The teacher stops and asks students to take a guess at what the end of the joke could be, by thinking of what they've discussed about the way mathematicians work.

Some students tell the end of the joke and the teacher asks each of them to argue their answer.

The teacher appreciates the students' endings for the joke and presents in front of the class the ending she knows:

The mathematician takes the fire extinguisher and gives it to the physicist to put out the fire, thereby reducing the problem to one solved previously.

Building knowledge: The teacher will introduce the notion of the square of a binomial through a **concept definition** activity and the derivation of the shortened computation formula for the square of a binomial through a **proof reading** activity followed by a **proof writing** activity. In the proof reading activity, the teacher will require students to examine an algebraic proof using a variety of representations: numerical, geometric, and algebraic.

Activity 3: Task: Please fill in, individually, the second column of your concept table after thinking what *the square of a binomial* means.

Concept	*What I think it means*	*Definition*
Square of a Binomial		

After filling in your concept table, discuss with your partner what you've written in the second column.

The students work individually and then discuss in pairs while the teacher oversees their activity. Some students are invited to present their answers in front of the class. The teacher then leads a brief discussion related to the concept definition and finally introduces the definition of a square of a binomial.

Activity 4: Task: Read the text on the worksheet carefully, write down the proofs, justifying every step in writing, and give another numerical example.

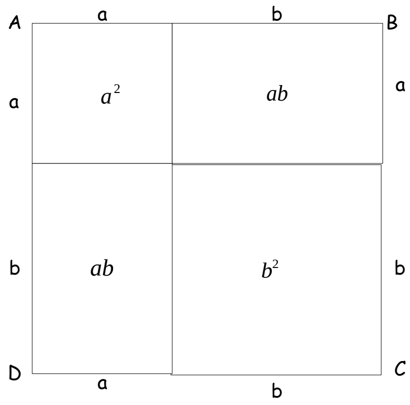
Worksheet 1 – The Square of a Binomial

We will show that $(a + b)^2 = a^2 + 2ab + b^2$

1. Algebraic Proof

$(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

2. Geometric Proof
Look at the figure:



The square ABCD has the length of a side equal to a + b, and thus will have area: $(a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

3. Numerical Example

If a = 2 and b = 5, we have $(2 + 5)^2 = 2^2 + 2(2)(5) + 5^2 = 4 + 20 + 25 = 49$ and $(2 + 5)^2 = 7^2 = 49$.

The students read the proofs on the worksheet, write them down while justifying their steps, and, finally, give a numerical example.

In this interval, the teacher moves around the classroom and monitors the students, offering help to those who need it.

Activity 5: Task: You will derive the formula for the square of a binomial containing the minus sign on your own. You will work in pairs and after ten minutes you will present the proof you have come up with. The pairs that finish early will also give a numerical example for the formula they have derived.

The students work in pairs. The teacher moves around the classroom and monitors the students' activity. Some students are invited to present their answers in front of the class.

Consolidation: In this lesson, consolidation will be achieved with the help of a **writing of solution steps (**it is called **method of operation** in Mower, P., 2003) activity. Students will use the formula for the square of a binomial in problems, writing down every step of their solutions.

Activity 6: Homework task: Solve the exercises on worksheet 2, answering the questions on the sheet.

Worksheet 2 – Applications of the Formula for the Square of a Binomial

A) Calculate $(2x – 3y)^2$ using the formula for the square of a binomial. Below, write each step of your solution:

1. ____

… Add as many lines as necessary for the solution.

B) Compute $(x + 1)^2 – (x + 2)^2 + (x – 3)^2 – 6$ using the formula for the square of a binomial. Below, write each step of your solution:

1. ____
2. ____
3. ____

… Add as many lines as necessary for the solution.

C) Write the following expression as the square of a binomial:
$4x^2 + 4x + 1$. Below, write each step of your solution:

1. ____

2. ____

… Add as many lines as necessary for the solution.

D) One thing we discovered today about formulas for shortening computation:

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Homework is checked and discussed in the next lesson.

Samples of the students' work

Activity 1: *What do we know about how mathematicians work?*

How do I work in mathematics?	*What do we know about how mathematicians work?*
I write for solving exercises and problems.	They write proofs.
I use writing when calculating.	They do mental calculations.
I write detailed solutions.	They choose the shortest/ most ingenious solutions and look for formulas already known.
I read the exercises/ problems and analyse them.	They read and analyse the context.
When I don't know how to solve a problem I get support or I give up.	When they need support they work with other colleagues. They never give up.

