Conjecturing (and Proving) in Dynamic Geometry after an Introduction of the Dragging Schemes

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Abstract
This paper describes some results of a research study on conjecturing and proving in a dynamic geometry environment (DGE), and it focuses on particular cognitive processes that seem to be induced by certain uses of tools available in *Cabri* (a particular DGE). Building on the work of Arzarello and Olivero (Arzarello et al., 1998, 2002; Olivero, 2002), we have conceived a model describing some cognitive processes that may occur during the production of conjectures and proofs in a DGE and that seem to be related to the use of specific dragging schemes, in particular to the use of the scheme we refer to as *maintaining dragging*. This paper contains a description of aspects of the theoretical model we have elaborated for describing such cognitive processes, with specific attention towards the role of the dragging schemes, and an example of how the model can be used to analyze students' explorations.

Introduction

Research has shown that the tools provided by dynamic geometry systems impact students' approach to investigating open problems (Silver, 1995) in Euclidean geometry (for example, De Villiers, 1998; Laborde, 2001; Mariotti et al., 2000; Arzarello et al., 1998), and also that such tools can be motivational for students in problem solving (Goldenberg, Cuoco & Mark, 1998; Hadas, Hershkowitz & Schwarz, 2000). The innovative aspect of dynamic geometry software with respect to the traditional paper-and-pencil, is that the figures are "dynamic". That is, points can be dragged along the screen, so that during the process the properties according to which the construction was made are maintained. In a DGE, dragging can be done by the user, through the mouse, which can determine the motion of different objects in two ways: direct motion, and indirect motion. The *direct motion* of a basic element (for instance a point) represents the variation of this element on the plane. The *indirect motion* of an element occurs when a construction has been accomplished. In this case, dragging the base points from which the construction originates will determine the motion of the new elements obtained through the construction (Mariotti, 2006).

The use of dragging allows one to feel "motion dependency", which can be interpreted in terms of logical dependency within the geometrical context. This becomes a key feature in the development of conjectures originating from the investigation of open problems in a DGE. In fact the use of *Cabri* in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between what will become hypothesis and thesis of a conjecture (Mariotti, 2006), a task which is not at all trivial. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the *Cabri* figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004).

Olivero, Arzarello, Paola, and Robutti (Arzarello, et al., 1998, 2002; Olivero, 2002) presented a theoretical model describing how conjectures are produced by experts and how experts manage the transition from the conjecturing to the proving phase, passing from what they call "ascending control" to "descending control", through abductive processes. Their model shows that abduction plays an essential role in the process of transition from ascending to descending control, that is from exploring-conjecturing to proving. Abduction guides the transition, in that it is the moment in which the conjectures produced are written in a logical 'if…then' form. Once the conjecture is produced through this type of exploration, all the ingredients necessary for the proof are already present, and therefore this model suggests an essential continuity in the process exploring-conjecturing-validating-proving, for experts. Moreover, Arzarello and his team classified subjects' spontaneous development of dragging modalities (Arzarello et al., 2002), which have been referred to as the "dragging schemes".

Building on the studies described above, we (in this paper "we" refers to myself, under the guidance of my advisor Maria Alessandra Mariotti) became interested in integrating the model (or potentially building a new one) in order to describe in as much detail as possible the nature of some cognitive processes that occur when dragging schemes are applied during the conjecturing stage of open problem investigations in a DGE. We further hypothesized that it might be possible to introduce students to certain dragging schemes through in-class activities aimed at fostering their appropriation of the schemes.

Dragging schemes introduced to students

We introduced students to four basic dragging modalities:

- wandering dragging (in Italian: "trascinamento a caso");
- maintaining dragging (in Italian: "trascinamento di mantenimento");
- dragging with trace activated (in Italian: "trascinamento con traccia");
- dragging test (in Italian: "test di trascinamento").

