Using ClassPad-technology in the education of students of electrical engineering (Fourier- and Laplace-Transformation)

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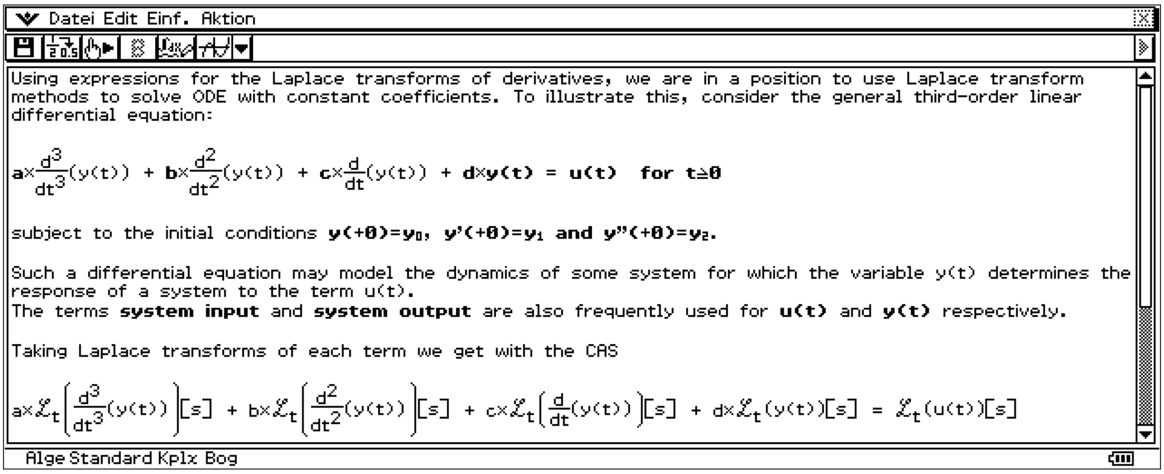
Abstract:

By the help of several examples the interactive work with the ClassPad330 is considered.
The student can solve difficult exercises of practical applications step by step using the symbolic calculation and the graphic possibilities of the calculator. Sometimes several fields of mathematics are combined to solve a problem.
Let us consider the ClassPad330 (with the actual operating system OS 03.03) and discuss on some new exercises in analysis, e.g. solving a linear differential equation by the help of the **Laplace transformation** and using the **inverse Laplace transformation** or considering the Fourier transformation in discrete time (the **Fast Fourier Transformation FFT** and the **inverse FFT**). We use the **FFT**- and **IFFT**-function to study periodic signals, if we only have a sequence generated by sampling the time signal.
We know several ways to get a solution. The techniques for studying practical applications fall into the following three categories: ***analytic***, ***graphic*** and ***numeric***. We can use the Classpad software in the handheld or in the PC (ClassPad emulator version of the handheld).

References:

http://edu.casio.com/products/classpad/cp_v302/
http://edu.casio.com/products/classpad/cp_v302/laplace.html
http://classpad.net/product/Classpad300/cp_manager_03.html
Glyn, James: **Advanced Modern Engineering Mathematics**, 3rd Edition, 2004 (repr. 2006), ISBN 978-0-13-045425-6
Burg, Klemens; Haf, Herbert; Wille, Friedrich; Meister, Andreas:
Höhere Mathematik für Ingenieure Band III: Gewöhnliche Differentialgleichungen, Distributionen, Integraltransformationen, 5th Edition, 2009, ISBN 978-3-8348-0565-2

Example of solving a linear ODE with initial condition, several ways of solution:



Using expressions for the Laplace transforms of derivatives, we are in a position to use Laplace transform methods to solve ODE with constant coefficients. To illustrate this, consider the general third-order linear differential equation:

$$\mathbf{a}\times\frac{d^3}{dt^3}(y(t)) + \mathbf{b}\times\frac{d^2}{dt^2}(y(t)) + \mathbf{c}\times\frac{d}{dt}(y(t)) + \mathbf{d}\times\mathbf{y(t)} = \mathbf{u(t)} \quad \textbf{for } t\geq 0$$

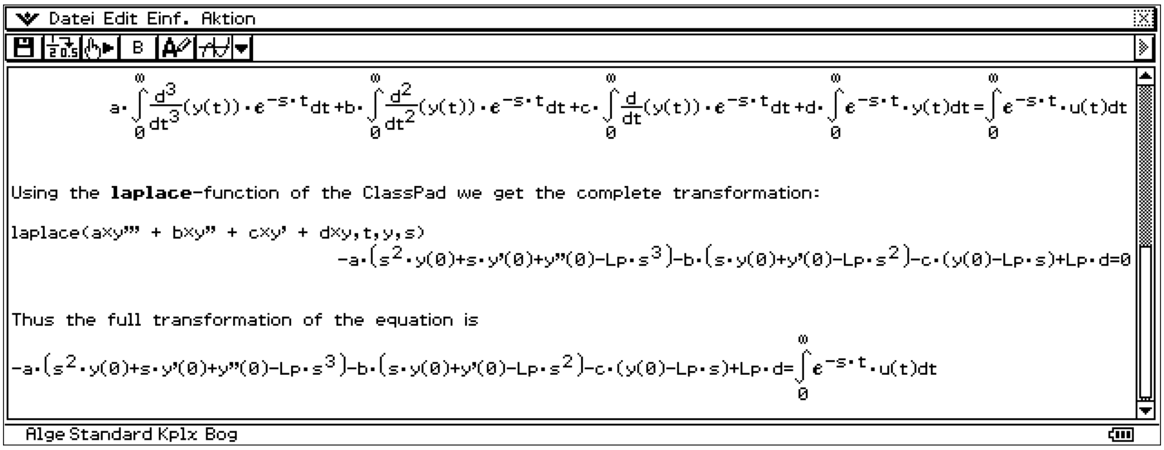
subject to the initial conditions $\mathbf{y(+0)=y_0}$, $\mathbf{y'(+0)=y_1}$ **and** $\mathbf{y''(+0)=y_2}$.

Such a differential equation may model the dynamics of some system for which the variable y(t) determines the response of a system to the term u(t).
The terms **system input** and **system output** are also frequently used for **u(t)** and **y(t)** respectively.

