Elementary Students' Construction of Proportional Reasoning Problems: Using Writing to Generalize Conceptual Understanding in Mathematics

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Abstract

This study engaged fourth and fifth graders in solving a set of proportional tasks with focused discussion and concept development by the teacher. In order to understand the students' ability to generalize the concept, they were asked to write problems that reflected the underlying concepts in the tasks and lessons. A qualitative analysis of the student generated problems show that the majority of the students were able to generalize the concepts. The analysis allowed for a discussion of problems solving approaches and a rich description of how students applied multiplicative reasoning in composing mathematics problems. These results are couched in a discussion of how the students solved the proportional reasoning tasks.

Introduction

Proportionality is one of those important mathematical topics that is not clearly defined as a set of ideas that build on each other. Proportional reasoning involves complex thinking involving a sense of co-variation and multiple comparisons and is concerned with inference and prediction involving both qualitative and quantitative methods of thought (Lesh, Post & Behr, 1988). While there is a wide range of studies on rational number, such research does not always emphasize ideas of proportional reasoning that are inherent in the concepts and/or the emphasis is often on the development of 'number sense' without explicit identification of potential ties to of proportional reasoning.

Proportionality permeates mathematics and is often considered as the foundation to abstract mathematical understanding. Analyzing students' thinking relative to their work with problems involving proportions can inform teachers so that their instruction is better suited to promote proportional reasoning. Lesh, Post & Behr (1988) believe that proportional reasoning is the capstone of children's arithmetic thinking and the cornerstone of their ensuing mathematical progress. The influence of instruction on the development of more sophisticated levels of proportional reasoning is well documented in the literature (Steinthorsdottir, 2005; Pittalis, Christou, & Papageorgiou, 2003; Lamon, 1995). Unfortunately, a coherent and well-articulated framework for how such reasoning develops has not been constructed. The lack of such models makes it difficult for teachers to design instruction so that concepts are accessible and students are moved forward in their thinking.

In conclusion, research and related literature on proportional reasoning provide helpful ideas related to problem features and how they relate to solving the tasks while also identifying key components and characteristics of students' thinking related to proportionality. Increasingly complex levels of proportional reasoning require relational understanding (Skemp, 1976) and conceptual knowledge. That is, students must know what to do and why as well as have knowledge of complex mathematical relationships (Hiebert & Lefevre, 1986).

Research Design

The study involved sixth grade students enrolled in a suburban elementary school. Six students were randomly selected from an advanced level mathematics course of 24 students. The stratified random selection allowed for an equal number of boys and girls with five White and one Asian student.

The focus of the classroom-based research project was to explore students' understanding of proportional relationships. Students began with a warm-up problem to get them thinking about proportional relationships. "Yan and David each pay $6 for a pizza. The pizza is cut into six equal slices. How many slices should each receive?"This was followed by two somewhat more difficult problems such as the lottery problem. “Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they share the money? Show all your work and describe what you did to solve the problem (Peled & Bassan-Cincinatus, 2005).

The focus of this project was to investigate students’ construction of problems as a means of demonstrating whether they generalized their understanding of proportional reasoning. Students were instructed to compose and solve a problem similar to the proportional problems they had solved.

Results

In general, students were constructed problems which reflected that they understood the underlying principles of proportional reasoning problems. Two types of problems were common: percentage applications and ratio problems that did not involve percentages. The majority of students also presented correct solutions for their problems. Half of the students (3 out of 6) used percentages in the solution methods to their problems. This is important to note as the use of percentages is reflective of multiplicative understanding, finding the answer by multiplying the base by the rate or percent.

As students shared their thinking about their problems, they negotiated the shared meaning of proportionality. Their work reinforced the concepts that they had discussed in solving the initial tasks. An analysis of the problem solving discussions showed that the approaches used by the students required a solid understanding of rational number principles and proportional reasoning. The problems clearly indicated an awareness of the multiplicative nature of proportions and did not depend on the use of pattern matching or build-up strategies which are more indicative of additive reasoning (Baxter & Junker, 2001). Four problems are discussed below to demonstrate students’ understanding.

The ‘jawbreaker’ problem was completed by Joe. Joe’s problem involved related rates as he describes a situation comparing number of gumballs to price. He first concludes that each jawbreaker costs 1¢. He presents an interesting way to show the number of gumballs for 1/6, ¼, and 1/8 of the total. Notice that he understands that he can multiply these ratios by a ratio equivalent to one to determine an equivalent ratio showing the number of gumballs out of 48.

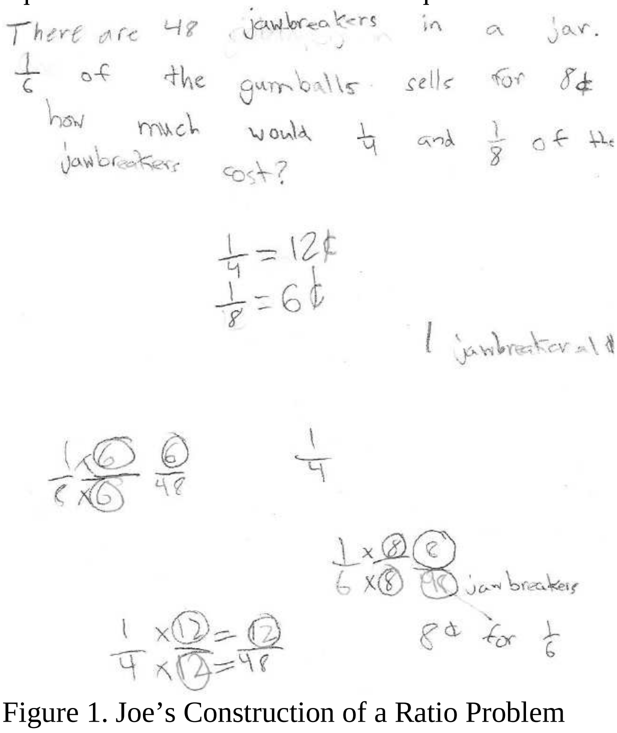

Figure 1. Joe’s Construction of a Ratio Problem

In the skateboard problem, Abbey illustrates how percentages are related to proportional relationships. Notice that she partitioned 60% into more easily manageable components of 50% and 10%. It follows then that 50% of the original price is $12.50 and 10% of the price is $5.00. Comparably, she shows that this process is analogous to finding ½ and 1/10 of the original price. She adds these two amounts to the original price of the skateboard to determine the new price that is 60% more than original.

