

The Mathematics Education into the 21st Century Project
The Future of Mathematics Education
Pod Tezniami, Ciechocinek, Poland
June 26th – July 1st, 2004

3D Computer Graphics and Analytic Geometry in Mathematics Education in Grammar Schools

A. Filler, F. Rieper

What do have computer games, movies or today's weather forecast in common? They are the result of mathematical calculations visualised with 3D computer graphics. Probably your students are fascinated by the realistic and, at the same time, virtual results of the computer graphic artists. It is even more fascinating to let students make computer pictures themselves. Apart from the highly motivating aspects, mathematical foundations of computer graphics are closely related to the classical contents of mathematics education in grammar schools. Teaching analytic geometry in the context of computer graphics can improve student's understanding because they can visualize their ideas and thoughts. And due to its obvious applications teacher will have – at least this time – a plausible answer to the notorious question “What is that good for?”

In this paper we discuss proposals for using the potential of computer graphics in mathematics education – starting with an analysis of aims and problems of teaching analytic geometry and going on with describing principles of 3D computer graphics and the usage of the 3D graphics software POV-Ray. Finally we present experiences we made in teaching computer graphics in analytic geometry classes with students of about 18 years.

Some problems with analytic geometry classes – potentials of including computer graphics

Analytic geometry is a compulsory subject in many national curricula. Especially in Germany – where the authors come from – the prescribed and generally accepted way of imparting this subject seems unsatisfactory: Students spend most of their analytic geometry classes calculating points and lines of intersection, distances and angles between lines and planes in a very formal way by repeating algorithms – without forming a picture of the geometry behind it. This situation is an after-effect of tendencies in the 60s and 70s, when in a lot of countries the didactics of mathematics subordinated geometric ideas to methods of linear algebra. In 1973 the Dutch mathematics didactician Freudenthal wrote: “The geometry which is possible in schools based on linear algebra, is a murky waste water.” ([3], p. 411) Although some “dry aspects” of linear algebra – e. g. the axioms of vector space – were removed from the curricula in the following decades, the lack of interesting geometric contents has remained until today. As a consequence the majority of teachers and mathematics didacticians in Germany consider analytic geometry the most problematic and unattractive subject of maths education in grammar schools.

For the development of analytic geometry instruction we deem it necessary to include visual, heuristic and experimental working methods, to look at a larger variety of geometric objects as well as to refer to interesting applications. Including 3D computer graphics into geometry classes can help to reach this goal because of two interesting aspects:

1. At the very heart of 3D computer graphics lies spatial analytic geometry, which is part of grammar schools curricula¹ – therefore, fundamental working methods of 3D computer graphics can be included as an interesting application into mathematics instruction.
2. Subjects of spatial geometry can be perfectly visualized with three-dimensional computer graphics software – consequently software of this kind can be used as a means to visual understanding and to treat more sophisticated topics of analytic geometry of space.

Both aspects can be combined, so that students see computer graphics as a *tool* for visualization *and* as a *subject* to which mathematics is usefully applied.

¹ For the mathematical foundations of 3D computer graphics compare e. g. [1], [4], [5] and [6].

The Mathematics Education into the 21st Century Project

The Future of Mathematics Education

Pod Tezniami, Ciechocinek, Poland

June 26th – July 1st, 2004

This goal can be achieved without replacing analytic working methods by methods of elementary synthetic geometry if students work with POV-Ray² – a computer programme that requires the description of geometric objects with coordinates. With POV-Ray students are able to produce photorealistic graphics and animations by using time-dependent parameters. Both possibilities are important to motivate students to spend time with the software.

Aspects of working with 3D computer graphics – relations to analytic geometry classes in grammar schools

There are three aspects of the creation of three-dimensional computer graphics:

1. The user represents geometrical objects or animations by (time dependent) coordinates or equations in spatial coordinates.
2. He goes on with defining and describing the visibility qualities of the objects, such as the surface properties like color and roughness, and general properties of the scene as a whole, e.g. the lighting.
3. It follows the mapping of a scene onto a plane represented by the computer screen using central projection.
 - For „taking a photograph“ in 3D computer graphics the user has to define the centre of projection (position of the camera) and the direction, which is orthogonal to the projection plane.
 - The computer calculates the final picture by physical lighting models (rendering) considering reflexions, refractions and other optical phenomena.

A simple three-dimensional scene (fig. 1) can be described in POV-Ray by the following code:

```
#include "colors.inc" #include "textures.inc"
background {White}
camera { location <0,8,-20>
         angle 12 look_at <0,-0.5,0> }
light_source { <-30,30,0> color White }
sphere { <-0.3,-0.5,1.5> 0.5
        texture {pigment {color Blue} } }
cone { <1.2,-1,0.5>, 0.5, <1.2,1,0.5>, 0
       texture {pigment {color Red} } }
torus {0.9, 0.15 translate <-0.6,-0.85,-0.6>
       texture {pigment {color Green} } }
cylinder { <0,-1.1,0>, <0,-1,0>, 2 texture { DMFWood6 } }
cylinder { <0,-10,0>, <0,-1,0>, 0.2 texture { Aluminum } }
```

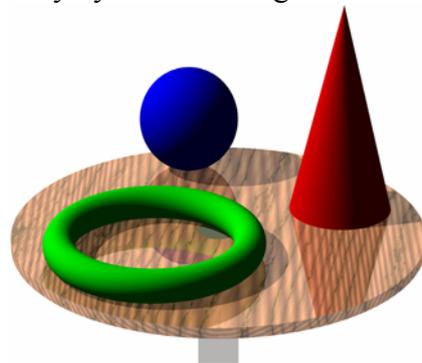


Figure 1

The `#include`-statements load files in which colours and surface textures (like `White` and `Aluminium`) are described; `background` defines the colour of the background and after the keyword `camera` the coordinates, the direction and the opening angle of the virtual camera have to be determined. To see the objects of the scene it is necessary to define at least one virtual `light_source` by its coordinates and the color of the emitted light. The statement `sphere` creates a sphere with the centre coordinates (0.3;-0.5;1.5) and the radius 0.5; `texture` defines surface properties of the sphere – in this example only a colour. Analogously the statements `cone` and `cylinder` create the other objects like the cone, the tabletop and the table-leg. A torus is by default

² POV-Ray is Freeware and can be downloaded from the website [9].

positioned at the origin of POV-Ray's coordinate system, making it necessary to **translate** and often also to **rotate** them into the desired position.

The example looks easy but to position light sources, camera and objects correctly, so that the camera "sees" the objects, requires complex considerations and is quite difficult for beginners. Since maths education should focus on *modelling geometric objects with coordinates and coordinate equations* and not on software subtleties, we developed templates with pre-defined cameras, light sources, textures and the possibility to visualize the coordinate axes for better orientation. We also provided a short manual for the students with the syntax of the most important statements to support their first steps in POV-Ray.³

Working with 3D computer graphics in analytic geometry classes – examples and first experiences

At the beginning of an analytic geometry course students should learn to work with spatial coordinates. This goal can be reached with some POV-Ray exercises in which students have to model simple objects by using predefined bodies like cubes, spheres and cylinders and position them with the coordinates of characteristic points. An illustrative example is:

Building a snowman.

Fig. 2 shows a typical result of a student after around 35 minutes of working with POV-Ray. To create the picture the student wrote the following code:

```

sphere{<3,2,2.5>,2 texture{blau_matt}}
sphere{<3,4.5,2.5>,1.5 texture{blau_matt}}
sphere{<3,-1.2,2.5>,2.7 texture{blau_matt}}
cylinder{<3,5.8,2.5>,<3,6,2.5>,2.5 texture{schwarz}}
cylinder{<3,6,2.5>,<3,9,2.5>,1.2 texture{schwarz}}
    
```

By skilfully varying the coordinates using sketches some students already achieved an almost realistic arrangement of the components of the snowman, including buttons and eyes.

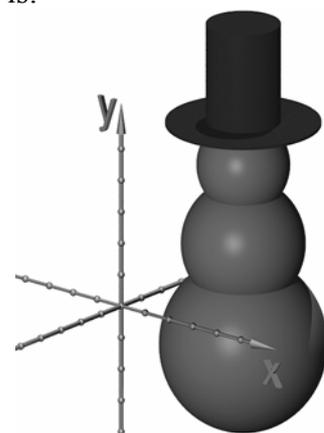


Figure 2

Ways to find out coordinates to build a well arranged snowman were discussed in the next lesson.

Students found out that it is useful to arrange components along POV-Ray's coordinate axes whenever this is possible or at least in coordinate planes because then the proper coordinate values can be found by making sketches and simple calculations. They also realized that the arrangement of the main components (trunk, body) only requires one-dimensional considerations while two-dimensional considerations are necessary to position the buttons and three-dimensional ones for the eyes (fig. 3).

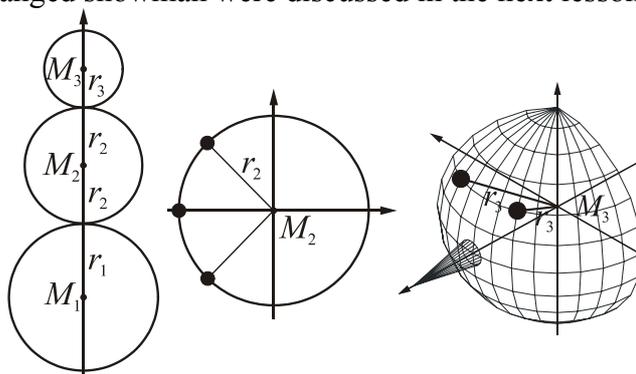


Figure 3

³ The templates, manuals and tutorials can be downloaded from the website [7].

The Mathematics Education into the 21st Century Project

The Future of Mathematics Education

Pod Tezniami, Ciechocinek, Poland

June 26th – July 1st, 2004

In the following two lessons the students improved their snowmen. Some of them found the best positions for the buttons and the eyes by stepwise changing the coordinates of the centres while others calculated the exact positions using the equations of circles (for the buttons) and spheres (for the eyes). Most students reached formidable results, like shown in fig. 4 and especially in fig. 5. A lot of them were fascinated by the work with POV-Ray and spent even some of their spare time improving their snowmen and creating other graphics.⁴

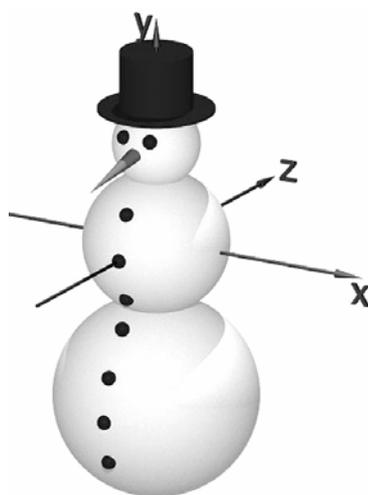


Figure 4



Figure 5

By arranging positions of simple geometric objects the students entered almost effortlessly the world of three-dimensional coordinates. This way they trained skills which are useful for the entire analytic geometry course that followed.

Alternative teaching arrangements – a POV-Ray project in mathematics classes 13 Gk

If the curriculum gives courses the freedom to choose topics of their own interest, the class can embark on more time consuming projects with POV-Ray. Especially students with little or no motivation for purely mathematical topics – like the “Grundkurse” (basic courses) in Germany – will get a boost for their interest in problems related to analytic geometry and linear algebra.