Taking Laplace transforms of each term we get with the CAS

$$a\times\mathcal{L}_t\left(\frac{d^3}{dt^3}(y(t))\right)[s] + b\times\mathcal{L}_t\left(\frac{d^2}{dt^2}(y(t))\right)[s] + c\times\mathcal{L}_t\left(\frac{d}{dt}(y(t))\right)[s] + d\times\mathcal{L}_t(y(t))[s] = \mathcal{L}_t(u(t))[s]$$

The last input line yields in the CAS software:



$$a\cdot\int_0^\infty\frac{d^3}{dt^3}(y(t))\cdot e^{-s\cdot t}dt+b\cdot\int_0^\infty\frac{d^2}{dt^2}(y(t))\cdot e^{-s\cdot t}dt+c\cdot\int_0^\infty\frac{d}{dt}(y(t))\cdot e^{-s\cdot t}dt+d\cdot\int_0^\infty e^{-s\cdot t}\cdot y(t)dt=\int_0^\infty e^{-s\cdot t}\cdot u(t)dt$$

Using the **laplace**-function of the ClassPad we get the complete transformation:

laplace(a×y''' + b×y'' + c×y' + d×y,t,y,s)

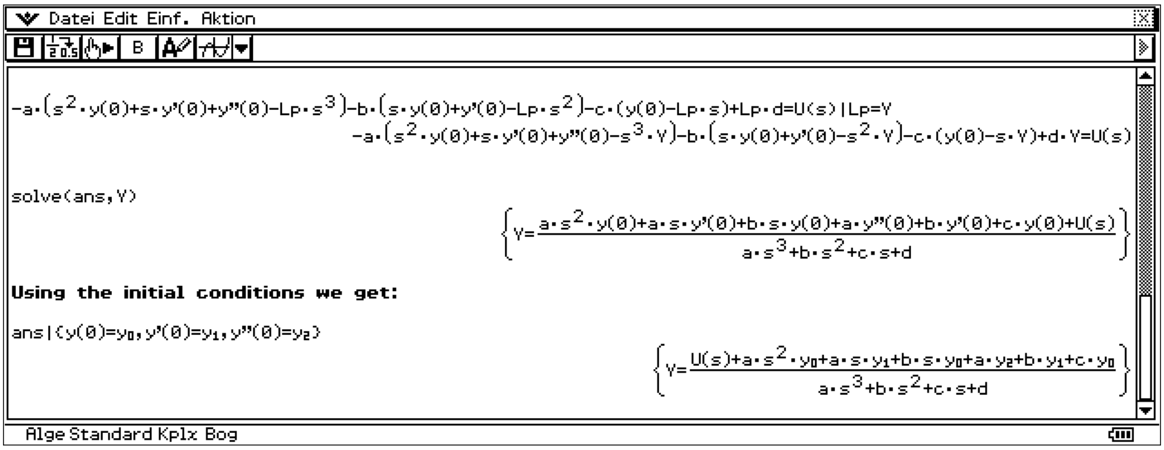
$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=0$$

Thus the full transformation of the equation is

$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=\int_0^\infty e^{-s\cdot t}\cdot u(t)dt$$

In the last line we have with **Lp** the Laplace transformation of y(t), sometimes we denote **Lp** by Y(s). Furthermore we discover the initial conditions y(0), y'(0), y''(0).

We denote the right hand side of the last equation by U(s) and solve this equation for Y(s):



$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-Lp\cdot s^3\right)-b\cdot\left(s\cdot y(0)+y'(0)-Lp\cdot s^2\right)-c\cdot(y(0)-Lp\cdot s)+Lp\cdot d=U(s)\,|\,Lp=Y$$

$$-a\cdot\left(s^2\cdot y(0)+s\cdot y'(0)+y''(0)-s^3\cdot Y\right)-b\cdot\left(s\cdot y(0)+y'(0)-s^2\cdot Y\right)-c\cdot(y(0)-s\cdot Y)+d\cdot Y=U(s)$$

solve(ans,Y)

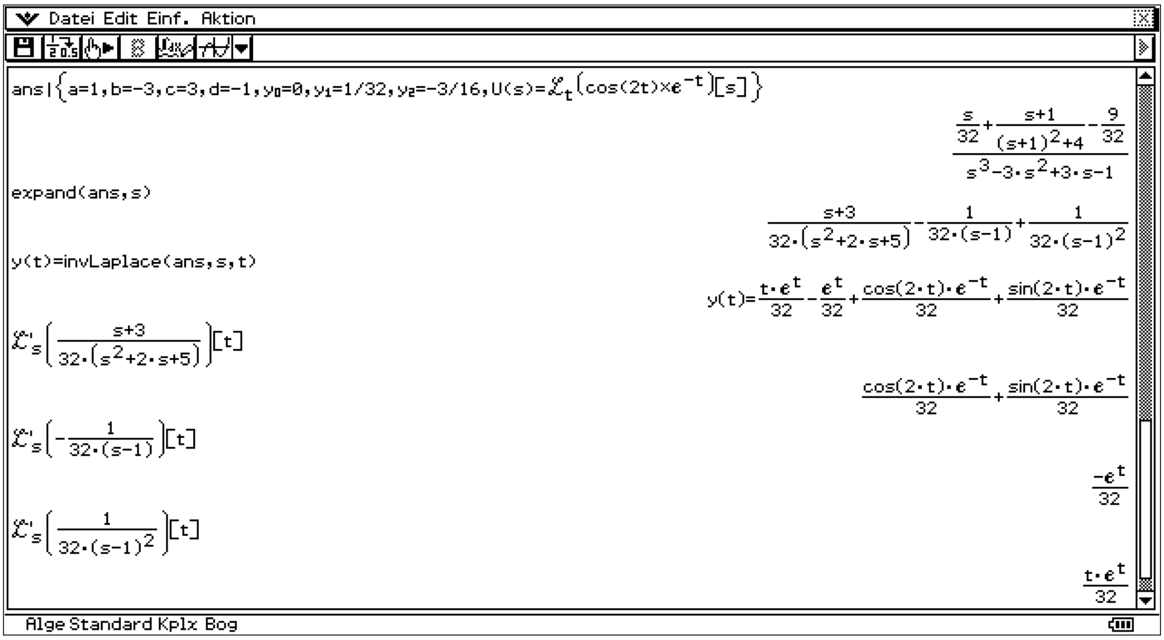
$$\left\{Y=\frac{a\cdot s^2\cdot y(0)+a\cdot s\cdot y'(0)+b\cdot s\cdot y(0)+a\cdot y''(0)+b\cdot y'(0)+c\cdot y(0)+U(s)}{a\cdot s^3+b\cdot s^2+c\cdot s+d}\right\}$$

Using the initial conditions we get:

ans|{y(0)=y_0,y'(0)=y_1,y''(0)=y_2}

$$\left\{Y=\frac{U(s)+a\cdot s^2\cdot y_0+a\cdot s\cdot y_1+b\cdot s\cdot y_0+a\cdot y_2+b\cdot y_1+c\cdot y_0}{a\cdot s^3+b\cdot s^2+c\cdot s+d}\right\}$$

Now we consider the following example: a = 1, b = -3, c = 3, d = -1, u(t) = cos(2t)*e^(-t) and y(0) = 0, y'(0) = 1/32, y''(0) = -3/16.



