

Problem Fields in Elementary Arithmetic

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Abstract Working with problems and making investigations is an activity one has to learn already very early. Therefore in primary school children should not only learn concepts and solve given tasks. They also should find out knowledge and reasons by themselves. Here you will find some problem fields in elementary arithmetic within children of primary school can make different investigations and find as well as give reasons for special statements. The topics concerned are partitions of numbers, sums of consecutive numbers, figured numbers, sequences and chains, table of hundred and number-walls.

Introduction

Working with problems and making investigations is an activity one has to learn already very early or better said one has to preserve it from early childhood. Therefore in primary school children should not only learn concepts and solve given tasks for example with using algorithms for computing with natural numbers. They also should find out knowledge and reasons by themselves and see that one can find different ways to work on a problem. The emphasis lies here on “find” and there are many things you can find out within mathematics already in primary school. Such discoveries may be not new for the teacher, it is only important that the children see it as discovery from their own. But sometimes even children can make discoveries nobody has done before or find a theorem for which nobody knows a proof until now. This all together can submit a more correct view of mathematics and support general aims like mathematical creativity or ability of structuring. Moreover from theory of learning we know that people keep things best by making discoveries and find connections by themselves.

In primary school mathematics there are many themes children can work on by themselves. Here there will be named some *problem fields* which start with a generating problem and by varying special aspects can lead up to different mathematical knowledge and conjectures.

1. Partitions of numbers in grade 1 or 2

One aim of arithmetic in grade 1 is to get good knowledge about the first natural numbers especially the connections between them. From empirical experiences we know that most children entering school know the numbers from 1 to 10. So after securing this knowledge already in the third or fourth week in school it might be a good idea to let the children work on a little problem field concerning sums. For this we give every child a set of 5 (later on 6 and 7) concrete elements with the task to *find all possible partitions* of this set and to find on this way (from enactive via iconic to symbolic working) *all sums the given number can be split into*. While discussing the results in the whole group with structuring them (by help of the teacher) we could get the following sums:

$$5 = 1+4 = 2+3 \quad \text{and} \quad 5 = 1+1+3 = 1+2+2 \quad \text{and} \quad 5 = 1+1+1+2 \quad \text{as well as} \quad 5 = 1+1+1+1+1$$

$$6 = 1+5 = 2+4 = 3+3 \quad \text{and} \quad 6 = 1+1+4 = 1+2+3 = 2+2+2 \quad \text{and} \quad 6 = 1+1+1+3 = 1+1+2+2 \quad \text{and}$$

$$6 = 1+1+1+1+2 \quad \text{as well as} \quad 6 = 1+1+1+1+1+1$$

$$7 = 1+6 = 2+5 = 3+4 \quad \text{and} \quad 7 = 1+1+5 = 1+2+4 = 1+3+3 = 2+2+3 \quad \text{and} \quad 7 = 1+1+1+4 = 1+1+2+3$$

$$\text{and} \quad 7 = 1+1+1+1+3 = 1+1+1+2+2 \quad \text{and} \quad 7 = 1+1+1+1+1+2 \quad \text{as well as}$$

$$7 = 1+1+1+1+1+1+1$$

We are completing this with the partitions of the numbers 2, 3 and 4 (here each child can be asked to find all sums by itself):

$$2 = 1+1$$

$$3 = 1+2 \quad \text{and} \quad 3 = 1+1+1$$

$$4 = 1+3 = 2+2 \quad \text{and} \quad 4 = 1+1+2 \quad \text{as well as} \quad 4 = 1+1+1+1.$$

By looking at the results the children can find out first that the number of sums is increasing if the basic number is rising up. This of course everybody would expect. But the amount of simple sums (i.e. partitions in only two parts) is the same for an even number and its following number and always half of the even number. A reason for this some of the children can find by themselves¹ and clear it with the others. The amounts of triple sums and so on we only can notice but not find reasons for them².

¹ The reason has to do with the rule of commutativity for the addition which some of the children already can explain on concrete examples I suppose.

² This is a problem for mathematicians within number theory. The above notations give a hint to find a solution e.g. for triple sums by looking out first all triple sums with 1 as beginning number, i.e. $1+1+(n-2)$, $1+2+(n-3)$, ...

- **square-numbers** are numbers like $1 \cdot 1, 2 \cdot 2, 3 \cdot 3, 4 \cdot 4, 5 \cdot 5, \dots$,

- **triangle-numbers** can be described as $1, 1+2, 1+2+3, 1+2+3+4, \dots$.

