

Does the parameter represent a fundamental concept of linear algebra?

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Abstract

In mathematics the parameter is used as a special kind of a variable. The classification of the terms "variable" and "parameter" is often done by intuition and changes due to different situations and needs. The history of mathematics shows that these two terms represent the same abstract object in mathematics. In today's mathematics, compared to variables, the parameter is declared as an unknown constant measure. This interpretation of parameters can be used in set theory for describing sets with an infinite number of elements. Due to this perspective the structure of vector spaces can be developed as a special structured set theory. Further, the concept of parameters can be seen as a model for developing mathematics education in linear algebra.

Introduction

What is a parameter? In mathematics this question is often answered by: *A parameter is almost the same object as a variable.* Due to this perspective it is characterized as a special kind of a variable. Consequently, the property of this variable depends on the way of application. For mathematicians "parameter" and "variable" represent only different names given to the same object. There is no need for an exact definition of the object "parameter", because the use of the word "parameter" is acquired by individual mathematical experience, implying a mathematician classifies a placeholder as variable or parameter by intuition.

This problem was transferred to school mathematics by adapting mathematical structures and their concepts exactly into the curriculum of mathematics education. Especially, emphasizing the fact of the absence of an accurate definition neither of the word nor of the object parameter. Consequently students are unable to distinguish "parameter" and "variable". They use the parameter in the same way like a variable, but they are convinced that there must be a difference between these two notions, which they can't explain¹.

Due to the different expressions, the question arises, if a model or a concept can be developed focusing on the idea of the parameter as an inferior kind of variable. This model must define or explain the use of parameters without distorting the general application. One way of realizing this complex task can be found in linear algebra.

The parameter and the description of sets

The fundamental concept of the following model is the classification of the parameter as an inferior variable by introducing the parameter as an auxiliary variable to describe all elements of a set. Due to the above, the parameter has found a somewhat exact definition. On the one hand, this declaration doesn't capture the entire meaning and use of this special term. But on the other hand students don't get misleading impressions which have to be modified later. This reduction represents only one possible perspective, how teachers can explain the complex matter. The point is that this perspective allows already a transmission to all mathematic applications employing parameters. Finally a comprehensible introduction of a parameter can't be given without some limitations.

The motivation is justified in enabling students to describe sets with an unlimited number of elements. For example: The linear equation

$$4x - 2y = -6$$

features an infinite number of possible solutions. By the method of trial and error students realize and accept very fast that it isn't possible to note all solutions in an explicit way. A parameter, for example λ , can be used to capture all solutions:

$$L = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (\lambda, 2\lambda + 3), \lambda \in \mathbb{R}\}$$

The parameter leads to a description of all solutions. The special function of the parameter λ can be pointed out by the dependence of the components x and y which belong to any solution of the equation. For any figure chosen as λ a certain result can be found.

¹ Preliminary results of video-interviews recorded at the University of Cologne in March 2009.

Finally the description of infinite sets represents a very abstract mode to motivate the introduction of the expression “parameter” for specially used kind of variables. Once this part is understood by students, the most difficult part of the entire model and its application has been achieved.

The parameter as a fundamental concept of linear algebra

The parameter is introduced as auxiliary variable supporting the capture of infinite sets. In linear algebra, this view can support the construction of a vector-space-structure as a special set structure. In this case the parameter-introduction is realized with systems of linear equations. For example the system

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & -1 \\ 2 & 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ -3 \end{pmatrix}$$

can be transformed to

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \\ 0 \end{pmatrix}.$$

The above representation of the origin system shows that it has more than one solution. Now, you’re interested in capturing all solutions as easy as possible. The parameter in the function of an auxiliary variable offers a model to describe all components (x,y,z) of all solutions:

$$L = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \lambda \in R \right\}$$

In this way the parameter represents a tool capturing finitely an infinite set in a special way. The parametrization of the solution set distinguishes already the structure of vector-spaces. By generalizing the idea of parametrization, whole vector-spaces or sub-vector-spaces can be described.

The model of set parametrization simplifies comprehensively the structures of adjoining mathematics disciplines like analytic geometry. Due to the perspective of simplification by parametrization, the concept of planes and lines as point sets will be understood quicker than without set-parametrization. Especially the link between the intersection of planes and lines on the one hand and the solution set of systems of linear equations on the other hand can point out the meta-structure and the correlation of mathematic disciplines. Consequently the parameter-concept represents a vivid part in contributing to experience the elementary structure of vector-spaces as well as its relevance for other mathematics disciplines.

The generalization of the parameter-concept in mathematics education

The introduction of parameters in linear algebra, as shown above, reverts to the description of infinite sets by parametrization. Given that set theory represents the basement of mathematics, the model capturing sets by parametrization can be transferred to all mathematic disciplines in which explicit descriptions of sets are needed.

In calculus functions which are reliant on an additional parameter, for example $f_k(x) = k \cdot x^2$, represent a famous example in school mathematics. Here the transmission is done by considering a whole class or set of functions. Concerning the mentioned example of $f_k(x) = k \cdot x^2$ the graphs of f_k represent a set of parabolas which all proceed the point of origin. The realization of a whole set of functions and graphs respectively is done by the parametrization.

Due to the perspective of an elementary concept of linear algebra, the parameter represents a special kind of variable which enables the capture of infinite sets, in this case especially the capture of vector-spaces and sub-vector-spaces. Otherwise the parameter can be pointed out as fundamental object of mathematics, because of the importance of sets. Of course, this definition of parameters doesn’t represent exactly what many mathematicians dealing with parameters figure or imagine. The main intention of this model is to show an easy way to explain the difference between the two expressions “parameter” and “variable” of placeholders in school mathematics.