ans|$\left\{a=1,b=-3,c=3,d=-1,y_0=0,y_1=1/32,y_2=-3/16,U(s)=\mathcal{L}_t\left(\cos(2t)\times e^{-t}\right)[s]\right\}$

$$\frac{\frac{s}{32}+\frac{s+1}{(s+1)^2+4}-\frac{9}{32}}{s^3-3\cdot s^2+3\cdot s-1}$$

expand(ans,s)

$$\frac{s+3}{32\cdot\left(s^2+2\cdot s+5\right)}-\frac{1}{32\cdot(s-1)}+\frac{1}{32\cdot(s-1)^2}$$

y(t)=invLaplace(ans,s,t)

$$y(t)=\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

$$\mathcal{L}^{-1}_s\left(\frac{s+3}{32\cdot\left(s^2+2\cdot s+5\right)}\right)[t]$$

$$\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

$$\mathcal{L}^{-1}_s\left(-\frac{1}{32\cdot(s-1)}\right)[t]$$

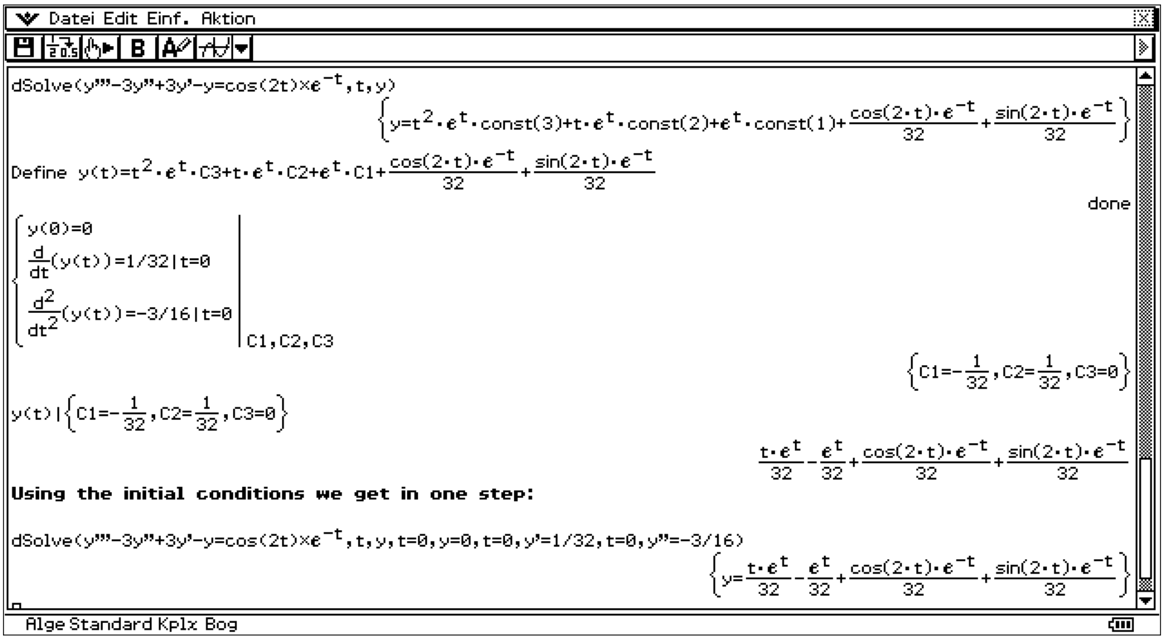
$$\frac{-e^t}{32}$$

$$\mathcal{L}^{-1}_s\left(\frac{1}{32\cdot(s-1)^2}\right)[t]$$

$$\frac{t\cdot e^t}{32}$$

Finally we see the inverse transformation of the several summands.

The direct way of solution consists in using the dSolve-function:



$\mathrm{dSolve}(y'''-3y''+3y'-y=\cos(2t)\times e^{-t},t,y)$

$$\left\{y=t^2\cdot e^t\cdot \mathrm{const}(3)+t\cdot e^t\cdot \mathrm{const}(2)+e^t\cdot \mathrm{const}(1)+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}\right\}$$

$\mathrm{Define}\ y(t)=t^2\cdot e^t\cdot C3+t\cdot e^t\cdot C2+e^t\cdot C1+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$

done

$$\left.\begin{cases} y(0)=0 \\ \frac{d}{dt}(y(t))=1/32\,|\,t=0 \\ \frac{d^2}{dt^2}(y(t))=-3/16\,|\,t=0 \end{cases}\right|_{C1,C2,C3}$$

$$\left\{C1=-\frac{1}{32},C2=\frac{1}{32},C3=0\right\}$$

$y(t)\,|\,\left\{C1=-\frac{1}{32},C2=\frac{1}{32},C3=0\right\}$

$$\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}$$

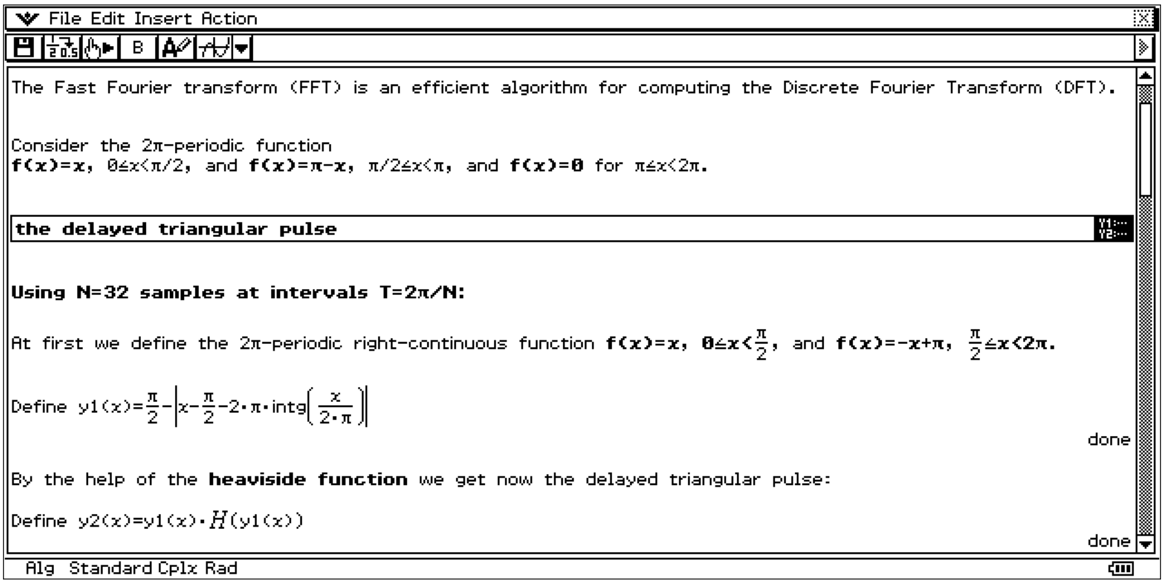
Using the initial conditions we get in one step:

$\mathrm{dSolve}(y'''-3y''+3y'-y=\cos(2t)\times e^{-t},t,y,t=0,y=0,t=0,y'=1/32,t=0,y''=-3/16)$

$$\left\{y=\frac{t\cdot e^t}{32}-\frac{e^t}{32}+\frac{\cos(2\cdot t)\cdot e^{-t}}{32}+\frac{\sin(2\cdot t)\cdot e^{-t}}{32}\right\}$$

This exercise shows how to work with the CAS to support the learning process of our students. We can see step by step in the handheld what happens to solve a given problem.

Example of computing a FFT of a sequence generated by sampling the time signal:



The Fast Fourier transform (FFT) is an efficient algorithm for computing the Discrete Fourier Transform (DFT).

Consider the 2π-periodic function
f(x)=x, 0≤x<π/2, and **f(x)=π-x**, π/2≤x<π, and **f(x)=0** for π≤x<2π.

the delayed triangular pulse

Using N=32 samples at intervals T=2π/N:

At first we define the 2π-periodic right-continuous function **f(x)=x, 0≤x<$\frac{\pi}{2}$**, and **f(x)=-x+π, $\frac{\pi}{2}$≤x<2π.**

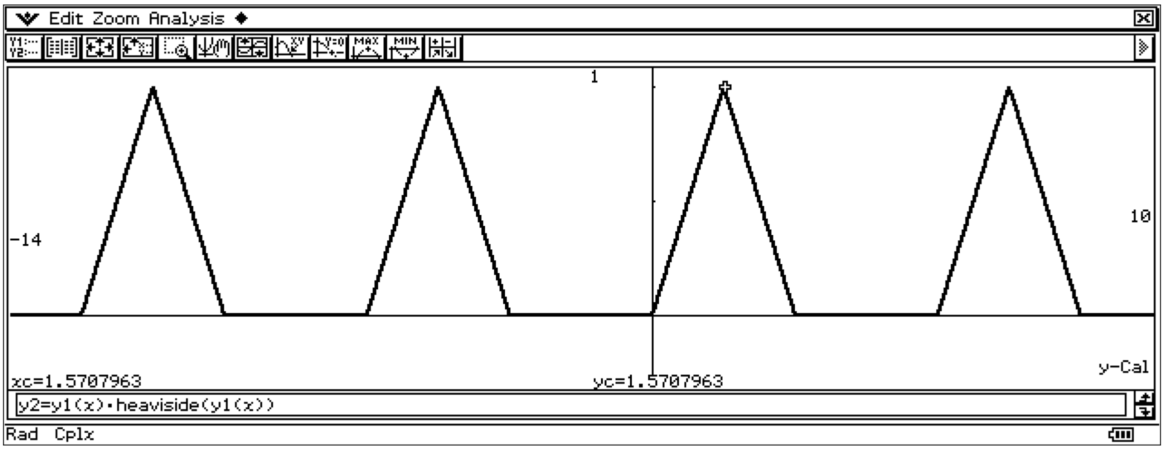
$\mathrm{Define}\ y1(x)=\frac{\pi}{2}-\left|x-\frac{\pi}{2}-2\cdot\pi\cdot \mathrm{intg}\left(\frac{x}{2\cdot\pi}\right)\right|$

done

By the help of the **heaviside function** we get now the delayed triangular pulse:

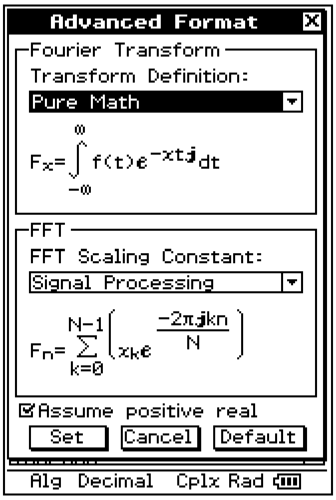
$\mathrm{Define}\ y2(x)=y1(x)\cdot H(y1(x))$

done



y2=y1(x)·heaviside(y1(x))

Now we have to setup the **Advanced Format** to compute the results in the wished manner:



Fourier Transform

Transform Definition: Pure Math

$$F_x=\int_{-\infty}^{\infty} f(t)e^{-xtj}dt$$

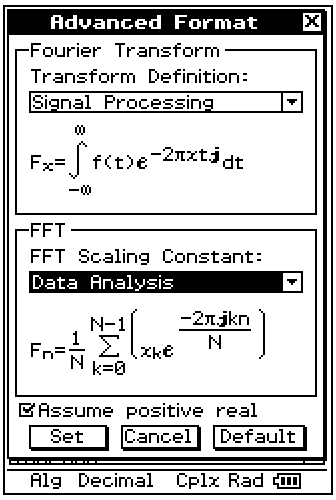
FFT

FFT Scaling Constant: Signal Processing

$$F_n=\sum_{k=0}^{N-1}\left(x_k e^{\frac{-2\pi jkn}{N}}\right)$$

☑ Assume positive real

or



Fourier Transform

Transform Definition: Signal Processing

$$F_x=\int_{-\infty}^{\infty} f(t)e^{-2\pi xtj}dt$$

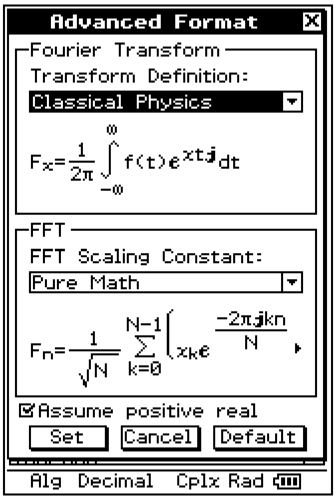
FFT

FFT Scaling Constant: Data Analysis

$$F_n=\frac{1}{N}\sum_{k=0}^{N-1}\left(x_k e^{\frac{-2\pi jkn}{N}}\right)$$

☑ Assume positive real

or



Fourier Transform

Transform Definition: Classical Physics

$$F_x=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{xtj}dt$$

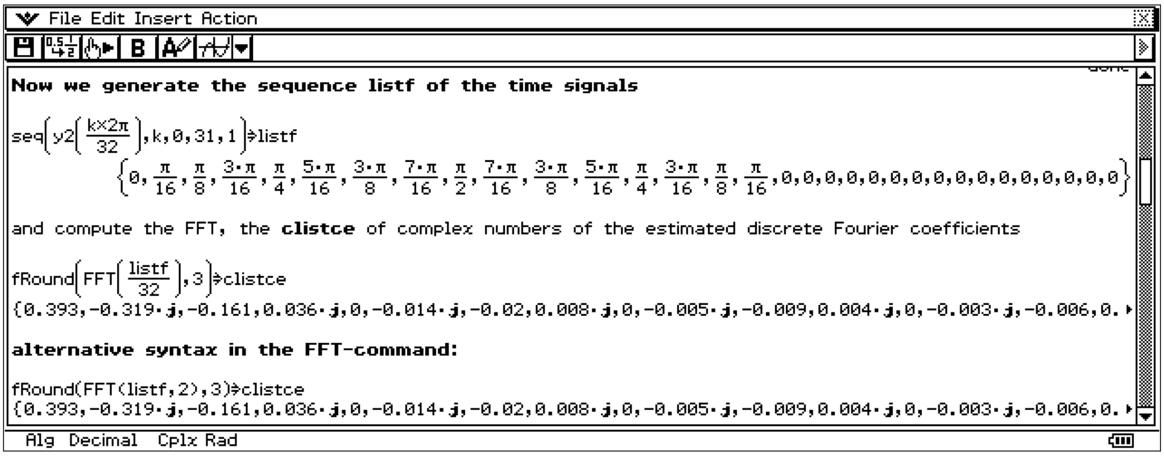
FFT

FFT Scaling Constant: Pure Math

$$F_n=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\left(x_k e^{\frac{-2\pi jkn}{N}}\right.$$

