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Hypothesis Aided Approach to the Instruction of the Limit of a Function

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Abstract

The concept of the limit of a function is undoubtedly the key to higher mathematics. With a view to very fine mathematical essence of the notion mathematics educators permanently deliberate what didactic method to take in order to reach relatively satisfactory level of its understanding. The paper presents an approach based on the aid of hypothesis that is put forward by means of calculator support.

Introduction

The concept of the limit of a function is a crucial issue of higher mathematics. Its extremely delicate mathematical essence brings about a lot of difficult didactic questions for the mathematics instructors when commencing calculus. We do not intend to open the problem from the viewpoint of timing of calculus teaching. We respect the diversified reality that calculus is taught both on secondary and tertiary level. On the other hand we consistently take into account that the entire matter concerns instructing of non-mathematicians who need the concept of a limit particularly for the purpose of introducing derivatives to investigate rates of changes (Strang [8]). The portfolio of such students is formed mostly by secondary school students and undergraduate university students of programs where a standard course of calculus is prescribed. Today's undergraduate students do not have the background and experience in rigorous thinking. They are unaccustomed to proofs and to the strict rules of mathematical logic. Our task is not just teaching these students some advanced mathematics, but also teaching them how to think. Despite a certain categorization of teachers on traditionalists, that seem to prefer the drill first and then developing understanding and reformers, favoring the reverse dogma (see Krantz [4]), mathematics instructors constitute a heterogeneous group. This heterogeneity is even more expressive when discussing our topic. In any case we will join the group of reformers favoring (also) drill, but that drill should be built atop a bedrock of understanding. Notice, that the problem of didactics of limit has not a unique solution even in case of instructing of mathematics specialists, when the full rigorousness is claimed. The discussions on whether to start with Heine definition using sequences and then to continue with Cauchy ϵ - δ definition or vice versa will probably never end. Although some modernizers tried to use this way of presenting the limit also for non-mathematicians, the resulting positive pedagogical outcome did not turn up. Moreover, it is time consuming and as we know, there is no spare time in teaching hours for mathematics, the subject belonging to the theoretical foundations of the program. Profile disciplines usually need derivatives as early as possible. Our approach (Mezník[5]) may be characterized as possessing the orientation "from Cauchy to Heine", ie. we will start with the limit of a function while the limit of a sequence becomes a special case of the limit at infinity, yet the support of working with sequences of numbers will be evident due to intuitive reasoning. As a matter of fact, we will formulate the notion of a limit of a function in a verbal form containing the moment of approaching or closeness respectively leading to the ability of putting the hypothesis about the limit or about its nonexistence and consequently to the intuitive understanding of the concept. Besides to put a hypothesis requires the use of calculators which is extremely contributing symbiosis. The ability to carry out the hypothesis proved to be a basic test of understanding the concept. Most of students in the first year at the

university have some conception of a limit and possess certain skill in calculation of limits including L'Hospital's rule. We found that many of them have difficulties to associate their knowledge with the mechanism of making hypothesis. It signals the absence of comprehension. Our experience with the hypothesis method is very positive from the viewpoint of the understanding of the limit. The idea of guessing the limit is not quite original (see for instance Buck[1]). We will insert it in the algorithm of limit calculations, which will be one of the main objectives of our further considerations.

Hypothesis about the limit of a function

In the sequel the following formulation for the limit of a function will be employed:

A function $f(x)$ has the limit a at the point c , if for the points close (from both sides) to c , but different from c , the corresponding values of function $f(x)$ are close to the unique value a .

Then we write $\lim_{x \rightarrow c} f(x) = a$. Alternatively we write $f(x) \rightarrow a$ as $x \rightarrow c$ and say that $f(x)$ approaches a as x approaches c .

With a view to the fact that our formulation is not "ε-δ definition" of limits several aspects should be recalled. Partly, approaching of x the number c is from both sides of c , partly approaching of $f(x)$ the number a as to a unique value. Further, the mechanism of approaching of x the number c means that the distance of x from c goes to zero, more precisely that the distance of x from c eventually goes below an arbitrarily small positive number ε and stays there. All the mentioned seeming details should be at least occasionally stressed. Also it is important to realize that no definition of the limit provides the algorithm of its calculation and that the limit of the function at a point ignores whether or not $f(x)$ is defined at this point itself.

We put a hypothesis about a limit of the function $f(x)$ at the point c in the way that we substitute values of x closer and closer (from both sides of c) to c into $f(x)$ and from the trend of the resulting values of $f(x)$ we judge the value of limit or its nonexistence.

Example We put a hypothesis about $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$. Denote $f(x) = \frac{x^2 - 1}{x - 1}$. Choose numbers close to 1, eg 1.1, 0.9, 1.01, 0.99. Computation gives $f(1.1) = 2.1$, $f(0.9) = 1.9$, $f(1.01) = 2.01$, $f(0.99) = 1.99$. From the resulting values we put number 2 forward as a hypothesis about the limit.

Example We put a hypothesis about $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. In a similar way as in Example 1 denoting

$f(x) = \frac{\sin x}{x}$ computation gives $f(0.1) = f(-0.1) = 0.998$, $f(0.05) = f(-0.05) = 0.995$, $f(0.001) = f(-0.001) = 0.9998$. Now, we may put number 1 forward as a hypothesis about the limit.

Example We put a hypothesis about $\lim_{x \rightarrow 0} \frac{1}{x}$. Denoting $f(x) = \frac{1}{x}$ we get $f(0.1) = 10$, $f(-0.1) = -10$, $f(0.01) = 100$, $f(-0.01) = -100$. We obviously come to the conclusion, that approaching 0 from the right the resulting values of the function are increasing positive numbers and approaching 0 from the left the resulting values are decreasing negative numbers. So the resulting values can not approach any unique value. Hence we put a hypothesis that the limit does not exist.

Obviously, putting a hypothesis about the limit requires the use of a calculator. We must be careful of possible breakdowns caused by representation of numbers and numerical aspects of calculations with approximate numbers. We will comment on it in the sequel.

Calculation of limits

We have already pointed out that the formulation of the limit does not provide a straightforward way how to find it. Since we have resigned to use standard mathematical tools, our minimal task is to suggest the way of correct calculations and handling with limits of “well-behaved” functions given as elementary functions, that are mostly emerging in current applications. In other words, the aim is to achieve certain rigor without rigor mortis. For this purpose we propose two algorithms- *Algorithm 1* and *Algorithm 2*, while from practical reasons *Algorithm 2* is used when *Algorithm 1* is not applicable.

Algorithm 1 This algorithm is applied in case if the function satisfies the condition of continuity. The concept of continuity is usually (but not necessarily) introduced in terms of the limit, i.e. after limits. Since we work with elementary functions we may at this stage state the rule:

The limits of elementary functions at the points of their standard domains are calculated by substituting the point into the function. In symbols, under given conditions

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Algorithm 2 This algorithm has a general use but if algorithm 1 may be used, then by its application the calculation of limit is much simpler, as mentioned above. It proceeds in two steps, while Step 1 may be omitted, though it is from several reasons valuable.

Step 1 We put a hypothesis about the limit as described in the previous paragraph. We may skip this step when the calculation of the limit is quite evident (experience, knowledge of formulas, etc.). The hypothesis is very important if it signalizes the nonexistence of the limit. In that case the finding of the limit is unreasonable and the negative result follows. In the affirmative the hypothesis may control the correctness of further calculations performed in the next step. Besides, in the event that our calculation in the next step is not successful, the hypothesis may serve as an orientation in the possible result.

Step 2 We state three basic theorems on limits describing the behavior of limits with respect to arithmetic operations, namely the limit of the sum, the product and the ratio of functions and give the list of important formulas on limits. Usually the following formulas are mentioned:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0, \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

We refer to the fact, that the given limits can not be calculated using *Algorithm 1* because although they are elementary functions, they are not defined at the limiting point 0 . Also draw the attention to the frequent situation making the use of *Algorithm 1* impossible- after substituting the limiting point (in our case 0) into f we come to the undefined expression. In

our case we formally get $\frac{0}{0}$; notice of intention is desirable that a gross mistake is often

made, namely that the result is 1. It seems to be sensible at this moment to present more examples leading to similar undefined expressions. Further, for a given $\lim_{x \rightarrow c} f(x)$ we present a

method (or a trick) of replacing a function f by another function, say g , that agrees with it near a point c , but which is elementary and defined at c , which makes the calculation easy applying *Algorithm 1*. It is absolutely necessary to give convincing argument of the correctness of such replacement to find the limit, namely to stress the fact that according to our definition the limit at c does not depend on the value $f(c)$ or on its nonexistence, respectively. Typical instances of such functions are rational functions that may be simplified by means of factorization in such a way that *Algorithm 1* is applicable. As we know, not a few instructors like this method, which in any case contributes to the flourishing of creativity. It is questionable how much time should be devoted to drill this technique.

Finally, let us make a note that in the event that when using Algorithm 2 step 1 was omitted and step 2 did not yield a correct result, it is necessary to return back to step 1, to put a hypothesis about the limit, advance again to step 2 and try a new attempt to find the limit.

Other types of limits

There is a strong analogy between limits and their mostly treated variants-limits at infinity, infinite limits and one-sided limits. For the above specified purpose seems to be sufficient only to modify the formulation of the limit admitting that real numbers c and a may be replaced by the symbols $\pm\infty$ explaining intuitively their meaning and similarly for one-sided limits changing approaching from both sides by approaching from either left or right side only. Symbols $\pm\infty$ evoke at students an impression of mystery and the possibility to get over the borders of finity. In fact this infinite version fills the concept of a limit with full sense. To show that we do not abandon the world of reality some convincing motivation from real processes is desirable. “Economics for all” supplies a grateful suggestion as the following example illustrates.

