

LEARNING WITH EXAMPLES AND STUDENTS' UNDERSTANDING OF INTEGRATION

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In this paper, I report on an investigation designed to reveal a range of responses regarding conceptual understanding of integration. I consider the use of a framework referred to as 'structure of a topic' to study students' understanding. Students seem to have different structures of attention that dominate their understanding compared to some experienced users. Their responses to a single construction task reveal their awareness with regard to the understanding of integration. Encouraging students to generate examples may help them gain a deeper appreciation and understanding of a mathematical concept.

Introduction

I became interested in the difficulties students face with calculus topics, particularly integration, for two principal reasons:

- 1) Differentiation is a *forward* process; the difficulties faced by students in this concept are not as complicated as those in the reverse or *backward* process of integration;
- 2) The many natures of integration: it is both the inverse process of differentiation and a tool for calculation of area and volume and length.

A review of the Malaysian Certificate of Education curriculum and the United Kingdom A level curriculum considering the topic of integration indicates assumptions that students will develop sophistication and awareness of mathematical concepts through practice with techniques, link concepts with their properties when solving problems and transfer the knowledge to different contexts routinely and automatically. As a result, students may be well-equipped to work on the usual, textbook problems. However, when faced with questions that are slightly different, they can become incapacitated.

In this paper, I report on an investigation which explores the effectiveness of the 'structure of a topic' framework to probe students' understanding of integration. I also look at example generation to reveal something of students' awareness of mathematical objects; example generation may provide insight into the students' sense of generality and the constituents which comprise their understanding of the concept.

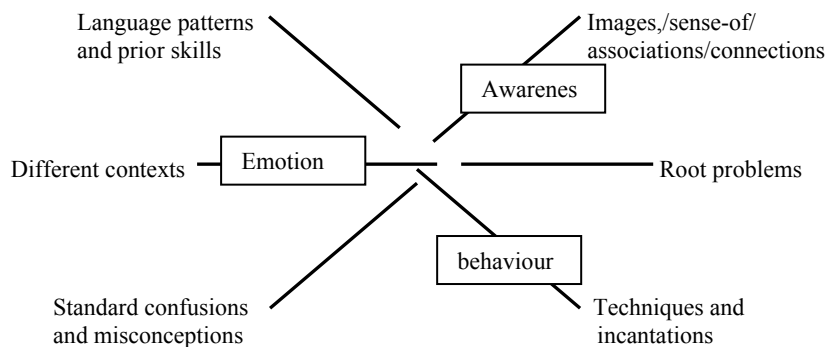
Review of Literature

A number of researchers have considered the topic of integration. However, these researchers report on students' lack of flexibility, their inability to make necessary links/connections between concepts/ideas and their lack of understanding of underlying principles but without looking into the causes of such problems. For example, Orton (1983) studied the understanding of integration of 110 students majoring in mathematics and explained how students have problems with the reasoning behind integration methods, particularly when calculating areas bounded by curves. Ferrini-Mundy & Graham (1994) reported inconsistencies between performance on procedural items and conceptual understanding in that conflicting conceptions are held comfortably and routinely in the development of calculus concepts, with separate understandings for geometrical and algebraic contexts. Students' preference to use procedural skills and their apparent reluctance to use geometric interpretations may explain their strong inclination to move to algebraic context (Dreyfus & Eisenberg, 1991). Norman & Prichard (1994) suggest that geometric intuitions about integration could become cognitive obstacles to the understanding of the concept. Selden, Selden & Mason (1994) reported that even good calculus students often cannot solve non-routine problems. Their study showed that students exhibited a tendency not to use calculus, preferring arithmetic and algebraic techniques for solving calculus problems, even though the use of elementary calculus methods would be inappropriate.