…

We use the Advanced Format for FFT with Signal Processing and divide the listf by N or we use the optional Parameter 2 (=Data Analysis) without division the listf by N:



Now we generate the sequence listf of the time signals

$$\mathrm{seq}\left(y2\left(\frac{k\times 2\pi}{32}\right),k,0,31,1\right)\Rightarrow \mathrm{listf}$$

$$\left\{0,\frac{\pi}{16},\frac{\pi}{8},\frac{3\cdot\pi}{16},\frac{\pi}{4},\frac{5\cdot\pi}{16},\frac{3\cdot\pi}{8},\frac{7\cdot\pi}{16},\frac{\pi}{2},\frac{7\cdot\pi}{16},\frac{3\cdot\pi}{8},\frac{5\cdot\pi}{16},\frac{\pi}{4},\frac{3\cdot\pi}{16},\frac{\pi}{8},\frac{\pi}{16},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\}$$

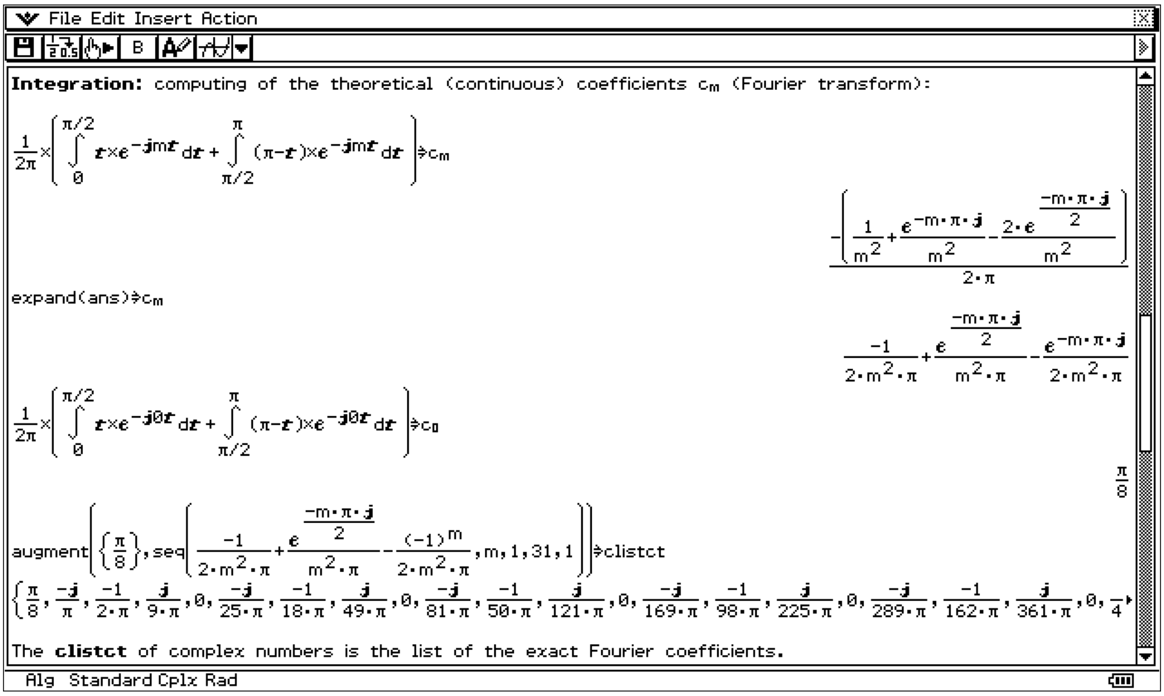
and compute the FFT, the **clistce** of complex numbers of the estimated discrete Fourier coefficients

$$\mathrm{fRound}\left(\mathrm{FFT}\left(\frac{\mathrm{listf}}{32}\right),3\right)\Rightarrow \mathrm{clistce}$$

```
{0.393,-0.319·j,-0.161,0.036·j,0,-0.014·j,-0.02,0.008·j,0,-0.005·j,-0.009,0.004·j,0,-0.003·j,-0.006,0.
```

alternative syntax in the FFT-command:

```
fRound(FFT(listf,2),3)⇒clistce
{0.393,-0.319·j,-0.161,0.036·j,0,-0.014·j,-0.02,0.008·j,0,-0.005·j,-0.009,0.004·j,0,-0.003·j,-0.006,0.
```



Integration: computing of the theoretical (continuous) coefficients c_m (Fourier transform):

$$\frac{1}{2\pi}\times\left(\int_0^{\pi/2} t\times e^{-jmt}dt+\int_{\pi/2}^{\pi}(\pi-t)\times e^{-jmt}dt\right)\Rightarrow c_m$$

$$-\frac{\left(\frac{1}{m^2}+\frac{e^{-m\cdot\pi\cdot j}}{m^2}-\frac{2\cdot e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2}\right)}{2\cdot\pi}$$

$$\mathrm{expand}(\mathrm{ans})\Rightarrow c_m$$

$$\frac{-1}{2\cdot m^2\cdot\pi}+\frac{e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2\cdot\pi}-\frac{e^{-m\cdot\pi\cdot j}}{2\cdot m^2\cdot\pi}$$

