Workshop

Jean Michel Hanna
EGYPT, jeanmichelhanna@yahoo.fr

The idea of this workshop is to show that doing mathematics is not only solving set problems given by others, but also (and especially) creating new problems and asking good questions.

In the domain of teaching (or training), in order to enable students (or trainees) to do this, teachers (or trainers) have to do an extra effort in choosing rich and open situations which can motivate and lead them to think about conjectures, experiment these conjectures and find a good reason (or a need) to give a proof, either for a self conviction or to convince others (or for the two reasons together).

In such situation, we can't expect what can be obtained as reactions or products of students (or trainees). Teacher's (or trainer's) treatment of these reactions or products plays a considerable role in achieving the objectives of the whole situation.

One of these situations is the following:

Using identical cubes, we can construct solids, such as rectangular solids or pyramids or …

I) How many cubes can be used to construct a rectangular solid with a square base which can be transformed in a cube?
Are there different answers?

II) How many cubes can be used to construct a rectangular solid with a square base which can be transformed in two cubes (or more)?
Are there different answers?

III) What other questions could be asked in this situation?
Try to give answers to each of these questions.

I experimented this situation in one of my training sessions. While preparing this session, I discovered the following numerical properties:

- For any positive integer n, there are n consecutive odd numbers where their sum is equal to the cube of their number (i.e. n^3).

- If there are n consecutive odd numbers where their sum is equal to n^3, the sum of the next (n+1) consecutive odd numbers equals $(n+1)^3$.

It is easy to find a proof for each of these two properties, and then we can use them to find (in an original way) the sum (in terms of n) of the cubes of all the integers from 1 to n.
(i.e. $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3$)

Proofs

<u>FIRST PROPERTY</u>

Let a be the first term of an arithmetic series with constant difference 2,
The sum of the first n terms of this series $= n/2\ (2a + 2(n-1)) = n(a + n - 1)$
This sum equals n^3 if, and only if, $a + n – 1 = n^2$ i.e. $a = n(n – 1) + 1$
For any positive integer n, a is an odd number.

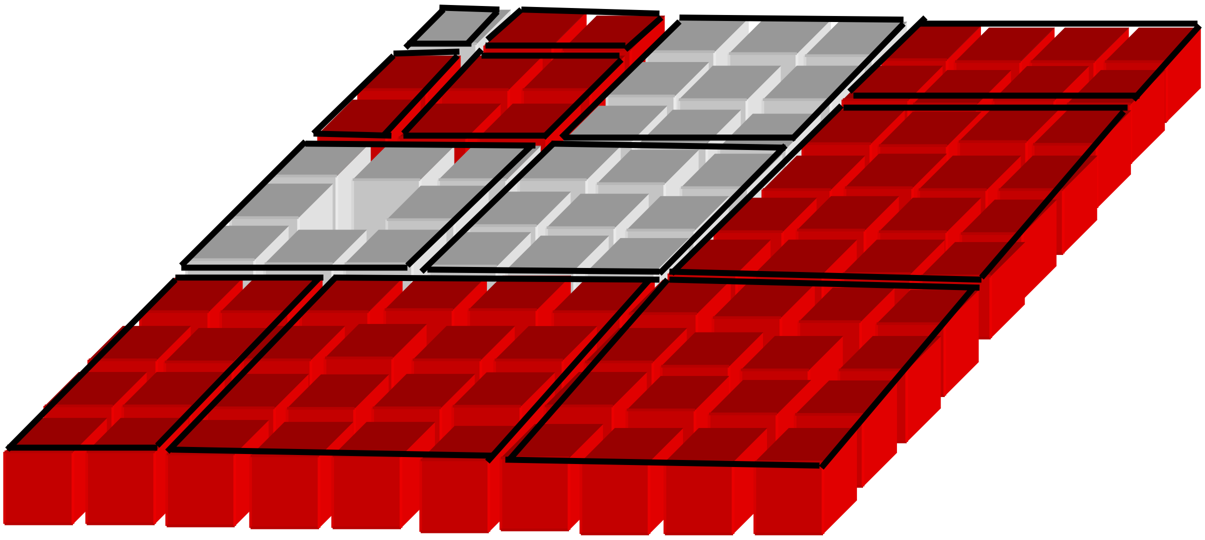
<u>SECOND PROPERTY</u>

From the first property, we found that for any positive integer n there are n consecutive odd numbers where their sum is equal to n^3
These odd numbers begin with $n(n-1) + 1$,
Their last number equals $n(n-1) + 1 + 2(n-1)$ i.e. $(n^2 + n – 1)$
The next (n+1) consecutive odd numbers begin with $(n^2 + n + 1)$
The sum of these numbers $= (n+1)/2\ (2(n^2 + n + 1) + 2n) = (n + 1)^3$

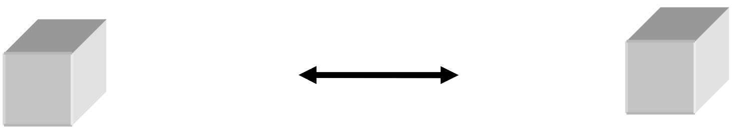
<u>The sum of the cubes of all the integers from 1 to n</u>

Using these two properties, we can say that:

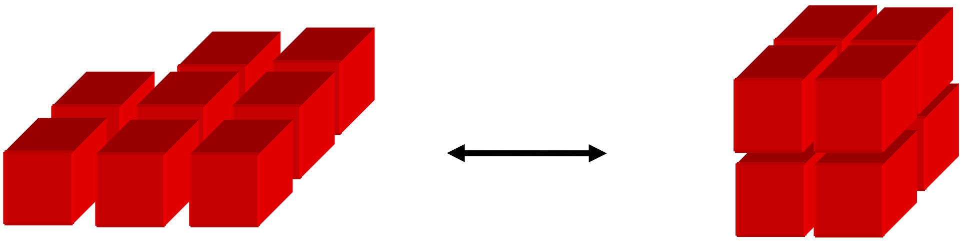
$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = 1 + (3 + 5) + (7 + 9 + 11) + \ldots\ldots + n^2 + n - 1$
= the sum of (1 + 2 + 3 + ………+ n) consecutive odd numbers which begin from 1



$1 = 1^3$



$3 + 5 = 2^3$



$7 + 9 + 11 = 3^3$

$13 + 15 + 17 + 19 = 4^3$

$1 + (3+5) + (9+11+13) + \ldots\ldots\ldots = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots$