Some Initiatives in Calculus Teaching

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Abstract
In our experience of teaching Calculus to engineering undergraduates we have had to grapple with many different problems. A major hurdle has been students' inability to appreciate the importance of the theory. In their view the theoretical part of mathematics is separate from the computing part. In general, students also believe that they can pass their exams even though they do not have a real understanding of the theory behind the problems they are required to solve. In an effort to surmount these difficulties we tried to find ways to make students better understand the theoretical part of Calculus. This paper describes our experience of teaching Calculus. It reports on the continuation of our previous research.

Introduction

From our experience we know that many students have difficulties in learning theory. To try to solve the problem we developed an approach called "self learning method" (SLM). We originally described the method in an earlier paper (2007). The method is intended to help students better understand the meaning of a theorem by involving them in the learning process, step by step. The way in which the material is presented encourages them to take part in formulating, discussing and proving a theorem. The following is a description of the process of learning a theorem:

1) *Reviewing concepts:* The first step is to have students review the concepts, definitions and theorems that are needed to learn a specific theorem.

2) *Formulation of a conjecture*: Normally, teachers present a theorem in a way that is clear to mathematicians, but is too formal for students. The SLM method aims to turn students from passive receivers of knowledge into active partners in the learning process. Students are given self learning material that includes assignments. The aim is for students to learn the concepts under review by completing the given assignments in an informal way and as a result, to be ready for formulation of a theorem, hopefully on their own.

3) *Formulation of the theorem*: At this stage of the learning process we state the theorem in a schematic way, emphasizing what is assumed and what is concluded.

4) *Exploring assumptions and conclusions*: We provide students with exercises and problems that focus on the following questions: What are the assumptions of a theorem and what are the conclusions? What is the geometrical meaning of a theorem? What happens when one or more of the theorem's assumptions are not fulfilled? What assumptions are necessary and which are sufficient? Generally speaking, what does the theorem mean?

5) *Proving the theorem*: This is the final step of the process. We use different ways to check whether the students really understand the material or whether they have simply memorized it.

In our earlier papers (2007 and 2009) we discussed the fourth SLM step using the three basic theorems of the differential part of Calculus: Lagrange's Mean Value Theorem, Fermat's Theorem and Rolle's Theorem. We had three groups of about one hundred engineering students in total try out the instructional materials. We were satisfied with the results of our experiment, as shown in the first paper. We, accordingly, decided to develop SLM for other Calculus theorems. We know from our teaching experience that students have many problems in understanding the Basic Theorem of Calculus:

If a function $f(x)$ is continuous on the interval $[a,b]$, then the function $F(x)=\int_a^x f(t)dt$ is differentiable on the interval $[a,b]$ and $F'(x)=f(x)$.

Our students have the following problems with this theorem:

1) In the definition of the definite integral there are constant limits of integration. When we define the function $F(x)$, one of the limits is a variable; this confuses students.
2) Students see the function $F(x)$ as very abstract, and do not understand its practical meaning.
3) Students get used to elementary functions, and it is difficult for them to accept other types of functions.

The above problems underscore how important the first and the second steps of SLM are. Students need to deal with the relevant concepts before formulating a theorem. This is in line with Tall's idea (2004) about the three worlds of mathematics used to distinguish among different modes of mathematical thinking. The three worlds are the embodied world, the proceptual world and the formal world. They reflect a development from just perceiving a concept through actions to formal understanding of a concept. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered. If we, the teachers, want to take students immediately to the formal world, they are very likely to get lost on the way. Therefore, we should invite the students into the embodied world in order to become acquainted with relevant concepts and statements. We concluded that the best way to implement this is to teach or learn "with examples" (Hazzan & Zaskis, 1999).

In this paper we demonstrate how we used SLM Steps 1 and 2 to teach the Basic Theorem of Calculus.

Reviewing concepts

The following concepts and statements were given to students to review:

1) Continuous function $f(x)$ at a point $x=a$ (on interval $[a,b]$);
2) Differentiable function $f(x)$ at a point $x=a$ (on interval $[a,b]$);
3) Antiderivative;
4) Definite integral;
5) Intergrable function $f(x)$ on interval $[a,b]$.

Statement 1. *If function $f(x)$ is constant ($f(x)\equiv C$), then* $\int_a^b f(x)dx = C(b-a)$.

Statement 2. *If function $f(x)$ is integrable on interval $[a,b]$, and $f(x)\geq 0$, then the integral* $\int_a^b f(x)dx = S(D)$, *here $S(D)$ is the area of the domain:*

$D=\{(x,y)\mid a\leq x\leq b, 0\leq y\leq f(x)\}$.

Statement 3. *If function $f(x)$ is integrable on interval $[a,b]$, then for every point c , $c\in[a,b]$,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Statement 4. *If function $f(x)$ is integrable on interval $[a,b]$,* and

$$g(x)=\begin{cases} f(x)\ , & x\neq c \\ C\ , & x=c \end{cases},\ a\leq c\leq b,$$

then $g(x)$ *is integrable on interval* $[a,b]$ *and* $\int_a^b g(x)dx = \int_a^b f(x)dx$.

It must be noted that the students had not yet learned the Newton-Leibniz Theorem.
In order to introduce the integral function $F(x)$ from a geometrical point of view, we gave the students the following definition:
Let $f(x)$ *be an integrable function on interval* $[a,b]$, $f(x) \geq 0$ *, then the area function* $S(x)$ *is the area of the domain* $D = \{(t, y) \mid a \leq t \leq x, 0 \leq y \leq f(t)\}$.