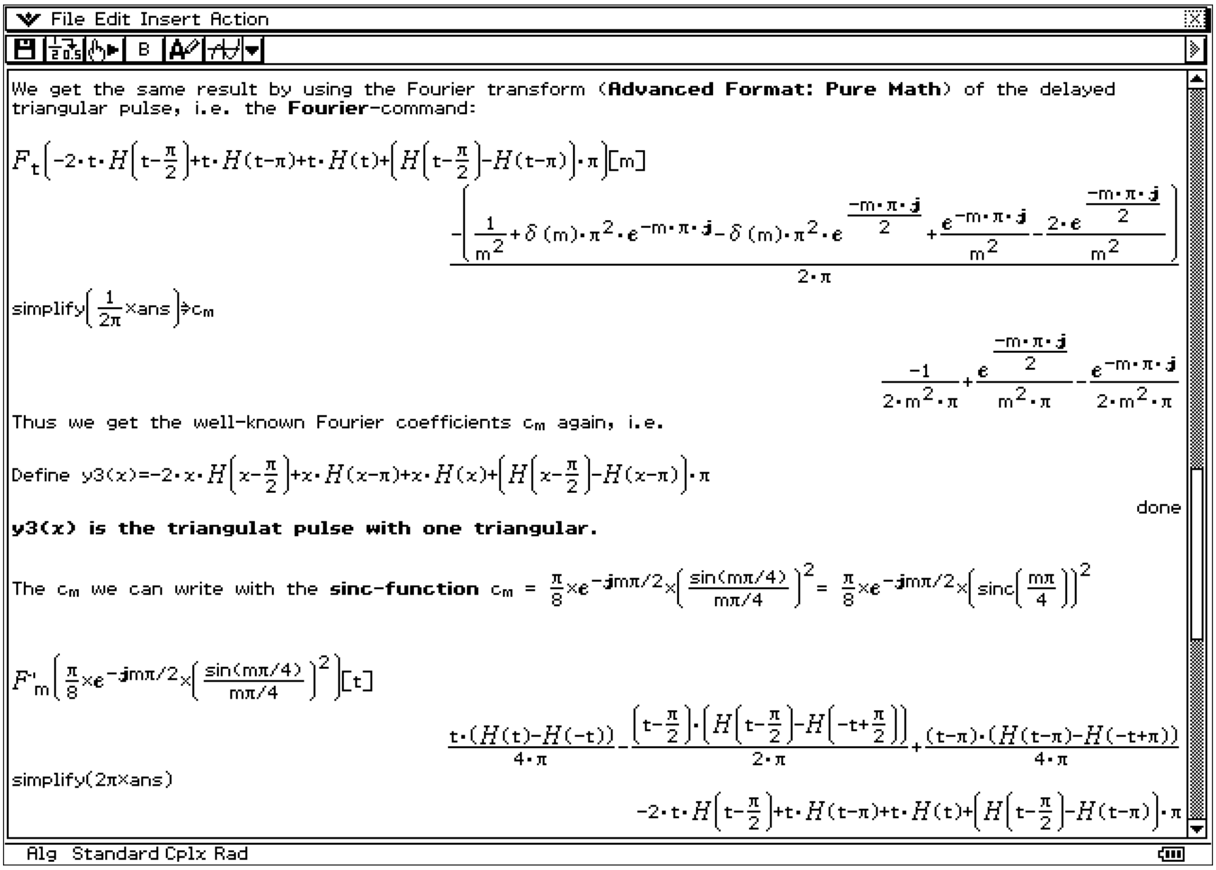
$$\frac{1}{2\pi}\times\left(\int_0^{\pi/2} t\times e^{-j0t}dt+\int_{\pi/2}^{\pi}(\pi-t)\times e^{-j0t}dt\right)\Rightarrow c_0$$

$$\frac{\pi}{8}$$

$$\mathrm{augment}\left(\left\{\frac{\pi}{8}\right\},\mathrm{seq}\left(\frac{-1}{2\cdot m^2\cdot\pi}+\frac{e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2\cdot\pi}-\frac{(-1)^m}{2\cdot m^2\cdot\pi},m,1,31,1\right)\right)\Rightarrow \mathrm{clistct}$$

$$\left\{\frac{\pi}{8},\frac{-j}{\pi},\frac{-1}{2\cdot\pi},\frac{j}{9\cdot\pi},0,\frac{-j}{25\cdot\pi},\frac{-1}{18\cdot\pi},\frac{j}{49\cdot\pi},0,\frac{-j}{81\cdot\pi},\frac{-1}{50\cdot\pi},\frac{j}{121\cdot\pi},0,\frac{-j}{169\cdot\pi},\frac{-1}{98\cdot\pi},\frac{j}{225\cdot\pi},0,\frac{-j}{289\cdot\pi},\frac{-1}{162\cdot\pi},\frac{j}{361\cdot\pi},0,\frac{-}{4}\right.$$

The **clistct** of complex numbers is the list of the exact Fourier coefficients.



We get the same result by using the Fourier transform (**Advanced Format: Pure Math**) of the delayed triangular pulse, i.e. the **Fourier**-command:

$$F_t\left(-2\cdot t\cdot H\left(t-\frac{\pi}{2}\right)+t\cdot H(t-\pi)+t\cdot H(t)+\left(H\left(t-\frac{\pi}{2}\right)-H(t-\pi)\right)\cdot\pi\right)[m]$$

$$\frac{-\left(\frac{1}{m^2}+\delta(m)\cdot\pi^2\cdot e^{-m\cdot\pi\cdot j}-\delta(m)\cdot\pi^2\cdot e^{\frac{-m\cdot\pi\cdot j}{2}}+\frac{e^{-m\cdot\pi\cdot j}}{m^2}-\frac{2\cdot e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2}\right)}{2\cdot\pi}$$

$$\mathrm{simplify}\left(\frac{1}{2\pi}\times ans\right)\Rightarrow c_m$$

$$\frac{-1}{2\cdot m^2\cdot\pi}+\frac{e^{\frac{-m\cdot\pi\cdot j}{2}}}{m^2\cdot\pi}-\frac{e^{-m\cdot\pi\cdot j}}{2\cdot m^2\cdot\pi}$$

Thus we get the well-known Fourier coefficients c_m again, i.e.

$$\mathrm{Define}\ y3(x)=-2\cdot x\cdot H\left(x-\frac{\pi}{2}\right)+x\cdot H(x-\pi)+x\cdot H(x)+\left(H\left(x-\frac{\pi}{2}\right)-H(x-\pi)\right)\cdot\pi$$

done

y3(x) is the triangulat pulse with one triangular.

