Paper Folding Squares

Kwan Shi-pui, Sze Chong-lap
Lecturers, The Hong Kong Institute of Education

Origami and Mathematics Education

The term "origami" originates from the Japanese art of paper folding in which "ori" means folding and "gami" means paper. Asian children begin learning origami at their mothers' knees. Nowadays European children are learning it at schools. Researches have shown that paper-folding, particularly in the junior primary school years, is a unique and valuable addition to the curriculum. Origami is not only a craft. It is also an invaluable approach for developing mathematical skills and logical minds.

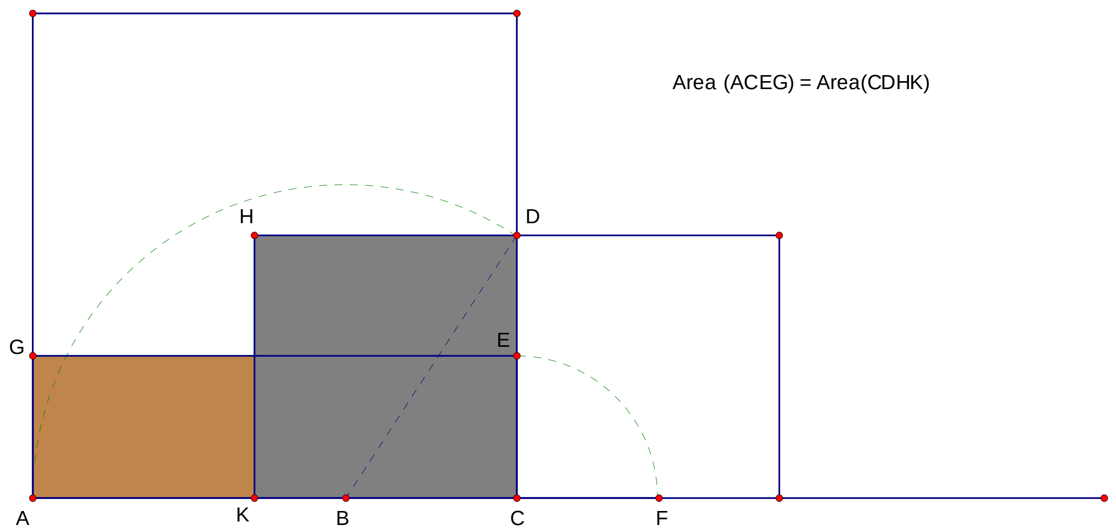
Paper Folding Squares

Given a piece of square paper, how to fold a smaller square (a) a quarter; (b) a half of its original area? These questions are obvious for children, particularly those familiar with origami. But how about folding a new square $1/n$ of its original area in general? Are there any systematic methods of doing so?

Swan M. (2002) discusses the method of folding a number line to show $1/n$ for all n. Leung K.S. and Sze C. L. (2004) extend the idea to fold m/n of a line. We claim that their findings pour light in solving our problem in mind.

The 'Squaring a Rectangle' Approach

Let us first recall a famous geometrical problem: squaring a rectangle. The solution is illustrated in the diagram below. To allow paper folding done within the square piece of paper, the smaller square is reflected along the right side of the larger one.



The proof is straight forward. Let AB and BC be r and x respectively.

$AC = (r + x)$

$CE = CF = (r - x)$

$\text{Area}(ACEG) = AC \times CE = (r + x)(r - x) = r^2 - x^2$

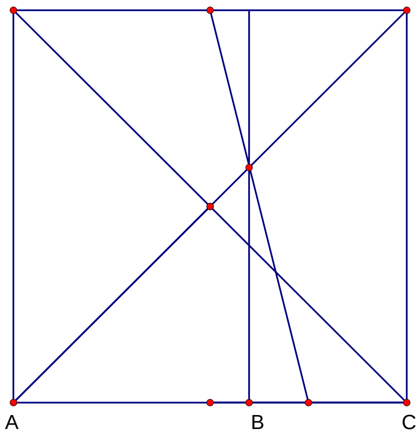
In ΔBDC, $r^2 - x^2 = DC^2 = \text{Area}(CDHK)$

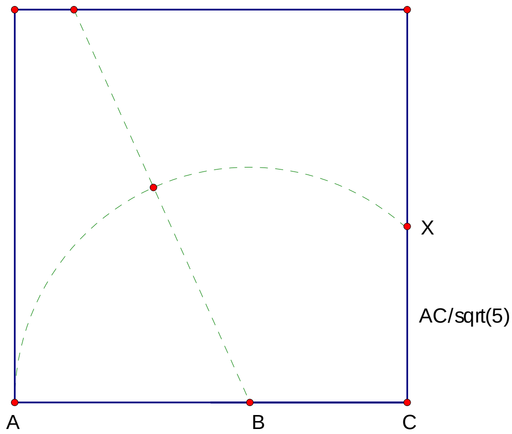
To fold a new square $1/n$ of its original area, we first apply Leung and Sze's method to divide its side in the ratio $p : q$, depending on the value of n. Then we transform the rectangle into a square by the method above. Below are some illustrations.

Examples

For $n = 5$, $p : q$ (i.e. AB : BC) = 6 : 4

$$CX = \frac{1}{\sqrt{5}} AC$$





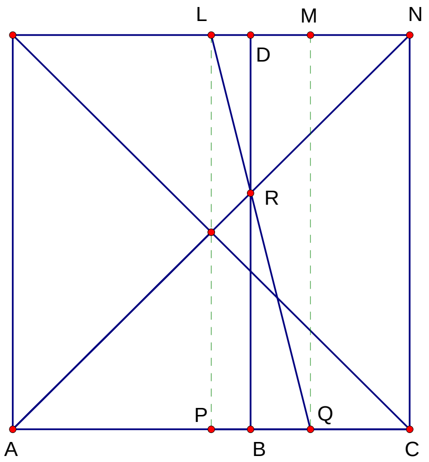
In the diagram sqrt (5) denotes $\sqrt{5}$ and the point X is obtained by folding A onto the right vertical side.

Proof:

$\Delta ARQ \sim \Delta NRQ \quad AQ : NQ = 3 : 2 = RB : RD$

$\Delta QRB \sim \Delta LRD \quad RB : RD = BQ : DL = BQ : BP = 3 : 2$

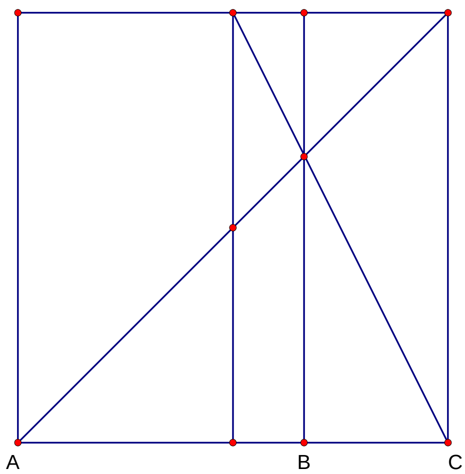
$\therefore AB : BC = 12 : 8 = 6 : 4$

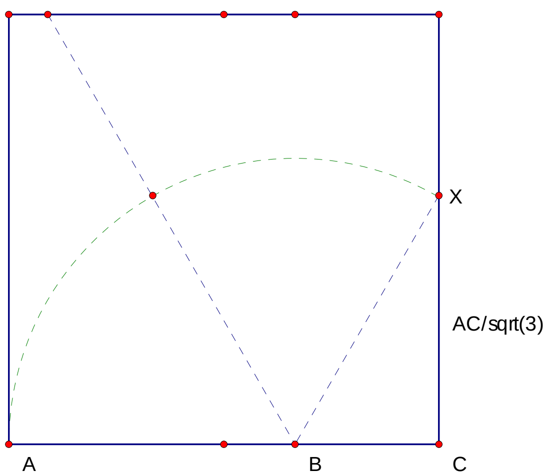


With $\frac{1}{\sqrt{5}}$AC as a side the required square is obtained.

For $n = 3$, $p : q$ (i.e. AB : BC) = 4 : 2

CX = $\frac{1}{\sqrt{3}}$AC



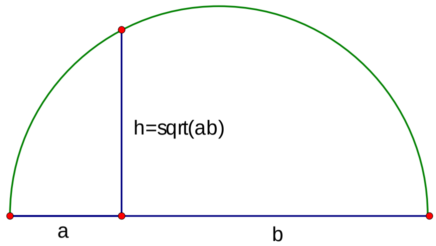


The proof follows the same line of thought. Although the method works theoretically for all values of *n*, it does have its own drawbacks. Can you suggest some practical problems encountered?

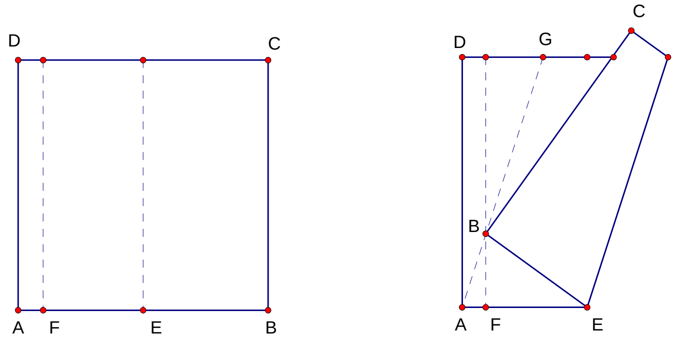
<u>Alternate Approach</u>

In search of a better approach, we find that the 'intersecting chords' property helps in obtaining a new way of folding. This property is actually a restatement of the squaring the rectangle construction.

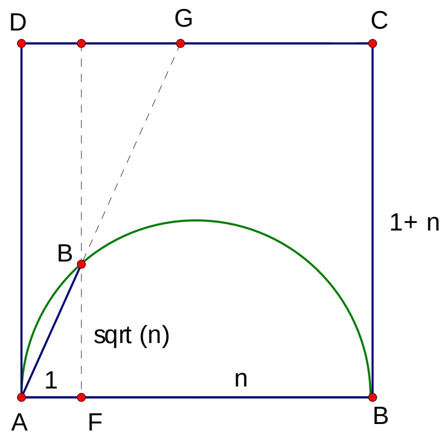
$h^2 = ab \therefore h = \sqrt{ab}$



In particular when $a = 1$, $h = \sqrt{b}$. It leads to a very intersecting way of folding:



If AE = $\frac{1}{2}$ AB and AF = $\frac{1}{n+1}$ AB we assert that DG = $\frac{1}{\sqrt{n}}$ AB.



The proof based on the properties of similar triangles.

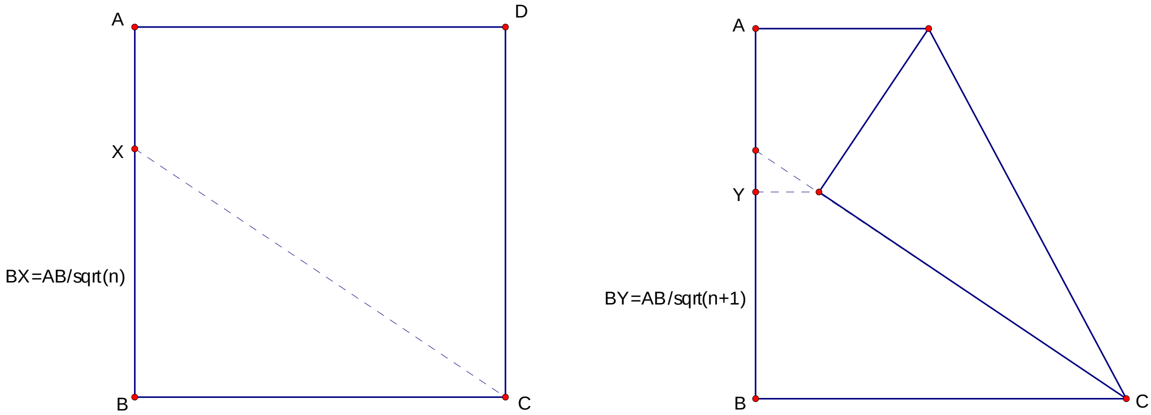
ΔAFB ~ ΔGDA, 1 : $\sqrt{n}$ = DG : (1 + n), $\therefore$ DG = $\frac{1}{\sqrt{n}}$ AB

