How do rabbits help to integrate teaching of mathematics and informatics?

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Abstract

Many countries are reporting of difficulties in exact education at schools: mathematics, informatics, physics etc. Various methods are proposed to awaken and preserve students' interest in these disciplines. Among them, the simplification, accent on applications, avoiding of argumentation (especially in mathematics) etc. must be mentioned. As one of reasons for these approaches the growing amount of knowledge/skills to be acquired at school is often mentioned.

In this paper we consider one of the possibilities to integrate partially teaching of important chapters of discrete mathematics and informatics not reducing the high educational standards. The approach is based on the identification and mastering general combinatorial principles underlying many topics in both disciplines. A special attention in the paper is given to the so-called "pigeonhole principle" and its generalizations. In folklore, this principle is usually formulated in the following way: "if there are $n + 1$ rabbits in n cages, you can find a cage with at least two rabbits in it". Examples of appearances of this principle both in mathematics and in computer science are considered.

Introduction: education and competitions

As we can learn from many written and oral sources the number of lessons devoted to exact disciplines in school is decreasing in many countries, especially in post-"socialist" ones. Of course, in some sense this can be compensated by introducing new technologies into education. Nevertheless, today not all teachers are ready to explore their advances. So the official curriculum, e. g., in mathematics today is far from the level of 1980-ies.

In this situation olympiads appeared to be a very strong consolidating factor. Olympiad "curricula" wasn't changed; it was developed further in an essential way. The standards that were elaborated in olympiad movement during many years in some sense became the unofficial standards for advanced education in mathematics. There is a lot of topics that are not included in any official school program but nevertheless are discussed regularly with all students interested in mathematics.

The other positive feature of contests is their stability. Teachers are aware that the olympiads will be held, and they can organize their activities and encourage their students to work additionally for a clear and inspiring aim.

So competitions, which had often been characterized as "elitarian", "discouraging", "far-from-life" etc., appeared to be the strongest support to advanced education at schools in a lot of countries.

All this sets new tasks to olympiads. The competitive factor is still extremely important, but also the educational factor of them has become very significant.From the previous it is clear that math contests today cover broad spectrum of mathematics. It is particularly important also because olympiad and contest problems from previous years are broadly used afterwards in everyday teaching practice.

Modern elementary mathematics

It is a tradition that the words "elementary mathematics" are connected with school only. It's not quite correct. Of course, no definition in the mathematical sense is possible. Trying to list the parts of elementary mathematics we include Euclidean planimetry and stereometry , linear operations with plane and space vectors, scalar, pseydoscalar and vectorial products, the greatest part of combinatorial geometry, elementary number theory, equations and systems solvable in radicals, algebraic inequalities, elementary functions and their properties, the simplest properties of sequences and the combinatorics of finite sets. There are many mathematicians, however, who include also elements of graph theory, simplest combinatorial algorithms, simplest functional equations in integers, etc. There are parts of mathematics which definitely should not be included: we can mention the methods which are effectively used only by a small amount of mathematicians as well as methods which, though used widely, demand a specific and advanced mathematical formalism.

We can give the following approximate description of elementary mathematics. Elementary mathematics consists of:

1) the methods of reasoning recognized by a broad mathematical community as natural, not depending on any specific branch of mathematics and widely used in different parts of it,
2) the problems that can be solved by means of such methods.

Evidently, such a concept of elementary mathematics is historically conditioned. Many new areas of mathematics, especially in the discrete and algorithmic parts of it, are still today exploring elementary methods as the **main tool**. Obviously it can be explained at least partially with the fact that the natural questions there have not yet been exhausted, and natural approaches are therefore effective.

The movement of mathematical contests, especially of mathematical olympiads, has made an important service to elementary mathematics. Becoming a mass activity, the system of math competitions created a large and constant demand for original problems on various levels of difficulty. Clearly school curricula couldn't settle the situation, and the organizers of the competitions turned to their own research fields where they found rich and still unexhausted possibilities.

One of important results that originated from the "olympiad mathematics" was the identification of the so called general combinatorial methods (mean value method, invariant method, extremal element method, interpretation method).(1)

Mean value method

Informally speaking, the mean value method helps to make precise the thousands of years long opinion "to reach considerable results, serious efforts should be concentrated in at least one direction". What does the words "considerable results", "serious effort", "direction" mean depends on each individual problem. In the simplest version of it, the Pigeonhole Principle (or, otherwise, Dirichlet Principle of Box Principle), "considerable result" should bet "$\geq n+1$ elements in n boxes"; "serious effort in at least one direction" in turn should be "≥ 2 elements in some box". From the fact that "considerable result" has been reached we conclude that "serious efforts" have been made.