By putting the figures of *two triangle-numbers together* they also can find out that e.g.

$2 \cdot (1+2) = 2 \cdot 3, 2 \cdot (1+2+3) = 3 \cdot 4, 2 \cdot (1+2+3+4) = 4 \cdot 5, 2 \cdot (1+2+3+4+5) = 5 \cdot 6, \dots$ or that the figures of *two consecutive triangle-numbers together* built a square-number like

$1 + (1+2) = 2 \cdot 2, (1+2) + (1+2+3) = 3 \cdot 3, (1+2+3) + (1+2+3+4) = 4 \cdot 4, \dots$

With looking at the square-numbers we e.g. can find out (and see at the figures) that the *difference of two consecutive square-numbers* built the sequence of odd numbers, at which any odd number is equal to the sum of the two basis numbers (or to the double of the smaller basic number plus one), i.e.

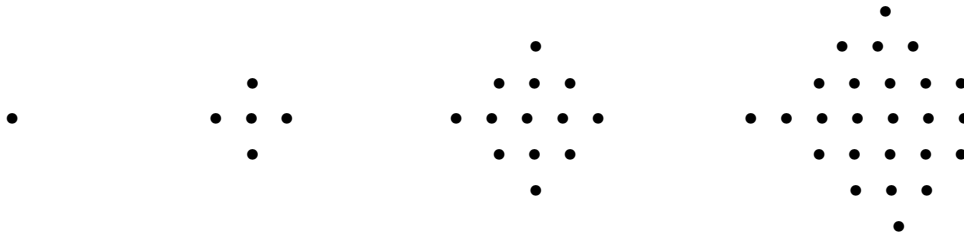
$2 \cdot 2 - 1 \cdot 1 = 3 = 2+1, 3 \cdot 3 - 2 \cdot 2 = 5 = 3+2, 4 \cdot 4 - 3 \cdot 3 = 7 = 4+3, 5 \cdot 5 - 4 \cdot 4 = 9 = 5+4, \dots$
 $= 2 \cdot 1 + 1, \quad = 2 \cdot 2 + 1, \quad = 2 \cdot 3 + 1, \quad = 2 \cdot 4 + 1,$

From this we can find out that a square-number can be seen as sum of odd numbers like

$1 \cdot 1 = 1, 2 \cdot 2 = 1+3, 3 \cdot 3 = 1+3+5, 4 \cdot 4 = 1+3+5+7, 5 \cdot 5 = 1+3+5+7+9, \dots$

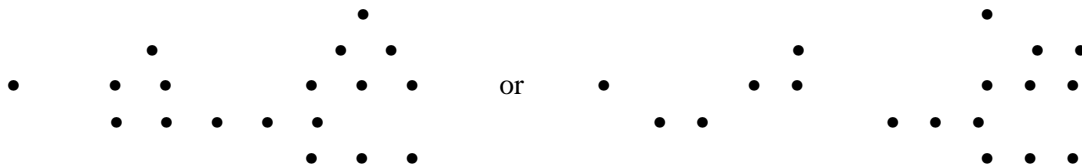
A *sum of two consecutive square-numbers* thus can be described also as sum of odd numbers which first increase and then decrease in the following way:

$1 \cdot 1 + 2 \cdot 2 = 1+3+1, 2 \cdot 2 + 3 \cdot 3 = 1+3+5+3+1, 3 \cdot 3 + 4 \cdot 4 = 1+3+5+7+5+3+1, \dots$ This symbolic description leads us to the following figured description:

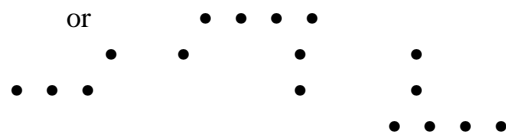


If we put *together a square-number and the triangle-number with side-length one less than that of the square-number* then we will get a **five-angled-number** (looking like a front of a house) or a **trapezium-number** (by using the right-angled form of the triangle-numbers). The amount of its dots can be computed in the following way:

$1 \cdot 1 + 0 = 1, 2 \cdot 2 + 1 = 5, 3 \cdot 3 + (1+2) = 12, 4 \cdot 4 + (1+2+3) = 22, 5 \cdot 5 + (1+2+3+4) = 35,$ and so on.



Building the *difference of a square-number and the square-number which lies two places before it* then we get a figured **squared frame** which $\bullet \bullet \bullet$ or $\bullet \bullet \bullet \bullet$ in symbolic way with n as side-length can be described as $n^2 - (n-2)^2$.



Parting these frames into the sides (each without one vertex) we can find out that the sequence of these frame-numbers build the multiples of 4 with formula $4 \cdot (n-1)$. Parting the frames into two opposite sides (each including both vertices) and the two “rest-sides” (each without both vertices) then we get a third formula for the frames: $2 \cdot n + 2 \cdot (n-2)$.

Another task for the children³ could be the working and looking out for different descriptions of *triangle-frames* and *five-angled-frames*. Or they could look out for three-dimensional figured numbers like *cube-number* or *pyramid-numbers*.

³ In upper grades you can get algebraic equations e.g. with T_x as triangle-number and S_x as square-number you can find $T_{2n} = 2 \cdot T_n + S_n, T_{3n} = 3 \cdot T_n + 3 \cdot S_n$ and $T_{4n} = 4 \cdot T_n + 6 \cdot S_n, T_{5n} = 5 \cdot T_n + 10 \cdot S_n$ which leads to the conjecture $T_{kn} = k \cdot S_n + T_{k-1} \cdot S_n$ and $T_{n^2} = n \cdot T_n + T_{n-1} \cdot S_n$ or $T_n - T_{n-m} = 2nm - T_m$.

The children also can make investigations about *difference-sequences* of a sequence of figured numbers (i.e. the difference of consecutive elements). Interesting in this context is that the difference-sequence of the difference-sequence is constantly 1 for the triangle-numbers, constantly 2 for the square-numbers and constantly 3 for the five-angled numbers.

4. Sequences and chains of numbers in grade 2 or 3

With the question “How does it go on” or “Call the next three numbers” children in grade 2 or 3 can be introduced into the *concept of sequences*. Examples may be 1, 3, 5, 7,... or 3, 4, 6, 9,... or 2, 4, 8, 16, ... or 1, 2, 1, 2, ... or 5, 4, 6, 3, ... or sequences of figured numbers. Afterwards they can produce such sequences of natural numbers by themselves and let find out by their neighbours the next five numbers or the rule it was produced. There are a lot of possibilities to build such sequences by using only basic operations in different ways and there are also often different ways to describe the rule for getting some next elements of a sequence⁴. In the example 5, 4, 6, 3, ... e.g. you could say that the way from one element to next is $-1, +2, -3, +4, \dots$ but you also could say that the sequence is existing of two sequences (5, 6, 7, ... and 4, 3, 2 ...) mixed up. Moreover this sequence will stop with its eleventh element because the following subtraction has no solution within \mathbb{N}_0 .

A famous *chain of numbers* is the sequence of Fibonacci. It starts with two ones and then always the next number is the sum of the two ones directly in front of it. A chain of numbers in general is defined by its first two elements and then always the next one is the sum the two ones directly in front of it.

Problems for discovery are e.g.: “Given two starting numbers find the next five.” or “Find two starting numbers so that the fourth (or fifth or sixth) element is equal to 10 (or 50 or 100 or ...)?” “How does a given chain change if you add to (subtract from) the first (the first two) numbers a 1 (or 2 or 3 ...)?” “How does this chain change if you multiply the two starting numbers with 2 (or 3 ...)?”

For questions like these we first look at some examples whereat always the first two numbers⁵ are given and the five next numbers are computed.

- 1st chain: 1, 1, 2, 3, 5, 8, 13 (Fibonacci sequence)
- 2nd chain: 1, 2, 3, 5, 8, 13, 21 (Fibonacci without first 1)
- 3rd chain: 1, 3, 4, 7, 11, 18, 29 (Second chain plus Fibonacci with 0 in front)
- 4th chain: 1, 4, 5, 9, 14, 23, 37 (Third chain plus Fibonacci with 0 in front)
- 5th chain: 2, 2, 4, 6, 10, 16, 26 (Double of Fibonacci)
- 6th chain: 2, 3, 5, 8, 13, 21, 34 (Fibonacci without 1, 1)
- 7th chain: 2, 4, 6, 10, 16, 26, 42 (Double of second chain)
- 8th chain: 3, 3, 6, 9, 15, 24, 39 (Triple of Fibonacci)
- 9th chain: 3, 4, 7, 11, 18, 29, 47 (Eighth chain plus Fibonacci with 0 in front)
- 10th chain: 4, 4, 8, 12, 20, 32, 52 (Fourfold of Fibonacci)

- Looking at the 1st, 2nd and 6th chain we get the following conjecture⁶: “If in a chain there are two consecutive numbers equal to two consecutive numbers of the Fibonacci sequence then all following elements are equal to the following elements of the Fibonacci sequence too.” And it is not difficult to see the following a little bit more general conjecture “If two consecutive elements of one chain are equal to two consecutive elements of another chain then all following elements are also equal.”