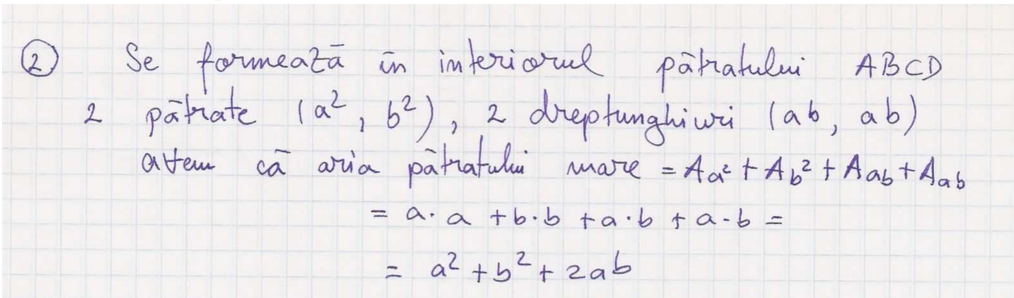
Activity 3: What I think it means the square of a binomial?

Student A: the square of a two figures number, the square of a two numbers product;

Students B: the square of "something with two";

Student C: the square of a two terms sum.

Activity 4: Geometrical proof

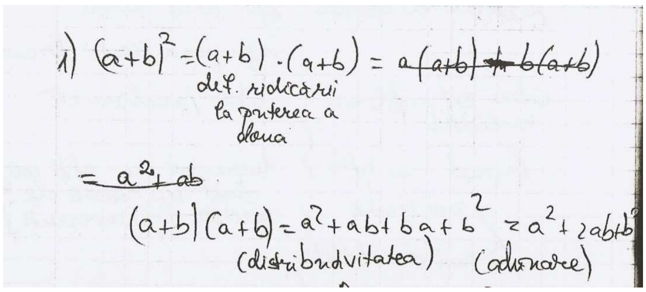


The square ABCD has side length a + b and thus area $(a + b)^2$ – because the area of a square is the length of its side squared.

The area of square ABCD is equal to the sum of the areas of the figures that make it up: the square of side length a, which has area a^2, the square of side length b, which has area b^2, and the two rectangles with dimensions a by b, which have area ab.

If we equate the two ways of writing the area of square ABCD, we have:

$(a + b)^2 = a \cdot a + b \cdot b + a \cdot b + a \cdot b = a^2 + b^2 + 2ab$.



Teacher's and students' reflections

During the lesson, each student actively participated. Activities 1 and 2 offered students good reasons for learning Strategies for Shortening Computation, as they didn't ask, as other students did in the previous years, "why are we studying these formulas?".

By being engaged in activity 4, the algebraic proof, many students had problems in identifying the propriety/law which allows writing the second and the third equal signs in the proof. They mixed up the distributive law with its reverse, factoring out the common factor, even if they are able to state the distributive property and to spontaneously use it in arithmetic calculations. We'll have to facilitate students' awareness of the distributive property.

For activity 5, all pairs, except one, gave an algebraic proof – geometry is still difficult for students or it is hard for them to use geometry during algebra lessons.

Checking the homework I observed that only 60% of the students used the formula for exercise A); the others solved the exercise by calculating, without using the formula. For exercise B), worksheet 2, 80% of the students used the formula. When discussing this issue with the class, some students who didn't use the formula for exercise A) and used it for the exercise B) told me that they used the formula for exercise B) because there was too much to calculate for exercise B).

Students' answers for D) – worksheet 2 – showed that they discovered about formulas for shortening computation that they help finish the calculations more quickly. Some students said that they discovered that "the formula for shortening computation is an equation with two elements. In some exercises we are given the right-hand element and asked to find the left-hand one, and in other we are given the left-hand element and asked to find the right-hand one." Three students said that they discovered that the square of a binomial can be used without proving it each time it is used.

I continued, for four lessons, to ask students to use the **writing of solution steps (method of operation)** when using the formula in exercises. After these four lessons, they could decide if they wanted or needed to continue using it. After two months, 15% of students were still writing the solution steps.

Approaching the lesson: *Difference of squares* by using almost the same writing strategies showed that students were quick and very good, this time, at defining the concept (difference of squares) and writing the proof. Finally, it is a learning process. Again, they were amazed by the geometrical proof but they considered it difficult.

I enjoyed the lesson; my main goal, as a math teacher, is to make students enjoy mathematics and to think, question and analyze mathematical issues as mathematicians do. During this lesson, we were on the right track to reach my main goal.

References:

Crawford, A., Saul, E.W., Mathews, S., Makinster, J. 2005.*Theaching and learning strategies for the thinking classroom*. New York, The International Debate Education Association

Mower, P. 2003. *Learning Mathematics through reading and writing activities.* San Francisco, Jossey Bass A Wiley Imprint

Polya, George. 1957. *How to solve it.* Princeton University Press.

Văcărețu, A.- S., 2007. [Romanian] *Lecții de matematică pentru dezvoltarea gândirii critice*, Cluj-Napoca, Eikon Educational