How to increase the understanding of differentials by using the Casio-calculator model 9860 G I/II to solve differential equations.

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Abstract

The major aims of this paper are to present how we can improve the students understanding and involvement in mathematics by using a programming/graphic calculator. I will use differentials as examples such as differentiation, integrals and differential equations, creating lines of slopes for differential equation of the type $y' = f(x,y)$. Find the solution of some differential equations by using regression and create the graph connected to the differential equation. As we have different approaches to solving a problem, it is a hope the students interest in mathematics will improve. The tools used will be programming, graphic commands as plot, f-line, etc. One goal is also to show how we can create small programs solving problems in mathematics. For many students this will be a stepping stone for further work with programming. The programs used can be copied using the program FA 124 that can be downloaded from Casios homepages. On request I can send you the programs.

Summary of workshop

An ordinary graphic calculator can be a very helpful tool to strengthen the understanding in mathematics. In this paper I will use differentials as examples.

Example 1: Differentiation of the function $f(x) = x^n$

The students are challenged to find a formula for derivative for the function finding local slopes for $x = a$ numerically

$$\frac{\Delta y}{\Delta x} = \frac{f(a + 0.01) - f(a - 0.01)}{0.02}$$

The program is short and put x-values from 0 to 4 in list 1

List 1: shows 0, 1, 2, 3, 4

and the local slopes in list 2.

List 1: shows 0, 1, 2, 3, 4

List 2 shows 0, 2, 12, 27, 48

We start by making the graph

$$y1 = x^n$$ and allow the students to make the program derive.

Later giving the proper proof, they easily recognize the result.

Making the graph combined with regression we find $x^3' = 3x^2$

With trying $n = 1, 2, 3, 4$ and 5, most of my students find the formula $x^n' = nx^{n-1}$

Later giving the proper proof, they easily recognize the result.

The next challenge is to find out if the formula is correct for $n$ being negative or a fraction.
I find that many students have big problems with functions like \( y = \frac{1}{\sqrt[3]{x^2}} \) and \( y = \frac{1}{\sqrt[3]{x^3}} \). It becomes easier if they can translate the expressions to \( x^{\frac{3}{2}} \) and \( x^{\frac{2}{3}} \) when they shall find the derivative.

**EXAMPLE 2:** Integral of the function \( y = x^n \)

We find the integral by sum up the areas of narrow columns with height \( h = f(x) \) and width 0.02 going from \( x - 0.01 \) to \( x + 0.01 \) that gives a good approximation for the real area.

Again a small program integral:

The program gives starting values for \( x, S = 0 \) and for the total area \( A = 0 \). For every \( x \) the area is added up and for every whole number of \( x \), the \( x \)-value goes to list 1 and the antiderivative of \( f(x) \) goes to list 2

list 1 : 0, 1, 2, 3  \quad List 2 : 0, 0.3333, 2.666, 9,

In this example we choose \( n = 2 \)

We make the graph and use regression:

By trying the students find regression of power 3 to fit: \( \int x^2 \, dx = \frac{1}{3} x^3 + C \) and trying \( n = 0, 1, 2 \) and 3 find the general formula:

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C
\]

What now when \( n \) is negative or a fraction?

The big question is also what happens when \( n = -1 \)?

It is good to have some questions unanswered before the proper proofs are given.

**EXAMPLE 3**

We shall find slopes of line for equations of type \( y' = f(x, y) \).

To make this we need the command F-line which creates the line between two given points.

F-line 1,1,3,2 makes a line between (1,1) and (3,2)

Diagram with lines of slopes for points \((x, y)\) with slope \( D \).
The angle A between tangent and x-axis is given by: \( A = \tan^{-1} D \) (SHIFT tan D)

A short line with length 0,6 will then join the point \((x-0,3\cos A, y-0,3\sin A)\)
with point \((x+0,3\cos A, y+0,3\sin A)\)  If \( A = 90^\circ \) The line is between \((x,y-0,3)\) and \((x ,y+0,3)\)

The example : \( y' = x + y \) ; \( x+y \rightarrow D \)
You can easily change to other equations.

The third program is also a short one.

The loops are governed by Lbl and Goto and conditions for breaking out of the loop is for example \( y>5 \) => Goto
The result is fascinating:

The line given by \( x + y = -1 \) is special. If we start with a point on this line, \( y' \) will be -1 all the time and we will follow this line.
\( y = -x - 1 \) is one solution of the differential equation \( y' = x+ y \)

EXAMPLE 4:
A GRAPHIC SOLUTION FOR THE EQUATION \( y' = x + y \) USING THE PLOT COMMAND.
In order to plot points following the graph we need a starting point, “startpoint”, \((A,B)\) and small increasements in \( x \) and \( y \); \( \Delta x \) and \( \Delta y \approx y' \Delta x \) I choose \( \Delta x=0.01 \) in this program.

The last program diffgr:
Using the program we get following graphs.
Startingpoint \((-4,3)\) is on the line \( x+y = -1 \) and the graph follows the line.

Startingpoint just a bow the line \((-5, 4.05)\) gives:

and just below the line \( x+y = -1 \)
We can solve the equation \( y' = x + y \)

\[ y' - 1 \cdot y = x \quad ; \quad y' + f(x) \cdot y = g(x) \]

The antiderivative to \( f \) is \(-x\), that gives an integrating factor \( e^{-x} \)

The solution is:

\[ y = e^x \int e^{-x} \cdot x \, dx + C \cdot e^x = e^x \left( -e^{-x} x - e^{-x} \right) + Ce^x = -x - 1 + C \cdot e^x \]

with \( C = \frac{x_s + y_s + 1}{e^{x_s}} \) \((x_s, y_s)\) is the starting point.

For \( x + y = -1 \) \( C \) becomes zero so one solution is \( y = -x - 1 \)

Startpoint \((-5, 4.05)\) gives \( C = 0.05 \cdot e^5 \approx 7.42\) with following graph.

\[ Y2 = -x - 1 + Ce^{-x} \]

\[ X = -2.004267693 \quad Y = 2.004267126 \]

We can make a lot of graphs having \( C \) vary from \(-7\) to \(+7\).

If we choose the mode simultaneous graph

we get this picture:

Let this remind us that Bloxberg is not far away from Dresden.

The programs used: