HOW TO SOLVE IT

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Abstract
This work is a reflection on the results of an experimentation carried out on secondary school students of between 16 and 18 from various classes. The experimentation aims at identifying the implicit ideas they use when asked to solve a certain mathematical problem. In particular, in giving them these problems an heuristic approach was suggested, and the differences between this and a purely deductive approach were measured. Analyzing the different approaches used by the students and the difficulties they had in distinguishing between argumentative and demonstrative operations has given rise to a reflection on the use of software such as Geogebra and Excel.

1. Introduction

The concept of “rigorous proof”, from an educational perspective, is still a problem. Demonstration, meant as the mathematical instrument “par excellence”, is often regarded as a hindrance to the development of the intuition and the capacities of exploration (Hanna, 2000).

Several teachers argue that the most convincing approach for a student is the one that requires investigation and exploration, which eventually stimulate intuition. They even maintain that the deductive demonstration may not be taught any longer (Hanna, 2000).

Furthermore, the increasingly frequent use of dynamic educational software pays ever greater attention to exploration. With this type of software it is possible to literally see various different representations of what is being studied in graphs that the students can easily create themselves.

From the point of view of maths teaching, we must underline that all this, as well as naturally encouraging the formulation of hypotheses, also implies the abandonment (intentional or not) of teaching-learning processes involving deductive proof, leaving the experimental approach as the sole mathematical tool brought to the students’ knowledge.

The software can examine a huge, or even, if we introduce the concept of continuity, infinite quantity of data (Mason 1991).

In effect, experimental speculation and rigorous mathematical proof probably make up the ideal combination, but I strongly believe that not distinguishing between the two approaches when teaching could be extremely dangerous. A distinction is necessary to be able to outline the contribution each culture may provide (Di Paola&Spagnolo, 2008; Spagnolo, 2005; Spagnolo&Ajello, 2008). This is why I suggest that a serious reflection on the inevitable use of computer sciences in teaching mathematics and physics today, is urgent.

The object of this research is to investigate the implicit ideas used by students when asked to prove a mathematical proposition. We will therefore investigate not only the type of proof used by the students, but also the type of reflections induced by a certain teaching approach and by the use of certain instruments.

The methodology used is “the theory of situations” (Brousseau, 1997, Spagnolo et alii, 2009) both in teaching and in analyzing the data.

Given that in Italy maths and physics are taught by the same teacher, we want to see if this encourages pre-comprehension and/or epistemological obstacles among students regarding the type of proof used in teaching the two different subjects.

2.1. Experimentation: first step

Experimentation was carried out on secondary school students of between 16 and 18 from various classes.

The students interviewed were given two problems:

1. Determine, among all rectangles with the same perimeter length, the one with the maximum area.
2. Determine, among all triangles with a given hypotenuse, the one that has the maximum ratio between the hypotenuse and the sum of the other two sides.

All students know analytical geometry, the concept of place, of points, and the concept of continuity in a function. Only a few of them know the problems regarding derivability.

The students in this experimentation were allowed to use Geogebra and Excel and were asked to write down the reasoning they used to solve the given problems. They were told to write simply and freely about anything that helped them formulate the hypotheses which led to their proof, when they found one. It was clearly stated that

they could use any non-specific terms that came to mind if they felt it necessary. They were also asked to express their impressions and feelings.
In the previous lessons a classroom experimentation on the motion of a mass on an inclined plane had been performed, the experimental data was collected and the curve most likely to describe the event was sought. The students, obviously, were asked to determine unknown quantities and approximate values, to repeat the experiments several times and to draw a conclusion. The teacher who followed them in this experimentation highlighted the fact that even before beginning the experimentation they already had an implicit idea of what should happen given that they had to look for some relation between time and space using certain objects: a trolley, an inclined plane, a position sensor and statistical analysis software. In other words, all they had to do in the lab was to confirm the following hypothesis: the distance covered by a trolley subjected to a constant force is proportional to the square of the time taken to cover this distance.
The problems were tackled by all students using Geogebra. All students were able to make hypotheses. Almost everyone formulated correct hypotheses (in the first problem, the rectangle of maximum area is the one with sides all of the same length; the second solution is that the triangle we are looking for is isosceles and the ratio is $\sqrt{2}$).
As for the proofs only a few students tried to write them down and submitted them.
Among those who tried to demonstrate the proof, a lot were satisfied with the example generated by Geogebra which was fairly clear and meaningful and helped the students in tackling the problem.
After this introductory part, mainly aimed at giving the problem to the students, there began the second part of the experimentation.

2.2. Experimentation: second step

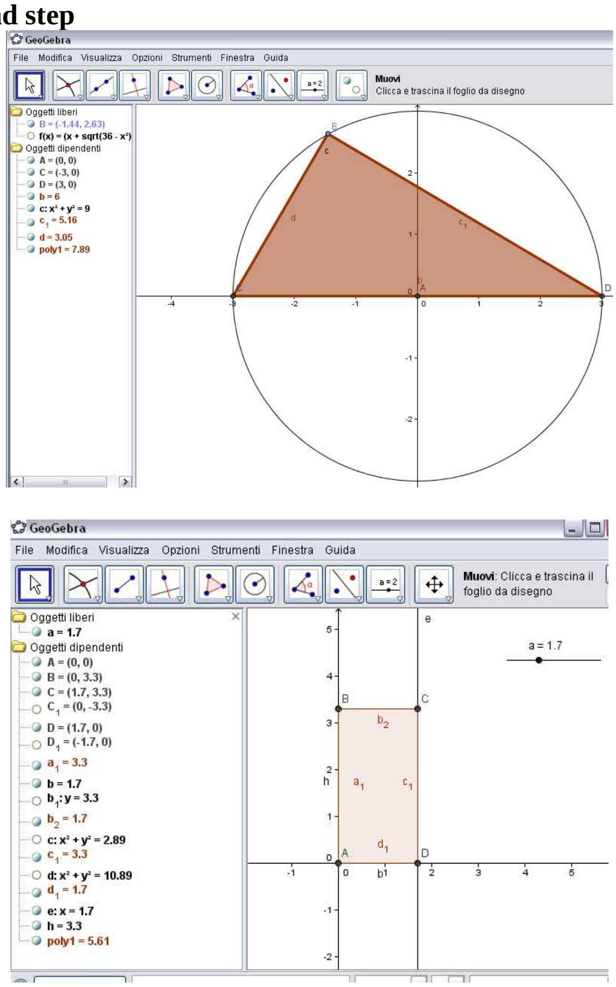

Figure 1

Two different types of strategies were provided to students.

The first strategy presented was a collection of data with the Geogebra, re-elaborated on Excel.
The reasoning processes are very simple since they are based on images generated by Geogebra and modified by "dragging"(Fig. 1). What is important is that this software, in addition to watching what happens moment by moment when the size of a side is modified, shows the dimensions of the segments and areas.
The proposed strategy involves nothing more than reporting the data obtained through real "sampling" in a two column Excel table: the first according to the values of the base, the second to the values of the area (Fig.2). It is a process which, together with measuring physical quantities, sounds familiar to the students because this table is also used to obtain pairs of values from a function using the concepts of a dependent and independent variable (*e.g.*: area and segment respectively). In this way it is easy to obtain points on a Cartesian plane representing not only the experimentation but also a curve which provides an estimated trend of intermediate values which have not been actually investigated. Excel is able to generate a graph connecting the points to give rise to a curve (in the first problem which resembles a parabola in every sense).
If this process was followed by each student we would obtain different graphs all for the same problem applying different values, but which would all concur in identifying the square in the first case and the isosceles triangle (and a ratio of 1.41) in the second.
The methods of solving the two problems are similar. The difference of the second exercise is that it requires the determination of an irrational number that is impossible to identify with absolute precision through data processing techniques.

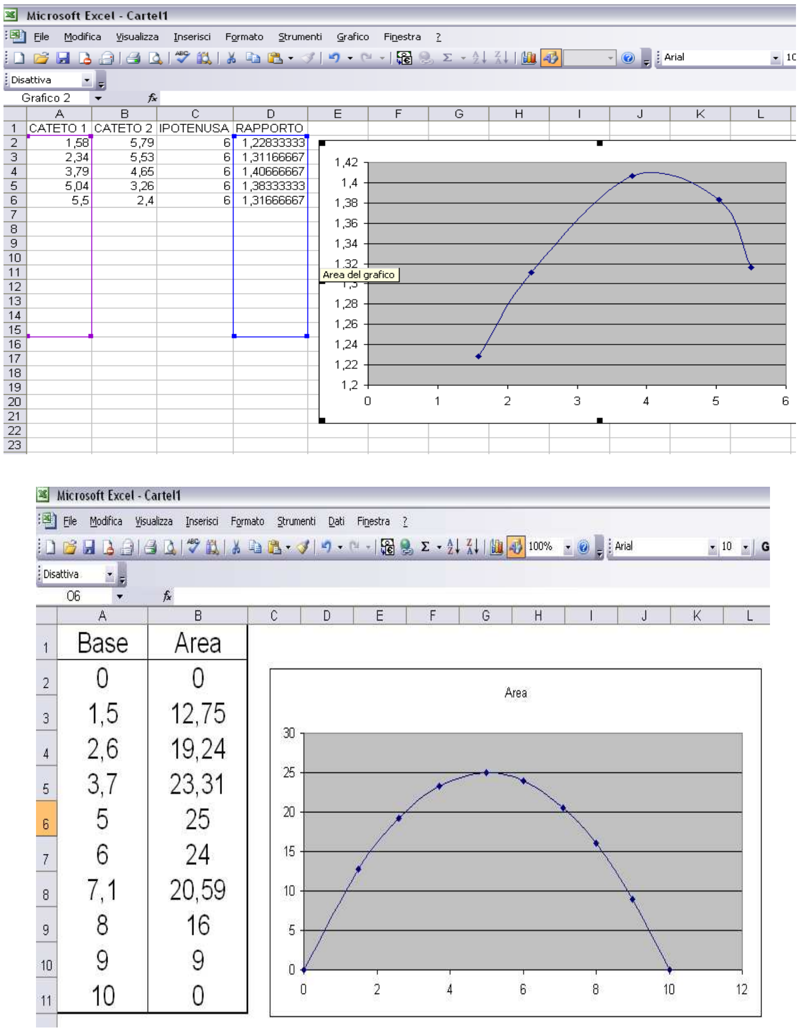

Figure 2

The second strategy is a typical textbook approach, modeling the situation with a function whose analysis provides the result.

2.3. Experimentation: step 3

Each step of this process was followed by personal notes. The students were asked to express their opinions and considerations: which approach did they prefer and why? The analysis of the data shows that most of the students:

- considered the heuristic approach more effective and easy to handle;
- considered the maths proof more elegant and provides the optimum response to the problems given;
- on the other hand, they themselves admitted that they could have never managed to produce a complete proof; the heuristic approach, however, does give the correct answer and lets them create the graph of the function quickly.

As for the comparison between the operating procedures confirming hypotheses on both the physical and mathematical problems:

- few students spoke of the difference between inductive and deductive procedures, thinking that the former is used more in physics and the latter in maths, but without going into details;
- most of the students found that the software gave more rapid solutions to this type of exercise: we can thus conclude that they prefer the inductive method and would like to use it preferentially in class.

3. Conclusions

The main thing that emerges, and on which it would seem necessary to dedicate more attention, is a certain contradiction:

- the software does help students formulate hypotheses;
- all the students found the empiric approach within their understanding, valid and effective;
- the approximations inherent in the use of this software seem acceptable to them (as it was in the physics problem);
- a simple data base like Excel immediately supplies a mathematical model of the solutions.

However, almost all the students found that the deductive procedure:

- is more elegant;
- is easy to apply in general;
- is the “correct” procedure, and would be a Good Thing to learn, develop and use.

Contradiction mentioned above suggest questions about what is the reason why the students consider “correct” and “beautiful” some particular types of reasoning. Therefore, I believe that a further development of the experimentation could be the analysis of the demonstration strategies of students from different cultures.

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