Conceptual understanding can be regarded not only as the ability to use situated knowledge to solve routine, textbook problems correctly but more importantly as the act of extending that

situatedness appropriately and efficiently to unfamiliar situations. Skemp (1976) distinguished between instrumental understanding and relational understanding, where the former is as a result of the training of behaviour in applying techniques and learning through rote memorization of rules. Relational understanding, on the other hand, describes a flexibility of thought to make sense of new situations and the ability to make necessary connections among units of thoughts and then apply them in new situations. According to Gattegno (1987), knowing means stressing awareness of something and Gattegno uses ‘awareness of that awareness’ to describe the knowing of a relative expert. Marton (Marton & Booth, 1997) referred to learning as making distinctions, as both discerning something from, and relating it to, a context. The ability to discern, to abstract structural properties and to construct new mathematical object by extending awareness of things that can vary could give insight into students’ conceptual understanding.

The ‘structure of a topic’ framework (Griffin et al., 1989; Mason & Johnston-Wilder, 2004) comprises three strands: *behaviour*, *emotion* and *awareness*. Basis for behaviour is trained through practice but training alone tends to render the individual inflexible. Flexibility arises from awareness which informs behaviour. Learning then involves educating awareness which in turn directs appropriate behaviour which provides motivation to learn.



Three strands that constitute the ‘structure of a topic’ framework

Placing emphasis slightly differently, Kilpatrick et. al. (2001) suggest five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These strands can be described in terms of awareness, behaviour and emotion. Conceptual understanding can be related to awareness as relevant actions are brought to attention when engaging in activities. Procedural fluency is attained by training of behaviour, although the training of behaviour alone may result in inflexibility of thought and a lack of adaptive reasoning. More appropriately training of behaviour must be driven by educating awareness so that students not only practice recently met ideas but also gain experience with some new concept. Productive disposition and strategic competence relate to emotion, as they deal with developing experience of identifying problem solutions and justifying conjectures.

The use of examples to illustrate mathematical concepts has been an integral part of mathematics instruction. While the teacher uses examples to clarify definitions and exemplify use of certain rules, students may develop the thinking that only those kinds of examples are appropriate and not see the generalities behind the chosen examples. Coming up with examples, on the other hand, requires different cognitive skills from working out given examples. Dahlberg & Housman (1997) showed how students who generated examples and reflected on the process attained a more complete understanding of mathematical concepts by refining and expanding their evoked concept image. Hazzan & Zazkis (1997) showed how students had difficulty managing degrees of freedom of generated examples. Encouraging students to generate examples of mathematical objects can expand their *example space* and shift their attention away from the examples to generalizations (Watson & Mason, 2005). Although students do not normally encounter this type of question in their learning of concepts, the process itself could bring out their ability to discern dimensions-of-possible-variation and reveal their awareness of the concept, which could reveal the structure of their understanding.

Methods

Semi-structured interviews were carried out with 11 pairs of students (Years 12 and 13) in a school in The United Kingdom. These students are doing A-level Mechanics and Statistics. The interviews were done in pairs, tape-recorded and time was allotted for each task so that interviewees had enough time to answer all the questions. Task 1 consisted of questions which seek to reveal something of students' understanding of integration. The questions in this task were intended to elicit learners' understanding in terms of what they are aware of, how their behaviour is developed as a result of the awareness of this awareness and the corresponding motivation and energy to learn. In Task 2, a problem of application of integration was given to the students to work on. In this task, what was studied is students' behaviour as demonstrated by their response to the task?. Task 3 consisted of a construction task in which students are asked to construct and generate different examples of a type of integration with specific properties. In this task, I wanted to find out students' responses to constructing and generating three different examples of the same type to a given example. The class teacher was also asked to predict the students' likely responses to the tasks to see whether the students' responses matched the teacher's expectation of what students had learned.

Data analysis

Responses to questions in the behaviour strand suggest that students associate the concept of integration with technical terms such as *limits, dx, functions, the arbitrary constant, area, and volume* and have knowledge of techniques of integration such as *integration by parts, by substitution, looking out for derivatives* and quite comfortably as *increasing the power and dividing with new power*. To questions regarding the awareness strand, responses suggest that students' awareness revolves around and is driven by techniques. Students' associations of integration as the reverse of differentiation and as area under a curve seem to structure their attention/awareness of the concept. The sign \int seems to command students to 'do something', triggering specific awareness of associated techniques in the learner who is working towards getting an answer. The students in this investigation also seem to have developed awareness of things to watch out for (like negative powers, expressions containing $1/x$ and fractional powers) when doing integration which could prevent them from making potential mistakes. This kind of active awareness of common errors suggests that students are aware of what they should *not* do when doing integration which could direct appropriate behaviour and prevent unnecessary frustrations. The fact that students do not see the potential use of the concept in real life appeared to affect the way they think of the concept and led them to engage in mechanical application of rules.

Responses to Task 2 indicate that students preferred using the formula to calculate distance travelled to using area under graph, even though geometrical solution would be more feasible. This lack of flexibility of thought could stem from not having an awareness/sense of the object in terms of imagery/connections, pushing it more to the background. Students also seemed uncertain of their ability to produce correct images of mathematical problems in that they preferred to use the formula. In constructing and generating examples, many respondents at first engaged in algebraic manipulation of the expression.

Researcher: Give me another example of $\int_0^2 (1-x)dx = 0$.

Student2 : You could have just x limits -1 to 1. What you do is you take the lower limit from the upper limit, so $1 - 1$ is zero. So $\int_{-1}^1 x dx$. No, that makes .2 Unless it's $\int_1^1 x dx$.

The students seemed to attempt this by simply putting in the values for limits into the expression, ignoring the \int sign and being unaware of the fact that the expression they have constructed is actually a single point. They seemed to manipulate the example superficially. The \int sign is seen as a command to *do something* and this turns into a command to plug in numbers. Students' awareness of imagery/associations (area under graph) seemed to be pushed to the background as they focused on techniques (behaviour) and were unaware of images of the object given as example and the example they constructed.

There appears to be a preference to use equations, even in cases where a geometrical solution is more feasible. This is suggestive that many students lack flexibility of thought and do not seem to make the necessary connections to get a sense of the concept. When encountering integration problem, students tend to focus on behaviour (technique) alone and push down awareness of the concept in terms of imagery and links/associations. After further prompting, some of them did generate examples and came up with a general expression for the example after becoming aware of some dimensions of possible variation.

Student1 : You integrate, no matter what value you put in, you still get zero. No, oh...you work backwards, $0 = x^2 - 4$.

Researcher: Give me another example.

Student2 : If you take $\int_{-\pi/4}^{\pi/4} \tan x dx$ that will come out to zero because it's equal area on top and at the bottom.

Student1 : If you take $\int_0^{2\pi} \sin x dx$, area above x-axis is equal to area below x-axis.

Researcher: Can you give me another example.

Student2 : If you had only a single term that you are integrating then you have a negative number and a positive number on either side of the \int sign that should always come up to zero. Oh no ...you have the x^2 one, it's never negative it's always positive. ...So you want odd power after you've integrated, so integrate even power. I think that would work. $\int_{-2}^2 4x^2 dx$, that comes out as plus.

Oh dear...it's the other way round, you want odd power. You want to integrate an odd power so that after you've integrated you have an even power.

Researcher: Can you give me a general example?

Student1 : $\int_{-a}^a kx^{2n+1} dx$

In generating examples, the students first attempted to change the numbers only but, by generating more examples, they realized they could change things that are variable and maintain that which is not permitted to change according to the criteria/situation. The students seemed confused as they tried to search for accessible objects and then constructed new examples from available information.

The teacher's responses to Task 1 and 2 were quite similar to those provided by her students. She generated examples by varying one dimension with a range of change, permitting a factor of 1 to be written as the quotient of two identical fractions. She provided examples such as $\int_0^2 \frac{x}{x} (1-x) dx = 0$

and a general example of $\int_0^2 a(1-x) dx = 0$, noting that some students may realize that a can be anything that can be changed.

Discussion

The 'structure of a topic' framework was used to reveal something about how students' understanding is structured in relation to awareness, behaviour and emotion associated with the concept. The investigation suggests students' tendency to downplay the significance of one or more of the strands in the framework and to direct their attention to techniques alone. This could demotivate them once a novel situation which does not appear to have a readily applicable technique is encountered. This kind of emotional state of "detachment" has the potential to undermine students' awareness. As a result, students gain impoverished experience to be able to appreciate mathematical concepts. Awareness of mathematical concepts in terms of imagery, relationships, connections and sense of the concept, knowledge of languages patterns/technical terms and efficiency with techniques and the corresponding emotion that accompanies students solving root and contextual problems are vital ingredients that make up conceptual understanding of mathematical ideas.

Example construction tasks offer possibilities for students to reveal their conceptual understanding of mathematical ideas and to enrich their repertoire of examples. Once students were prompted to produce more examples and recognized the freedom this offered, they generalized at least to some extent. The responses reveal some awareness of things that can vary but students tended to maintain the form of the original function. Realization of things that can vary and more importantly exercising of their freedom to change properties of mathematical objects can indeed educate students' awareness. Generating examples can not only enrich example space in terms of its content, but it can also provide a chance for students to explore its structure in terms of relationships among elements in the space, which in turn can reveal and alter students' sense of generality. Thus, it is important that students do not simply engage in tasks in order to get answers but to appreciate their autonomy and thus educate their awareness. Teachers must be explicit with their emphasis on structural properties of examples so that students can see the general through the particular. Unless they are provided with such opportunities, students are likely to be confined to ritualized habits of rote learning mathematics.

Reference

- Dreyfus, T. & Eisenberg, T. (1991) *On the reluctance to visualize in mathematics*. In W. Zimmermann & S. Cunningham (Eds.) *Visualization in Teaching and Learning Mathematics*, MAA Notes No. 19, 25 – 37.
- Ferrini-Mundy, J. & Graham, K. (1994) *Research in Calculus Learning: Understanding of Limits, Derivatives, and Integrals*. In J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning*, MAA Notes #33. Washington D.C.: Mathematical Association of America.
- Gattegno, C. (1987) *The Science of Education, Part I: Theoretical Considerations*. Educational Solutions, New York.
- Kilpatrick, J., Swafford, J. & Findell, B. (eds)(2001) *Adding It Up: Helping Children Learn Mathematics*. Mathematics Learning Study Committee, National Academy Press, Washington.
- Marton, F. & Booth, S. (1997) *Learning and Awareness*. Erlbaum, Mahwah, NJ.
- Mason, J. (1998) *Enabling Teachers To Be Real Teachers: Necessary Levels of Awareness and Structure of Attention*. *Journal of Mathematics Teacher Education*, **1**(3), 243 – 267.
- Mason, J. & Johnston-Wilder, S. (2004) *Fundamental Constructs in Mathematics Education*. London: RoutledgeFalmer.
- Norman & Prichard, (1994) *Cognitive Obstacles to the Learning of Calculus: A Krutetskiian Perspective*. In J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning*, MAA Notes #33. Washington D.C.
- Orton, A. (1983) *Students' understanding of integration*, *Educational Studies in Mathematics*, **14**, 235 – 250.
- Selden, J., Selden, A. & Mason, A. (1994) *Even Good Calculus Students Can't Solve Nonroutine Problems*. In J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning*, MAA Notes #33. Washington D.C.: Mathematical Association of America.
- Skemp, R. (1976) *Relational Understanding and Instrumental Understanding*. *Mathematics Teaching*, **77**, 20 – 26.
- Watson, A. & Mason, J. (2005) *Mathematics as a Constructive Activity: The Role of Learner-generated examples*. Erlbaum, Mahwah, NJ.