Example Total cost TC of a firm is given by $TC = TC(Q) = 10 + 4Q$, where Q is a production. Number 10 represents so called fixed cost and number 4 variable cost per unit of production.

Average cost AC is the ratio of total cost and production, $AC = AC(Q) = \frac{10}{Q} + 4$. It is

legitimate to investigate what happens with average cost when the production grows to large amount. In terms of the limit we ask for the limit of AC when Q approaches to infinity. It is quite evident that the larger Q is then $\frac{10}{Q}$ is closer to 0, ie. as Q increases then $\frac{10}{Q}$ approaches

0 and consequently AC approaches 4 (the graph of AC is very helpful). In practical formulation in economics it anticipates the trend of behavior of average cost as production grows, i.e. if production is sufficiently large, then average cost approximately equals variable cost per unit of production.

The advance from infinite type limits to one-sided limits is more convenient for one-sided limits have also infinite variants. Of course, some further calculation formulas and also rules (best in symbolic form) for “calculations” with $\pm\infty$ symbols are to be added. We did not get into difficulties with students comprehension when employing for other types of limits the above mentioned didactic approach for limits. For limits of infinite types the support of geometric tools is strongly recommended. Of an extraordinary importance is the instance of the limit at infinity leading to the irrational number $e = 2,71828\dots$, the base of the natural

logarithm, $e = \lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x$. It is also a quite exceptional case when putting a hypothesis about the limit. There is no other way then to take this limit for granted.

As a special case of the limit at infinity *the limit of a sequence* may be considered for a sequence is a special type of function defined on the set of natural numbers. With a view to existing theorem, we formally express the sequence as a function and the result of calculation of the limit at infinity we assign to the sequence.

Computing breakdowns

Working with hypothesis we may encounter computing problems that are caused by the representation of numbers in computers. It is a good opportunity to release students objective complications stemming from the use of calculators or more sophisticated computing devices. Many of them currently use mathematical software and they should be aware of possible so called legitimate (irremovable) errors that are theoretically very complicated. The realization

of mathematical software has variety of faces and may perform calculations on numbers differently. Particularly those based on the principle of symbolic calculations process numbers with a fixed number of digits after the decimal point. All those mentioned facts may considerably negatively influence the results as the following example illustrates.

Example We intend to put forward a hypothesis about $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x$. Rearranging we get

$\lim_{x \rightarrow \infty} (\frac{x+1}{x})^x$. Computation (with a standard concrete calculator) gives (for $f(x) = (\frac{x+1}{x})^x$)

$f(100) = 2.70481\dots$, $f(1000) = 2.71692\dots$, $f(10^6) = 2.71828\dots$, $f(10^{10}) = 1$. The obvious reason is that starting from some (sufficiently large) value of x the ratio $\frac{x+1}{x}$ is very close to

1 and due to technical limitations of a calculator is identified as 1 and therefore for 1^x yields 1. We realize that such hypothesis is out of use.

Conclusions

The objective of the paper is to propose an approach to in many ways discussed the problem of instruction of the limit. The following boundary conditions were regarded: (1) The instruction concerns non-mathematicians (2) Limited time is at the disposal for the issue (3) The understanding has the priority over drill calculations (4) The main purpose is to set up the tool to define a derivative. The stress is laid on putting forward a hypothesis about the limit which requires the use of calculators. This proved to be an active element in grasping the concept of the limit. We do not claim a definite way how to proceed to instruction of the limit. Didactic is still and would last forever an open and turbulent discipline.

References

- [1] Buck, R.C.: *Advanced Calculus*. McGraw-Hill Kogakusha, LTD., Tokyo 1978
- [2] ICMI : *The Impact of Computers and Informatics on Mathematics and its Teaching*. Cambridge University Press, 1986
- [3] ICMI : *School Mathematics in the 1990s*. ICMI Study Series (Ed. A.G. Howson and J.-P. Kahane), Kuwait University Press 1986
- [4] Krantz, G.S.: *How to Teach Mathematics*. American Mathematical Society, Providence, Rhode Island 1999
- [5] Mezník, I.: *Matematika I*. Faculty of Business and Management, Brno University of Technology in Academic Publ. Comp. CERM, Brno 2008 (in Czech)
- [6] Sierpinska, A: *Humanities Students & Epistemological Obstacles Related to Limits*. Educational Studies in Mathematics 18, 1987, 371-397
- [7] Sierpinska, A.: *On Understanding in Mathematics*. Montreal, Qué 1992
- [8] Strang, G.: *Calculus*. Wellesley – Cambridge Press, Wellesley 1991
- [9] Williams, S.R.: *Models of Limit Held by College Calculus Students*. Journal for Research in Mathematics Education 22(3), 1991, 219-236