The c_m we can write with the **sinc-function** $c_m = \frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\frac{\sin(m\pi/4)}{m\pi/4}\right)^2 = \frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\mathrm{sinc}\left(\frac{m\pi}{4}\right)\right)^2$

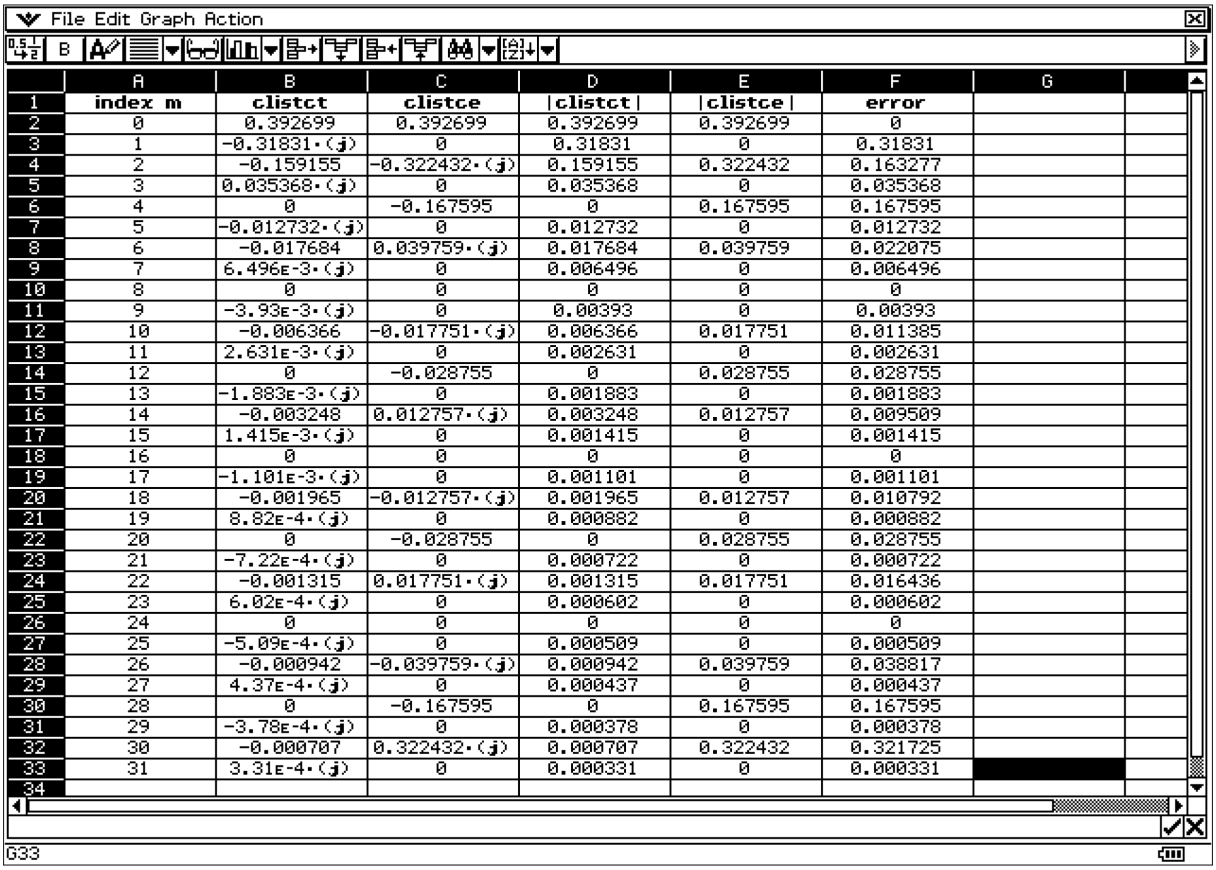
$$F'_m\left(\frac{\pi}{8}\times e^{-jm\pi/2}\times\left(\frac{\sin(m\pi/4)}{m\pi/4}\right)^2\right)[t]$$

$$\frac{t\cdot(H(t)-H(-t))}{4\cdot\pi}-\frac{\left(t-\frac{\pi}{2}\right)\cdot\left(H\left(t-\frac{\pi}{2}\right)-H\left(-t+\frac{\pi}{2}\right)\right)}{2\cdot\pi}+\frac{(t-\pi)\cdot(H(t-\pi)-H(-t+\pi))}{4\cdot\pi}$$

$$\mathrm{simplify}(2\pi\times ans)$$

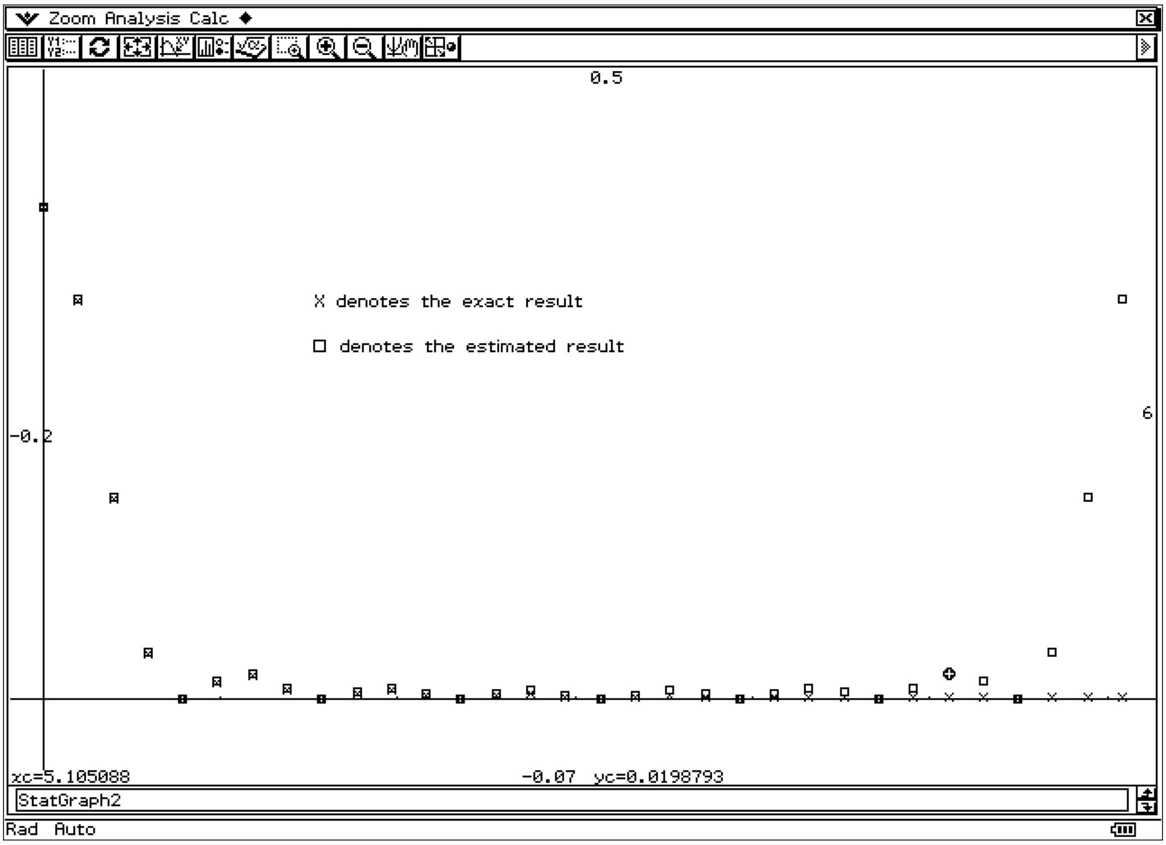
$$-2\cdot t\cdot H\left(t-\frac{\pi}{2}\right)+t\cdot H(t-\pi)+t\cdot H(t)+\left(H\left(t-\frac{\pi}{2}\right)-H(t-\pi)\right)\cdot\pi$$

Have a look into the lists (spreadsheet-application)



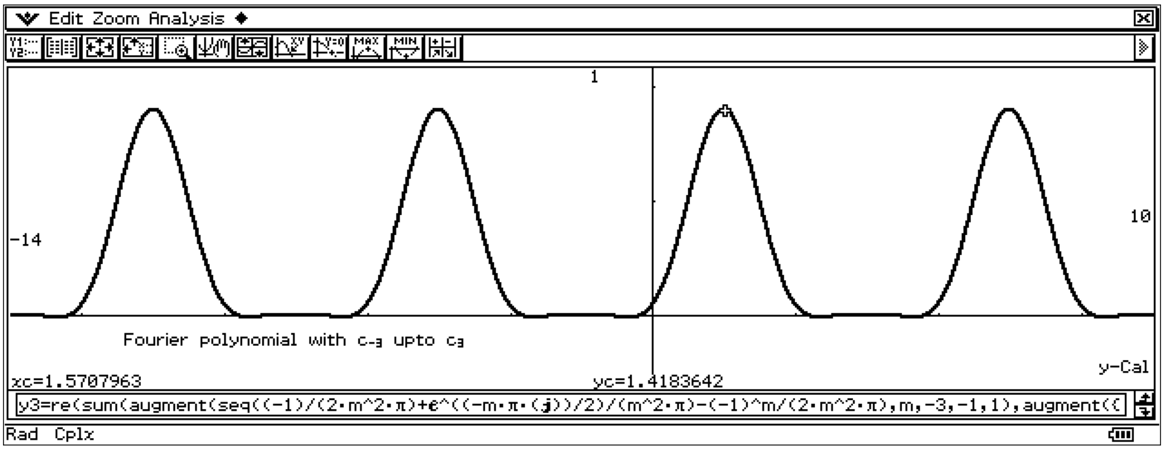
	A	B	C	D	E	F	G
1	**index m**	**clistct**	**clistce**	**|clistct|**	**|clistce|**	**error**	
2	0	0.392699	0.392699	0.392699	0.392699	0	
3	1	-0.31831·(j)	0	0.31831	0	0.31831	
4	2	-0.159155	-0.322432·(j)	0.159155	0.322432	0.163277	
5	3	0.035368·(j)	0	0.035368	0	0.035368	
6	4	0	-0.167595	0	0.167595	0.167595	
7	5	-0.012732·(j)	0	0.012732	0	0.012732	
8	6	-0.017684	0.039759·(j)	0.017684	0.039759	0.022075	
9	7	6.496E-3·(j)	0	0.006496	0	0.006496	
10	8	0	0	0	0	0	
11	9	-3.93E-3·(j)	0	0.00393	0	0.00393	
12	10	-0.006366	-0.017751·(j)	0.006366	0.017751	0.011385	
13	11	2.631E-3·(j)	0	0.002631	0	0.002631	
14	12	0	-0.028755	0	0.028755	0.028755	
15	13	-1.883E-3·(j)	0	0.001883	0	0.001883	
16	14	-0.003248	0.012757·(j)	0.003248	0.012757	0.009509	
17	15	1.415E-3·(j)	0	0.001415	0	0.001415	
18	16	0	0	0	0	0	
19	17	-1.101E-3·(j)	0	0.001101	0	0.001101	
20	18	-0.001965	-0.012757·(j)	0.001965	0.012757	0.010792	
21	19	8.82E-4·(j)	0	0.000882	0	0.000882	
22	20	0	-0.028755	0	0.028755	0.028755	
23	21	-7.22E-4·(j)	0	0.000722	0	0.000722	
24	22	-0.001315	0.017751·(j)	0.001315	0.017751	0.016436	
25	23	6.02E-4·(j)	0	0.000602	0	0.000602	
26	24	0	0	0	0	0	
27	25	-5.09E-4·(j)	0	0.000509	0	0.000509	
28	26	-0.000942	-0.039759·(j)	0.000942	0.039759	0.038817	
29	27	4.37E-4·(j)	0	0.000437	0	0.000437	
30	28	0	-0.167595	0	0.167595	0.167595	
31	29	-3.78E-4·(j)	0	0.000378	0	0.000378	
32	30	-0.000707	0.322432·(j)	0.000707	0.322432	0.321725	
33	31	3.31E-4·(j)	0	0.000331	0	0.000331	
34							

Consider graphical representations of the absolute values of coefficients:

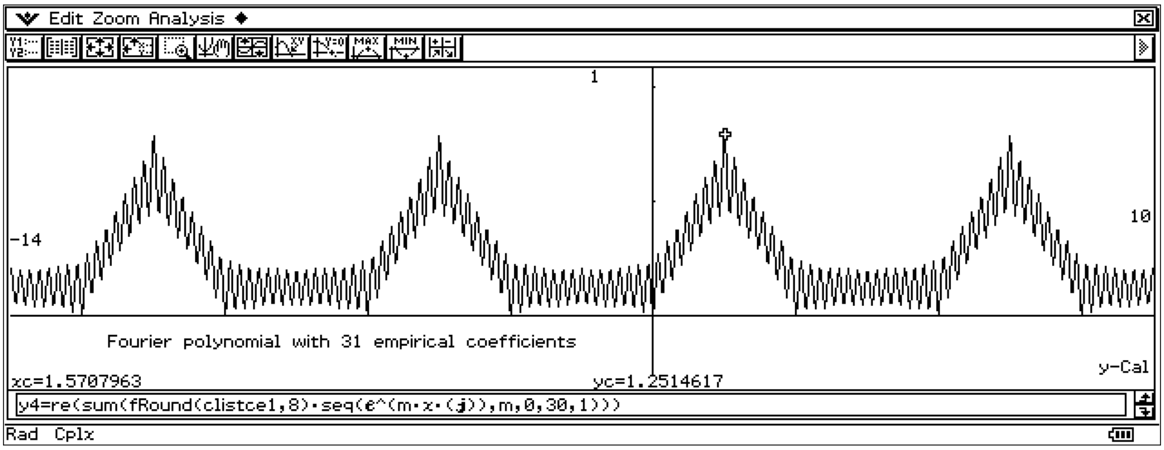


```
xc=5.105088          yc=0.0198793
StatGraph2
```

Graphical representations of the Fourier polynomial:



```
xc=1.5707963          yc=1.4183642
y3=re(sum(augment(seq((-1)/(2·m^2·π)+ℯ^((-m·π·(𝐢))/2)/(m^2·π)-(-1)^m/(2·m^2·π),m,-3,-1,1),augment((
```



```
xc=1.5707963          yc=1.2514617
y4=re(sum(fRound(clistce1,8)·seq(ℯ^(m·x·(𝐢)),m,0,30,1)))
```