- Coming out of the rule of building the chain elements we also easily can find the following conjecture⁷ “The difference sequence of a chain (difference between two consecutive elements) is equal to this chain without the first element”.

- Discussing the 1st, 2nd, 3rd, 4th chain on one hand and the 5th, 6th, 7th on the other hand we can see that adding +1 to the second element influences +1 at the third element, +2 on the fourth element, +3 on the fifth element, +5 on the sixth element and +8 on the seventh element (i.e. plus Fibonacci with 0 in front).

⁴ And more: A sequence is not defined exactly with only a few elements of it. So principally there are also different rules (with different following numbers) possible. But that is normally not a problem for this age.

⁵ Here we took the first two numbers a_1 and a_2 out of the set $\{1; 2; 3; 4\}$ with $a_2 \geq a_1$.

⁶ A proof or at least a plausible reason for this (and other conjectures) some of the children may find with words by their own. An algebraic way to handle with these chains then in grade 8 is a good way for deepening algebra. For example if we start with a, b then the next five numbers have the following algebraic description: $a+b, a+2b, 2a+3b, 3a+5b, 5a+8b$. It is interesting to find the numbers of the sequence of Fibonacci (twice) again here.

⁷ In algebraic form the building role of such a chain is $a_{n+2} = a_n + a_{n+1}$ and thus we get $a_{n+2} - a_{n+1} = a_n$.

- Comparing the 2nd chain with the 5th one and the 3rd chain with the 6th one and the 4th chain with the 7th one as well as the 6th chain with the 8th one and the 7th chain with 9th one and finally the 9th chain with 10th one we can come to the conjecture that adding +1 to the first element influences for the following elements +1, +1, +2, +3, +5 (which is the Fibonacci sequence again)⁸.

- Looking at the 1st, 5th, 8th, 10th chain as well as comparing the 2nd chain with the 7th one we find: “Multiplying the two starting elements with one natural number yields the multiplying of all elements with this natural number.” The 1st, 5th, 8th, 10th chain also can lead us to the conjecture that starting with two equal numbers we get the Fibonacci sequence multiplied with the starting number⁹.

With these ten chains we already could make a lot of discoveries. The children later on can *vary the rule of producing chains of numbers*. For example they can start with three numbers and then produce always the next number as sum of the three ones directly in front of it. Or they first multiply the last element with an integer n and the element before the last one with an integer m . The sum of these two products (instead of the simple sum) then is the next element. Or they use differences instead of sums. In grade 3 they also can change to multiplication¹⁰ instead of addition or mix up different operations.

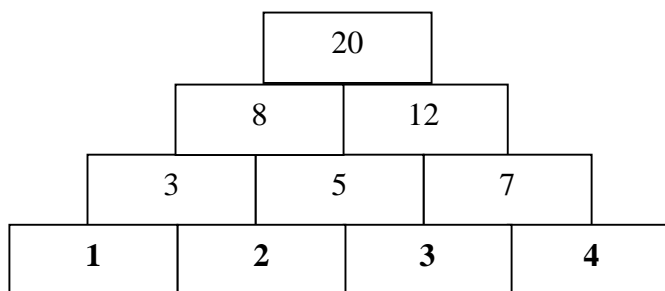
5. Discoveries within the table of the hundred in grade 2

Another problem field for grade 2 has to do with discoveries within the “table of hundred” (a table containing all numbers from 1 to 100 structured in rows of ten). With this the children can deepen the structure of the decimal system for example by walking one step (or more) down or respectively right or left with looking at the change of the numbers. They also could investigate all “horse-jumps” (jumps of a horse in chess – two steps in one direction plus one step perpendicular to it or one step in one direction plus two steps perpendicular to this). A look at a diagonal row or a parallel row to it also can lead to statements.

As variation of such discoveries we can use as basis a sheet of a calendar with seven columns or a table of natural numbers ordered in 5 or 6 or 8 columns. We also can use a table of only odd numbers instead of the normal “table of hundred”.