Wandering dragging consists in randomly dragging a base point (draggable point, from which the construction originates) on the screen. Once a particularly interesting potential property of a figure is detected (for example, the possibility that a certain quadrilateral, part of the dynamic figure, might "become" a square), the user can use *maintaining dragging* to try to drag a base point and maintain the interesting property observed. In other words, *maintaining dragging* involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant during dragging. Using Laborde's terminology such invariant would be denoted as a *soft* invariant (Laborde, 2005). With *dragging with trace activated* we intend any form of dragging after the trace function has been activated on one or more points of the figure. During the introductory lessons we only activated the trace on the base point that was being dragged. Finally the *dragging test* refers to a test that a figure can be put through in order to verify whether it has been properly constructed or not (Olivero, 2002; Laborde, 2005). During the introductory lessons we used the *dragging test* after having reconstructed the figure we were investigating, adding a new property (by construction) to it that we had hypothesized might induce the original interesting *soft* invariant to become a true (or *robust*) invariant. Thus the dragging test was applied to test whether the originally desired property was actually maintained during dragging. An expert might say we were using the *dragging test* to test a conjecture (even if it might not have been explicit at that point).

Model for the maintaining dragging scheme (MDS)

Let's consider a problem of the following type: "Given a certain step-by-step construction, make conjectures on a certain geometrical figure (that arises from such construction)". The model we constructed (and revised during the study) for the exploration of open problems of the type described above proceeds recursively through levels. In this paper we will concentrate on the phase that originates from experts' use of *maintaining dragging* (MD) to generate conjectures when solving an open problem of the type above. By "experts" we intend subjects for whom *maintaining dragging* (together with the other dragging schemes related with it in this phase) has become an acquired tool with respect to the task of formulating conjectures given an open problem situation. To frame these ideas and give meaning to the terminology used, we may refer

to Rabardel and Samurçay (2001) and consider the acquisition of the *maintaining dragging* scheme as an explorative tool that occurs through a process of instrumentation, leading to the formation of a principle that becomes part of the user's knowledge. The principle (or rule), in our case, consists in knowing that while dragging P (the base point in consideration) on some *path* (see next section) the *Intentionally Induced Invariance* (III) will be maintained. Thus, the "creativity," for an expert user, lies in the subroutine related to the "discovery" of a *geometrical description of the path* (GDP).
Let us assume that the solver has encountered an interesting configuration (frequently through *wandering dragging*), and decides to investigate "the conditions under which (or "when") the initial construction falls into this case," using *maintaining dragging*. The general exploration scheme can be described as follows.

1. Choice of an III (*Intentionally Induced Invariance*) to try to maintain during dragging;
2. Application of the maintaining dragging scheme, which presupposes the hypothesis (in the particular case being explored) of the existence of a *path* to be made explicit through the perception of a new regularity or invariance. We refer to such regularity as an IOD (*Invariance Observed during Dragging*).
3. If in fact during the exploration it seems possible to maintain the III (and therefore the solvers believe that a *path* exists and search for a geometric description of it), the solvers propose a GDP in one of two ways:
   a) the solvers interpret the IOD (potentially with help of the scheme *dragging with trace activated*) as a curve they recognize during dragging;
   b) the solvers are not able to visually perceive an IOD (even with help from the scheme *dragging with trace activated*) so they use abductive reasoning (calling into action known geometrical theorems, rules of which the particular figure they are interested in is a case of) to give a geometric description of the *path.*
4. At this point the solvers link the III and the IOD (which now has a geometric description) through a conditional link (CL), passing from a form such as "the figure is a …when…" to a form such as "the figure is a …if (and only if) this point belongs to this curve". In general, this link can occur in two ways:
   a) the solvers link the III and IOD through a conditional link (CL) expressing it as a conjecture (or we might say "hypothesis" in a broad sense) that in their opinion describes the "behavior" of the figure they have explored (the conjecture is not a known theorem).
   b) or they link the two invariants through a conditional link (CL) choosing from their bag of mathematical knowledge (known theorems) a rule of which the situation seems to be a case of.

In either case, the establishment of a CL is key in the transition from "dynamic" to "static": the final outcome of a dynamic exploration is a conjecture – that may be successively refined – which has been crystallized into a "static" statement.

The path and its origin

The application of the *maintaining dragging* scheme, for solvers who have appropriated it, leads to the search of an invariance or regularity in the movement of the base point being dragged. When solvers apply this scheme and verify visually (and manually) that it *is* possible to drag the base point they are interested in and maintain the III they have identified, they already have in mind the idea of a *path* to be found, that is, a set of points on the plane with the following property: *when the dragged base point coincides with any point of the path, the III is visually verified*. Notice that this notion is not associated to a particular geometric shape (or curve), nor does it (necessarily) coincide with the mathematical notion of locus – the set of *all* points of the plane that guarantee verification of the III when the base point is chosen from it (the *path* may, in fact, be a proper subset of such mathematical locus). The solvers then engage in the search of a

geometrical description of such *path* (GDP). During this stage they may choose to activate the *trace* on the dragged base point in order to visualize the movement in a different way. Depending on the student's (the one who is dragging) ability in dragging, the movement (and associated trace if activated) will appear more or less "regular".
In some cases from the movement (and trace if activated) solvers can easily "see" the regularity and they are able to give a *geometrical description* of the *path* they have hypothesized. In other cases the III is difficult for the solver who is dragging to maintain, and therefore regularities are difficult to perceive from the movement (or from the trace, if activated). A *geometrical description* of the conceived *path* is therefore given in a different way. The solvers look at the figure in a "theoretical" way, and through an abduction, according to Peirce's description of abductive inference (Peirce, 1960, p. 372), come to a GDP. The solvers may propose successive more and more refined GDPs leading (ideally) to one that is a *P-invariant*, if P is the base point dragged, that is invariant for dragging of the specific base point P (Baccaglini-Frank et al., in press). Once the solvers have reached a GDP, in order to visually (and manually) test its CL with the III, they construct it as a Cabri object on their figure. There now is a "concrete" geometrical object potentially representing the *path* and that can be used to either drag the base point along "by eye" ("*soft" dragging test*) or to link the base point to and verify (*"robust" dragging test*) visually (and manually) the GDP and the CL.
The *path*, and the GDP in particular, seem to also play a fundamental role in the proving phase. More precisely, the GDP can be seen as a "bridge" to proof: the new relationship(s) between the invariants (that can be translated into geometric properties of the figure), that the GDP has made explicit during the conjecturing phase, become key in the proving phase and solvers at this phase can link back to them in order to successfully construct their proof.

Analysis of two students' exploration

Activity: Draw three points A, B, C. Then construct the line through A and C, and construct the parallel to this line, through B, and call it *l*. Construct the perpendicular line to *l* through C and construct point D as the intersection of this perpendicular line and *l*. Drag the points and make conjectures about ABCD. Then try to prove your conjectures.
Two Students' Response (an episode of their exploration): After making the construction (Fig. 1), through *wandering dragging* of the base point A the students noticed that ABCD could "become" a rectangle, for different (discrete) choices of positions for A. Then they saw that there seemed to actually be "infinitely many" choices of positions for A. This led them to believe in the existence of a *path* (see next section), and to believe that applying the MDS would be possible and insightful. So, with the intention of maintaining the property "ABCD rectangle" (III), the students dragged A and successively continued to drag maintaining the III with the trace activated, as shown in Fig 2.

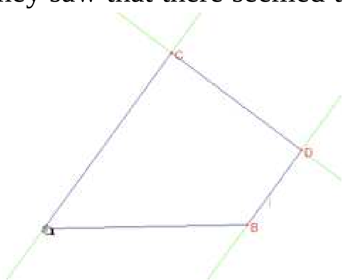

Figure 0: The students decide to drag base point A of the construction.