There are a lot of applications of this simple idea both in mathematics and in computer science, even on the middle and high school level. We mention only few classes here:

a) important classical lemma in number theory: among any $n+1$ integers you can find two which are congruent modulo n,
b) "overlapping lemma" in combinatorial geometry: if some figures of a common measure exceeding n are situated within a "home" of unit measure, then there is a point in the "home" which is covered by at least $n+1$ of these figures,
c) Levy's theorem: for any two finite systems of segments A and B, if the sum of projections of elements of A on any axis is greater then the sum of projections of elements of B on the same axis, then the sum of the lengths of elements of A is greater than that of the elements of B,
d) plane Ramsey theory (2),
e) classical Ramsey theorem for graphs (3):

    for any positive integers m and n there exists such a positive integer $R(m,n)$ that in any complete graph with $R(m,n)$ vertices and each edge coloured either white of black you can find either a clique consisting of m vertices with all edges white of a clique consisting of n vertices with all edges black. (In fact, usually by $R(m,n)$ we denote the smallest of all integers with this property.)

f) analysis of the lower bound of running time of an algorithm (4),
g) impossibility proofs for finite automata in computer science (5),
h) applications to coding theory.

A lot of applications on the school level see, e.g., in (6).

To illustrate the rich possibilities here let's mention only one example – some of more than 50 appearances of the classical result about Ramsey numbers (see above).

It is common knowledge that $R(3, 3)=6$. As far as we know, for the first time this fact was introduced into math contests in 1953, when it was proposed to the contestants of William Lowell Putnam Mathematical Competition:

"Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same colour."

The following list contains only some of theorems and contest problems the proofs/ solutions of which are based significantly on this result.

A. The inductive proof of general Ramsey theorem (see above).

B. There are at least two such “monochromatic” triangles in a two-coloured complete graph K_6.

C. In a two-coloured complete graph K_7 there are at least 4 “monochromatic” triangles.

D. (Goodman’s theorem) In a two-coloured complete graph K_n there are at least $f(n)$ “monochromatic” triangles where

$$f(n)=\begin{cases} 2C_k^3 \text{ for } n=2k, k\in \mathrm{N} \\ C_{2k}^3+C_{2k+1}^3-k \text{ for } n=4k+1,\ k\in \mathrm{N} \\ C_{2k+1}^3+C_{2k+2}^3-k \text{ for } n=4k+3,\ k\in \mathrm{N} \end{cases}$$

and these estimations are the best possible.

E. (6th IMO, 1964). In a group of 17 scientists each scientist sends letters to the others. In their letters only three topics are involved and each couple of scientists makes reference to one topic only. Show that there exists a group of three scientists which send each other letters on the same topic.

F. (20th IMO, 1978). An international society has its members from six different countries. The list of members contains 1978 names, numbered 1*;* 2*;* …*;* 1978*:* Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

G. (21st IMO, 1979). We are given a prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces. Each side of the two pentagons and each of the line segments A_iB_j for all $i; j = 1; ...; 5;$ is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of a different colour. Show that all 10 sides of the top and bottom faces are the same colour.

H. (33rd IMO, 1992). Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

I. (Latvian Olympiad). There are $2n$ points in the plane. Among their pairwise distances there are at least n^2+1 which don’t exceed 1. Prove that there are three points which can be cowered by a circle of radius $\frac{1}{\sqrt{3}}$.

J. (Folklore). There are 6 irrational numbers. Prove that there are 3 numbers among them such that all their pairwise sums are irrational too.

K. (Folklore). Six points are given in space such that the pairwise distances between them all are distinct. No 4 of these points are in the same plane. Consider the triangles with vertices at these points. Prove that the longest side in one of these triangles is at the same time the shortest side in another triangle.

This list can be prolonged very far. We see from this example that the same simple idea has served for many years even in such a sensitive area as high-level math competitions. So it is also a suitable tool to explain the idea of mean value method with purely educational purposes.

For more examples of the applications of mean value method on educational level see, e.g., [(6)].

Conclusions

The mean value method still remains one of the most powerful tools in composing and solving contest problems. It plays also important role in the research in mathematics and theoretical computer science. Acquaintance with it has great educational and aesthetical value. Therefore it must remain in the (official and unofficial) curricula of advanced mathematical education. New applications of it must be carefully collected. A monograph-type teaching aid on this method on college level should be most welcome.

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