Example: For $n = 3$, AM = MB and AN = NM, DG = $\frac{1}{\sqrt{3}}$ AB

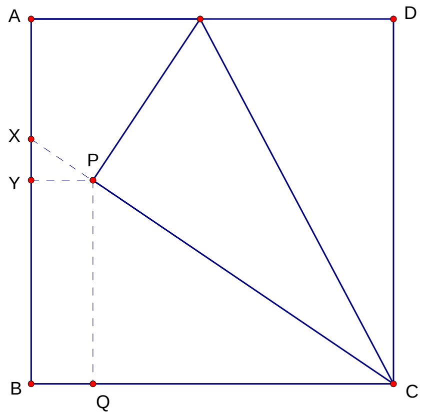
The method yields $\frac{1}{\sqrt{n}}$ AB when $\frac{1}{n+1}$AB is known. Since 1 : (n + 1) is required we can either apply Swan's method or Haga theorems (which may involves quite a number of folds) or we can select those special cases which $\frac{1}{n+1}$AB can be obtained easily. n = 3 is a good example. Can you give another one?

<u>An 'Iterative' Approach</u>

Can the above idea be modified and improved? How about the possibility of getting $\frac{1}{\sqrt{n+1}}$AB when $\frac{1}{\sqrt{n}}$ AB is known, like an iterative formula in mathematics? The following folding method yields a solution.



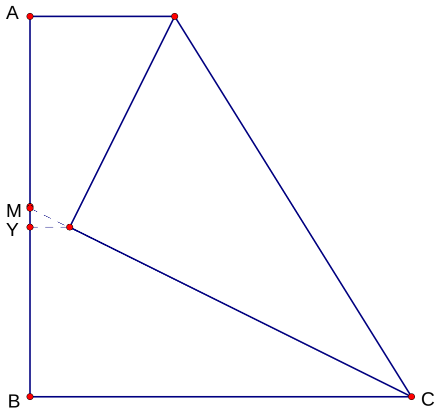
Once again the concept of similar triangles is used.



ΔBCX ~ ΔQCP
BX : CX = QP : CP
$\frac{1}{\sqrt{n}}$AB : $\frac{\sqrt{n+1}}{\sqrt{n}}$AB = BY : AB

$$BY = \frac{1}{\sqrt{n+1}}AB$$

Example: For $n = 5$,



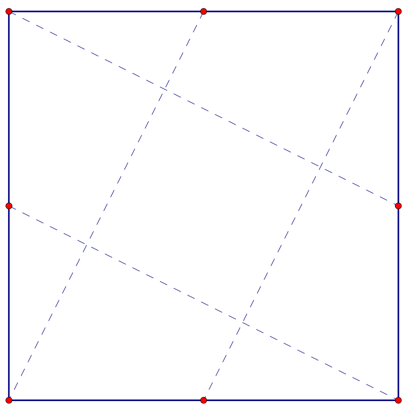
$$BM = \frac{1}{2}AB = \frac{1}{\sqrt{4}}AB, \therefore BY = \frac{1}{\sqrt{5}}AB$$

So start from $n = 4$, we have a neat and easy way of folding $\frac{1}{\sqrt{n}}AB$. What are the pros and cons of this 'iterative' method? What remedial measures can be implemented?

<u>Further Investigations</u>

In retrospection, we have found that there are other ways in tackling the problem, particularly for some specific values of n. One well known ancient method, known as the 'Greek Cross', in dealing with $n = 5$ is given below.

The folding method is simple, 'symmetrical' and provides self check. Does it shed light in developing a new approach in folding paper squares? We leave it for interested readers to carry out the investigation.



<u>Origami in Mathematics Classrooms</u>

Paper folding square is just an example amongst the many possible connections between origami and mathematics education. Through origami folding sequences, lots of basic mathematical ideas can be revealed and discussed. A folded model is both a piece of art and a geometric figure. Just unfold the model and take a look. We shall see a complex geometric pattern no matter how simple is the fold. Origami is an effective tool and interesting means for learning and teaching of mathematics.

<u>References</u>

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