The students were able to interpret the trace in a geometric way, providing two GDPs, which also described the regularity in the movement of point A (IOD) that they were observing during the *maintaining dragging*. The students then proceeded to construct the *path* according to their second (more refined) geometric description of it (circumference with diameter BC), as shown in Fig 3. Once they constructed this circumference they were not convinced that *all* of it necessarily represented their hypothesized *path* (one student proposed that maybe only dragging A along a part of it would make ABCD a rectangle). To investigate further and reach an answer to their uncertainty, the students performed a robust dragging test, redefining A on the constructed circumference. This led them to believe that the III was (visually)

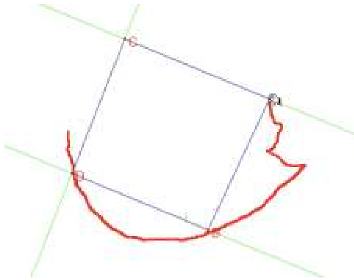

Figure 2: The students are applying the MDS with trace activated.

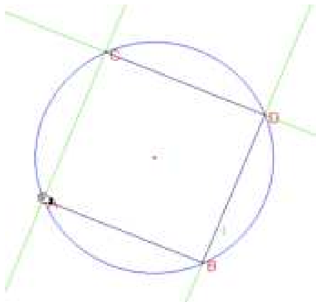

Figure 3: The students perform a robust dragging test.

maintained for the dragging of A along the *whole* circumference, thus confirming their CL between the III and the IOD. They finally formulated and wrote the following conjecture: "If A is on the circumference centered in O (midpoint of CB) then ABCD is a rectangle."

The students then successfully proved their conjecture, making fundamental use of the circumference (their GDP). This is how they proceeded:

1) D, together with A (by their hypothesis), B, and C, have to belong to the circumference, since the angle ∠CDB is right by construction (and so it is inscribed in the semi-circumference centered in O), and thus OB≅OD≅PC≅OA.

2) Triangles AOC and BOD are isosceles and congruent (∠ACO≅∠OBD because they are alternate interior with respect to the parallel lines AC and BD; and therefore also the other angles are respectively congruent), in particular ∠AOC≅∠BOD.

3) C, O, B are aligned by hypothesis; while A, O, D are aligned because ∠AOC≅∠BOD and thus vertically opposite angles (since C, O, B are aligned).

4) Therefore ABCD is a quadrilateral with diagonals that intersect at their midpoints and that divide one another in four congruent segments. Thus ABCD is a rectangle.

Conclusions

Our conclusions here are developed along two lines: considerations upon the model, and importance of the notion we introduce of *path*.

Our model, of which we introduced only a part in this paper, seems to describe cognitive processes that occur in connection with the MDS (and other dragging schemes as described). In fact we were able to interpret data generated by solvers who had appropriated the MDS in terms of phases of the model, as we did for the episode described above. Moreover having a model that seems to describe "experts' use" of the MDS is a very useful tool both for "catching" expert behavior (and thus complete appropriation of the MDS) in students, as well as for diagnosing where and how appropriation has failed. In fact many of the students' cognitive difficulties become describable in terms of "mismatches" between the model and the students' actual behaviors. Therefore the model becomes a powerful tool for designing new activities aimed at overcoming the diagnosed conceptual difficulties and at fostering complete appropriation of the MDS.

The *path* seems to play a main role in the generation of a proof, becoming a part of solvers' "reorganization and transformation" that occurs with abductive reasoning (Cifarelli, 1999). In particular the *path* and the GDP make explicit various new relationships between invariants of the dynamic figure, which can be translated into geometric properties. The *path* and the GDP therefore can become a powerful tool for the solvers to use in their proof, in order to link back to their reasoning and insights from the conjecturing stage, and thus bridge the potential gap between argumentation and proof. In this sense we believe it can foster cognitive unity (Boero et al., 1996; Pedemonte, 2003) and the production of proofs. Given these considerations, teaching students to consciously use certain dragging schemes, and make use of what we describe a *path*, can potentially accelerate and facilitate the entire process of making a conjecture and proving it. This seems to be a significant step towards the achievement of an important goal that the mathematics education community has set for mathematics teaching and learning.

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