The project Mr Rieper undertook with students of the age of 18 (Mathematik-Grundkurs 13) started with an introductory talk about 3D computer graphics, its importance for CAD/CAM and film animations and the principles of ray tracing. In order to make sure that all students were familiar with the main features of POV-Ray a manual with little problems was compiled allowing students to learn the basics of POV-Ray in a self study way. Following the introduction students were asked to choose a project theme out of a list or to find one on their own. After two weeks of work students presented their “multi media results” to the class.

One project topic was called “programming short animation films”.⁵ Some students simply programmed rotations of a scene or parts of it with the `rotate` command, using the time variable `clock` as a rotational angle. The results were e.g. a rotating table with geometrical objects on top (fig. 6), the movement of the earth around the sun (fig. 8) or other planetary movements (fig 7).

⁴ Especially the female student which created fig. 5 worked a lot on it at home to model the arms (using frustums and spheres) and the broom. Her snowman consists of 25 geometric objects that she positioned by coordinates. The POV-Ray-file which she wrote to create the snowman is available on the website [7] under the column „Raytracing-Praxis“.

⁵ A short tutorial for creating animations and videos with POV-Ray can be downloaded on the website [7]. A lot of examples are described in [2].

The Mathematics Education into the 21st Century Project
The Future of Mathematics Education
Pod Tezniemi, Ciechocinek, Poland
June 26th – July 1st, 2004

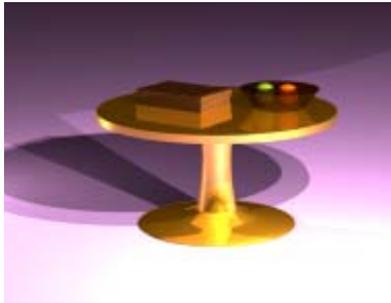


Figure 6

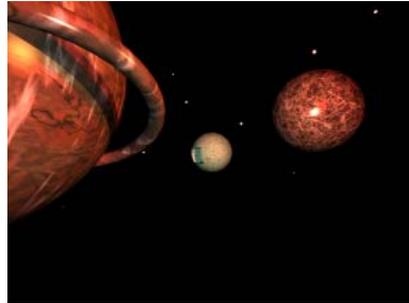


Figure 7

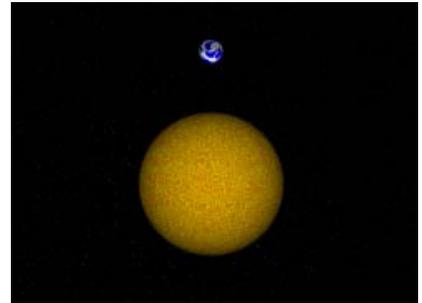


Figure 8

Things become more complicated when one movement superposes the other. To programme a ball rolling down a plane the rotation of the ball is interrelated with its translation along the plane. One student animated a light house with a rotating spot light (fig. 9). Since a spot light in POV-Ray needs a point in space where it points at, a curve in space with a `clock` parameter had to be programmed:



Figure 9

```
light_source { <20,60,15> color White
  spotlight
  point_at <-50*sin((3.14*clock+20)/180),1,-50*cos((3.14*clock+20)/180)>}
```

A female student implemented a wall clock with a pendulum (fig. 10). She faced – and finally managed – the problem of synchronizing different movements in space. On one side there is the periodic movement of the pendulum, whose period corresponds to something like two seconds. On the other side are the circular movements of the clock's minute and hour hand. If one relates POV-Ray's `clock` parameter to one second, than 60 seconds correspond to 30 periods of the pendulum, a rotation by $360^\circ/60$ of the minute hand and a rotation of $36^\circ/3600$ of the hour hand.



Figure 10

Since the centres of rotation are not identical the rotated pendulum has to be translated after its rotation.

The most fascinating project result was an animated scene of a medieval room (figures 11 and 13).

The female student programmed a curve through space along which the camera panned, whereas the focus of the camera was fixed to the centre of the scene. The panning was a superposition of a rotation around and a translation towards the centre of the scene giving the impression the viewer approaches the table while walking around it. Fascinatingly realistic were the flames of the candles. They were composed of different coloured droplets – realized by use of surface of revolutions – which rotated around each other.



Figure 11

The Mathematics Education into the 21st Century Project
 The Future of Mathematics Education
 Pod Tezniemi, Ciechocinek, Poland
 June 26th – July 1st, 2004

Throughout almost all projects students used surfaces of revolution (SOR). Rotational bodies were already analysed in calculus classes and thus the object exploration continued in analytic geometry. In order to construct an SOR you need to know the coordinates of the outline of the body. With the help of the command `sor` POV-Ray interpolates the points using splines to a continuous curve and rotates it around the z-axis to receive a rotational body. Students had to measure real objects, simplify and adapt the real data in order to model a wine glass or a wooden table. The results were convincing as can be seen in the figure 13.

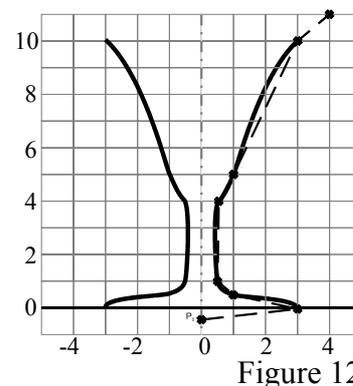


Figure 12

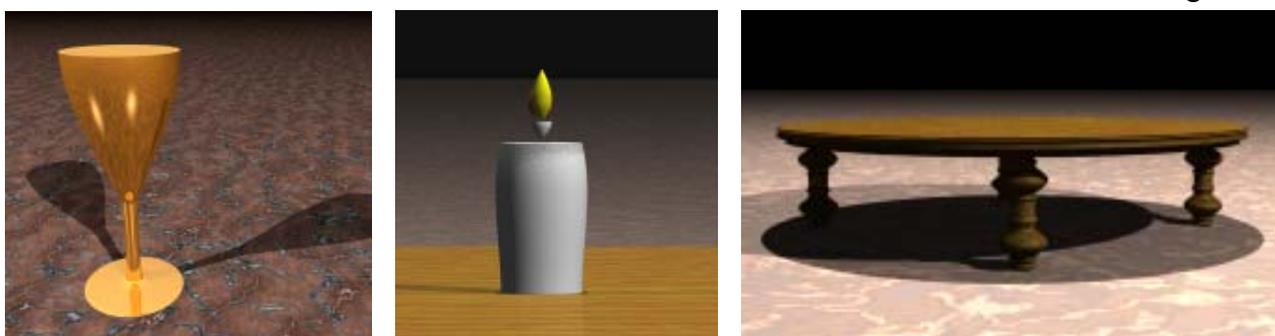


Figure 13

Another group of students did the project “conical intersections”. What is difficult to do in practice because one cannot cut a cone a thousand times with different angles is done rather easily in POV-Ray. The plane is constructed as a very thin, horizontal and half transparent box using the command

```
box {<-5,2,-4>, <5,1.99,4> texture {...} rotate <0,0,20>}
```

A cone is programmed with

```
cone{ <0,3,0>, 2 <0,0,0>, 0 }
```

Later the box is rotated about the origin by a certain angle. With the command `intersect` POV-Ray computes – in a new picture – the intersection between cone and box. Varying the angle of inclination of the box allows the students to analyse the different intersections. For advanced students an animation showing cone and the conical intersections in one picture at constantly changing angles is a challenging task.

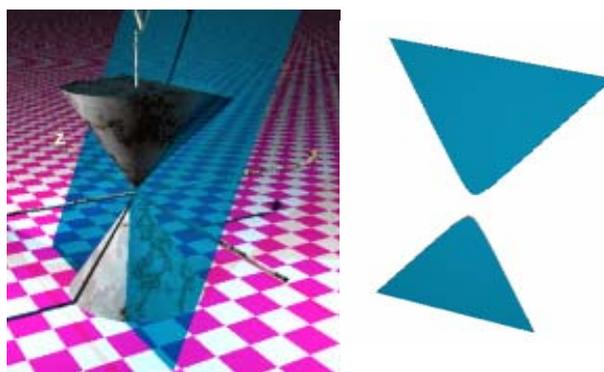


Figure 14

But other projects in mathematics education or even other school subjects are thinkable. An interesting physics project analyzes the focusing properties of thick spherical and aspherical lenses. Since POV-Ray supports refraction of light inside glass material it is easy to construct lenses as intersection of spheres or paraboloids with different thickness and index of refraction. Students can be asked to analyze the focusing properties and the dependence between thickness of the lens, focal distance and size of the focal area. Another interesting question is which surface a thick lens should have to focus parallel light into a single point.

The Mathematics Education into the 21st Century Project

The Future of Mathematics Education

Pod Tezniami, Ciechocinek, Poland

June 26th – July 1st, 2004

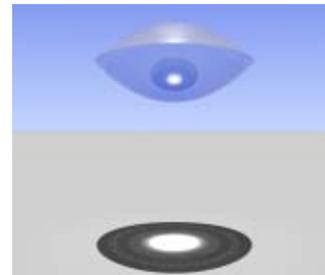
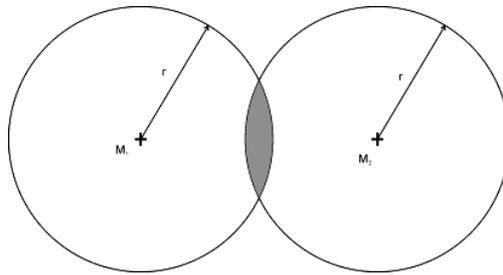


Figure 15

We conclude with some general observations concerning the project. The student's results were evenly spread on a qualitative spectrum from very simple to highly complex scenes reflecting the different mathematical talents. Obviously all students created scenes which modelled objects of real life from planetary movement, representation of rooms to still lives. The majority of the students animated their scenes using camera panning or moving objects of the scene thus challenging themselves with new problem types in analytic geometry.

Although some students, especially girls, disliked mathematics and the work with computers, the project turned out to be great fun for all of them. They gained confidence in their capability of learning a computer programme and improved their imagination of geometric problems. Last but not least they understood the importance of mathematics behind 3D computer graphics, which in turn is part of every day life.

Bibliography

- [1] FILLER, A.: Dreidimensionale Computergrafik und Analytische Geometrie. In: *mathematica didactica* 24 (2001), Nr. 2, S. 21-56
- [2] FILLER, A.; GIEDING, M.; RIEPER, F.: *3D-Computergrafik*. To appear 2004, Cornelsen, Berlin.
- [3] FREUDENTHAL, H.: *Mathematik als pädagogische Aufgabe*, Klett Verlag, Stuttgart, 1973.
- [4] FOLEY, J. D.; VAN DAM, A.; FEINER, S. K.; HUGHES, J. F.: *Computer Graphics. Principles and Practice*. Addison Wesley, Workingham, Reading, 1993.
- [5] JONES, H.: *Computer Graphics through Key Mathematics*. Springer, London, Berlin, Heidelberg, 2001.
- [6] WATT, A.: *3D Computer Graphics* (3rd Ed.). Addison Wesley, Workingham, Reading, 1999.

Websites

- [7] Filler, A.: 3D-Computergrafik und die Mathematik dahinter:
www.afiller.de/3dgcg.html
- [8] Mathematics-Project Raytracing at the School "Fürst-Pückler-Gymnasium", Cottbus:
www.fpg-cottbus.de/faecher/mathematik/povrayprojekt.html
- [9] POV-Ray Homepage: www.povray.org