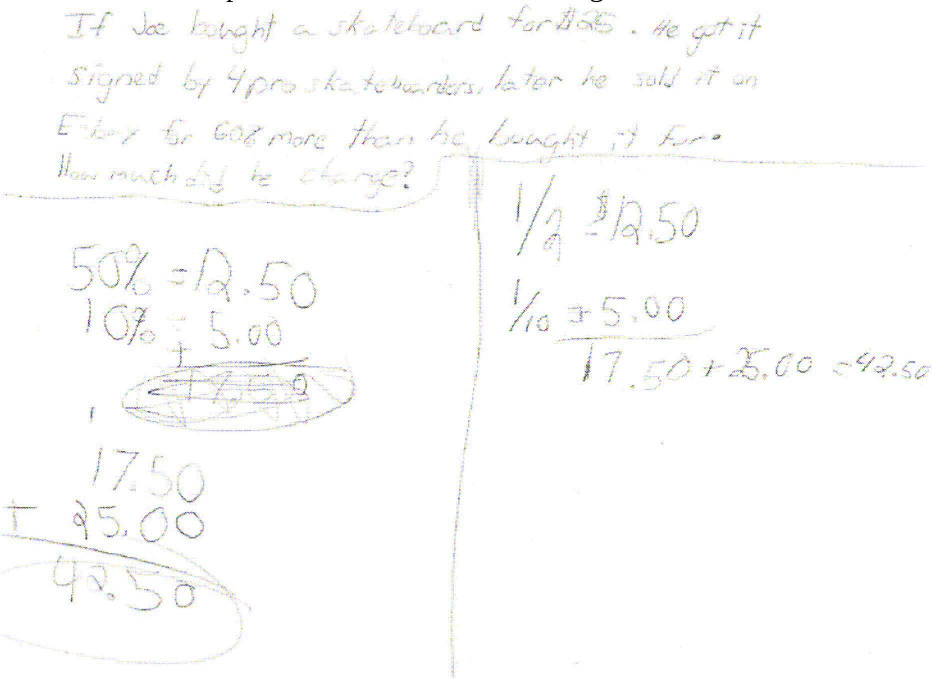

Figure 2. Abbey's Construction of a Percentage Problem

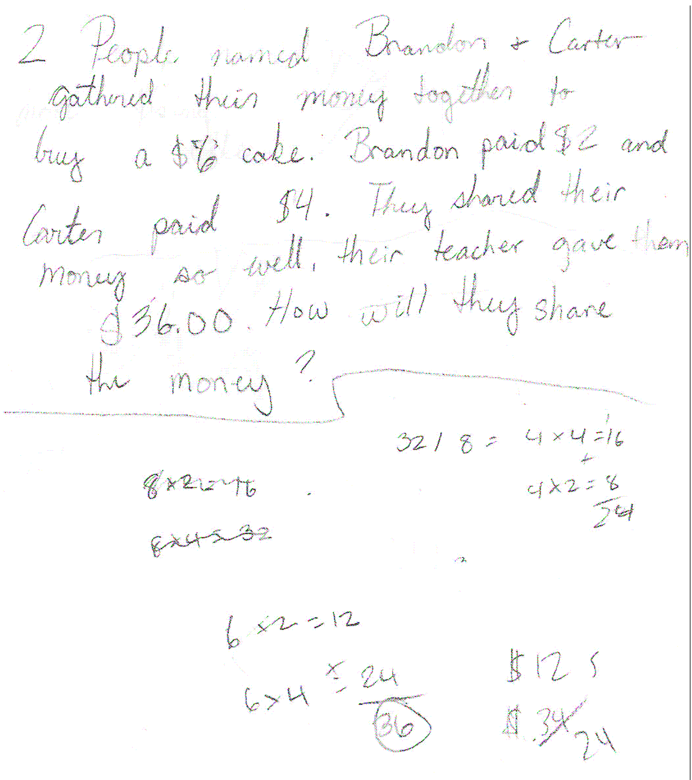

Figure 3. Anna's Related Rates Problem

In the cake problem, Anna presents a related rate problem that requires multiple comparisons. It is similar in many ways to the lottery problem that was the focus of the initial investigations. Though not explicit in her work, Anna realizes that she can multiply the original cost of the cake by the proportional amounts of each contributor (Brandon with $2 and Carter with $4). This gives us 6 X 2 and 6 X 4. The resulting amounts of 12 and 24 are verified as summing to $36. It isn't clear why Anna initially used 8 but it is obvious that she concluded that this approach did not work (resulting in $24 not the required $36).

Conclusions

The open ended nature of the proportional reasoning tasks allowed the researchers to make inference about students' thinking as they composed and solved problems related to those they had worked on initially as part of the project. Writing, as a generative act, was a powerful way for students to express their understanding and think deeply about the nature of proportional relationships. As they modeled these situations in their solutions to their constructed problems, the multiplicative nature of their proportional reasoning was evident. The analysis of the problems created by the students served as evidence that they had generalized skills in solving proportional problems and could illustrate the underlying relationships of such problems in their own novel applications. Writing and solving problems that reflect important mathematical concepts is a valuable learning tool for students and a powerful means for teachers to assess what their students really know and can do relative to the mathematics as well as a providing direction for additional instruction.

References

Baxter, G.P. and Junker, B. (2001, April). Designing cognitive-developmental assessments: A case study in proportional reasoning. Paper presented at the annual meeting of the National Council for Measurement in Education, Seattle, Washington.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lamon, S. J. (1995). Ratio and proportion: elementary didactical phenomenology. In: J. T. Sowder, & B. P. Schappelle (Eds.), *Providing a Foundation for Teaching Mathematics in the Middle Grades* (pp. 1 67–198). Albany, NY: State University of New York Press.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), Number concepts and operations in the middle grades (pp. 93–118). Reston, VA: National Council of Teachers of Mathematics.

Peled, I. & Bassan-Cincinatus, R. (2005). Degrees of freedom in modeling: Taking certainty out of proportion. In Chick, H.L. & Vincent, J.L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, 4,* 57-64. Melbourne: PME.

Pittalis, M., Christou, C., & Papageorgiou, E. ((2003). Students' ability in solving proportional problems. *Proceedings of the 3rd European Research Conference in Mathematics Education*, Bellaria: Italy, 3, http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG3/TG3_list.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 1-7.

Steinthorsdottir, O.B. (2005). Girls journey toward proportional reasoning. 2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 225-232. Melbourne: PME.