6. Number-walls in grade 2 and 3

Number-walls represent a special format of tasks. You start with a first row with three or four (or more) numbers. The sum of always two neighbouring numbers then built the second row. In the same way you built the third row, and so on until you have only one number in the last row. The following example starts with the numbers 1, 2, 3, 4 given in the first row.



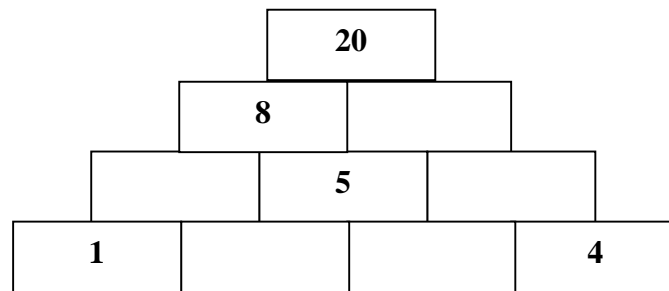
If you start in this way with numbers in the first row this is only a special form for routine tasks. But if you start with numbers in different rows you open a wide field of different problems. In the following

⁸ In algebraic form in a general case we find $a+n$, $b+m$, $a+b+(n+m)$, $a+2b+(n+2m)$, $2a+3b+(2n+3m)$ and so on.

⁹ Children in primary school can give with examples and general arguments in words reasons for this statement. In algebraic form we can look at $a \cdot n$, $b \cdot n$, $(a+b) \cdot n$, $(a+2b) \cdot n$, $(2a+3b) \cdot n$, ... [or with $a=b=1$ we get n , n , $2n$, $3n$, $5n$, ...].

¹⁰ In grade 4 or 5 we e.g. can find out that a chain working with multiplication instead of addition and with starting numbers 1, n (n any natural number) does produce a sequence with powers of n whereat the exponents are equal to the elements of the sequence of Fibonacci. The same (without the beginning 1) we get when we start with the two numbers n , n . If we take the starting pair 2 , n then we will get the following sequence (noticed in an algebraic way): 2 , n , $2n$, $2n^2$, 2^2n^3 , 2^3n^5 , 2^5n^8 , ... (again with the numbers of Fibonacci)!

example you only have to work back. You can make different considerations by starting with less of these given numbers.



- If you erase the 1 in the last row the number-wall also is determined with the same solution because in the first field of the second row you find that there must stand 3 and below of the three you can only have 1, 2 or 2, 1. If you take 2, 1 then the bottom row comes out with 2, 1, 4, 4 which leads to a 8 right of the 5 in the second row and a 13 right of the 8 in the third row which stands in contradiction to the 20 in the top row. Without using an argument of contradiction you also will come to the same solution if you work back and find first the 12 right of the 8 in the third row then a 3 and a 7 in the second which produces first a 3 in the third field of the bottom row and then a 2 in the second field of the bottom row and finally the erased 1 in the first field of the bottom row. (Similar consideration you can make if you erase the 4 instead of the 1 in the bottom row.)
- If you erase 1 and 4 in the bottom row then you have two different solutions. There are of course also different solutions if you erase two other numbers or even all numbers except the 20 in the top row. If you only replace the first 1 with another number then - as already discussed - you will get no solution.

You see that by varying the given task you get a problem field with different types of problems including also tasks which may have more than one solution or no solution. And you can differentiate for different abilities of the children by using different givings as well as different large number-walls. Good children also can find variations by themselves.

Moreover the children can make experiences in the direction of *functional connections*. For this we will discuss some examples:

- If you add in the given bottom row of a number-wall a fixed number to the first (or last) number so this fixed number also has to be added in the solution in the top row. But if you add this fixed number to the second (or third) number in the bottom row so you have to add the triple of the fixed number to the solution in the top row.
- If you double (or triple ...) all numbers of the bottom row then all other numbers are doubled (or tripled ...) too.
- If you take in the bottom row a fixed number at all positions then in the second row you will find always the double of this number, in the third row the fourth of the bottom number and in the top row the eighth of the given number in the bottom row (notice the powers of 2).

Another *variation of these number-walls* you get if you use multiplication instead of addition. We then will speak from "multiplication-walls". This is of course possible only in grade 3 or later after being familiar with multiplication of bigger numbers because the numbers rise up very fast. In the case of given numbers not only in the bottom row you will get a lot more tasks which have no solution because many divisions have no solution within the set of natural numbers.

It is of course also possible to mix addition, subtraction and multiplication. But then it is better to change the wall to a tree (often used together with word problems) where we have beside the rectangular fields for numbers also round fields for the operation.