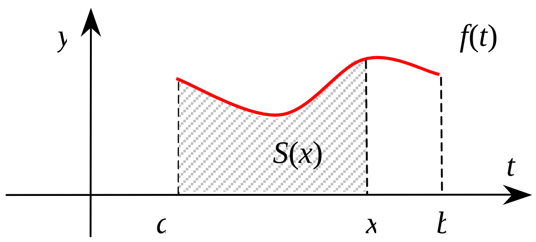

Fig. 1. The Area Function

<u>Remark</u>: As follows from Statement 2: $S(x) = \int_a^x f(t)dt$, $a \leq x \leq b$.

The following assignments were given to the students.

Assignment 1.
Below are several examples of functions defined on the interval $[0,2]$. For every function fill in Table 1 and try to find out the formula of the area function. We recommend that you apply the statements given above.

$f(x)=1$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}1, & x \neq 1\\ 2, & x = 1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}1, & x \leq 1\\ 2, & x > 1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \leq 1\\ \_\_\_\_, & x > 1\end{cases}$
$f(x)=x$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x) =$ ______
$f(x)=\begin{cases}x, & x \leq 1\\ 1, & x > 1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \leq 1\\ \_\_\_\_, & x > 1\end{cases}$
$f(x)=\begin{cases}x, & x \leq 1\\ 2, & x > 1\end{cases}$	$S(0)$	$S(0.5)$	$S(1)$	$S(1.5)$	$S(2)$	$S(x)=\begin{cases}\_\_\_\_, & x \leq 1\\ \_\_\_\_, & x > 1\end{cases}$

Table 1. Examples.

The students were also provided with graphs of the functions and area formulas for a rectangle, a triangle and a trapeze.

Table 2 shows the answers to the questions, which were also given to the students. This table can also be used for the next assignment.

Example 1	Example 2	Example 3
$S(x)=x$	$S(x)=x$	$S(x)=\begin{cases} x & , x\le 1 \\ 1+2(x-1) & , x>1 \end{cases}$
Example 4	Example 5	Example 6
$S(x)=\frac{x^2}{2}$	$S(x)=\begin{cases} \frac{x^2}{2} & , x\le 1 \\ \frac{1}{2}+(x-1) & , x>1 \end{cases}$	$S(x)=\begin{cases} \frac{x^2}{2} & , x\le 1 \\ \frac{1}{2}+2(x-1) & , x>1 \end{cases}$

Table 2. Answers.

Assignment 2.

Answer the following questions (see Table 1):

1) Which $f(x)$ functions are integrable on the interval $[0,2]$?
2) Which $S(x)$ functions are continuous on the interval $[0,2]$?
3) Can the following statement be true?

Statement:

If function $f(x)$ is integrable on $[a,b]$, then the function $S(x)$ is continuous on this interval.

Formulation of a conjecture

We used the above examples and the results of the assignments for the next SLM step, in which the students continued to learn about the relation between the functions $f(x)$ and $F(x)=S(x)$. We also provided additional examples for the integral function in the general case.

Assignment 3.

Using the previous assignments, fill in Table 3. Answer: “Yes” or “No”.

Example	$f(x)$ is continuous on $[0,2]$	$S(x)$ is differentiable on $[0,2]$	$S'(x)=f(x)$ on $[0,2]$
1	Yes	Yes	Yes
2	No	Yes	No
3	No	No	No
4	Yes	Yes	Yes
5	Yes	Yes	Yes
6	No	No	No

Table 3. Assignment 3 and Answers.

Assignment 4.

Which of the following statements cannot be true?

1) *If function $f(x)$ is integrable on $[a,b]$, then the function $S(x)$ is differentiable on this interval.*
2) *If function $f(x)$ is continuous on $[a,b]$, then the function $S(x)$ is differentiable on this interval and $S'(x)=f(x)$.*
3) *If the function $S(x)$ is differentiable on $[a,b]$, then the function $f(x)$ is continuous on this interval.*

At this point the students were able to answer the above questions easily since they had worked their way through enough examples, including counterexamples to false statements.

Assignment 5.

Definition: *Let* $f(x)$ *be an integrable function on interval* $[a,b]$ *, then the integral function* $F(x)$ *is defined in the following way:* $F(x)=\int_a^x f(t)dt$.

<u>Remark</u>: If $f(x)\geq 0$, then $F(x)=S(x)$. In the general case $F(x)$ is not an area of a domain.

Find the function $F(x)$ for each of the following functions:

Example 7	Example 8	Example 9
$f(x)=-1$	$f(x)=\begin{cases}-1, & x\leq 1\\ -2, & x>1\end{cases}$	$f(x)=\begin{cases}-1, & x\leq 1\\ 1, & x>1\end{cases}$

Table 4. Additional Examples.

We recommend using the questions of Assignment 2 and the properties of the definite integral. Table 5 provides the answers to Assignment 5.

Example 7	Example 8	Example 9
$F(x)=-x$	$F(x)=\begin{cases}-x, & x\leq 1\\ -2x+1, & x>1\end{cases}$	$F(x)=\begin{cases}-x, & x\leq 1\\ x-2, & x>1\end{cases}$

Table 5. Answers.

Assignment 6.

Formulate two statements, similar to the statement in Assignment 4, about the integral function: a false one and a true one.

Conclusions

The main points of our approach are:

1) Students are given a bank of comparatively simple examples, in which they can also find counterexamples, a task which they usually find to be difficult.
2) When solving seemingly simple problems the students also solve theoretical problems, and arrive at theoretical conclusions. They are thus co-opted into the research process.
3) Students' learning becomes motivated by their success.

By using the above examples we tried to introduce the concept of an integral function before attempting to formulate the Basic Theorem of Calculus. The SLM method allowed the students to absorb the concept of an integral function into their embodied world, so they were able thereafter to draw it into the formal world.

Our hope is that, in the end, students will be able to formulate a theorem independently.

References

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