Understanding Probability Using Algebra

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Abstract
Many teachers would agree that probability is a difficult topic to learn and to teach. It is often said that one of the reasons for difficulties is the fact that at the beginning the probability problems seem to be somewhat counterintuitive. Providing students with probabilistic experiences and involving them in probabilistic activities can help them develop intuition that will lead to reasonable theoretical conclusions. The following probability game and its unexpected connections to algebraic ideas provide opportunity to use algebra as a tool for generalizing basic ideas of probability.

The game
Students can play this game in class or at home. It is played by two people. To play, you need some identical objects (for example, tiles) of two different colors. A certain number of objects are put into a bag. Without looking, the first student takes one object, records its color and (keeping the first one) takes another object from the bag and records the result. Then the objects are returned to the bag and the second student does the same. This is repeated 10 times. The person who has more pairs of the same color wins. There are two reasons to play this game: students will have a chance to collect and analyze some data; at the same time while playing the game students will gain a better understanding of the experiment under consideration which will help them draw reasonable theoretical conclusions. Figure 1 shows the table that can be used to record results of each student. Let's say that we have objects of blue and red color.

Figure 1

# of trial	red red	red blue	blue red	blue blue
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

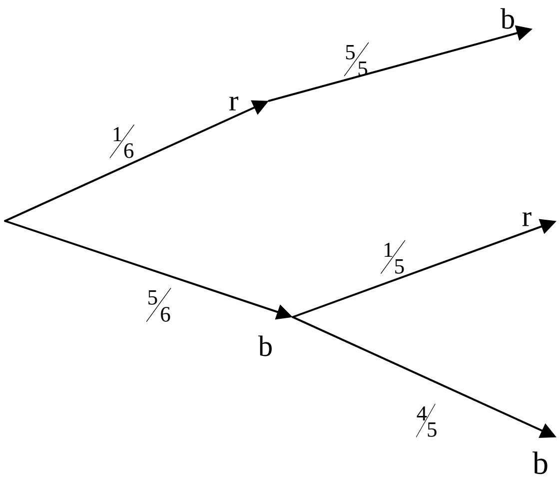
The game begins by placing six objects in the bag. One third of the class can have a combination of 1 blue and 5 red or 5 blue and 1 red; one third of the class can have a combination of 2 blue and 4 red or 4 blue and 2 red; and one third of the class can have combination of 3 blue and 3 red.

Having sufficient data, students may notice that the likelihood of the same color versus the likelihood of different colors depends greatly on the ratio of the different colors.

Before the game starts, the teacher may ask students which group in their opinion would get the pair of the same color more often than different colors.

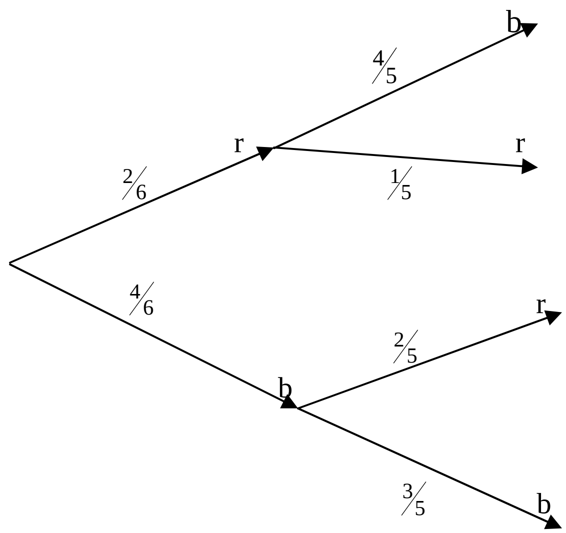
After the game is played, the results for the three different groups should be compiled. This data will give students some idea concerning which group is more likely to have a pair of the same color. Now the number of objects can be increased (for example, up to 10) and new groups will be formed to play with combinations: 1 and 9, 2 and 8, 3 and 7, 4 and 6, 5 and 5.
Looking at the results, students will notice that the same color pairs appear less often in the group with an equal amount of different colored objects than in other groups. On the other hand, in the group where one object is red and the rest of them are blue the same color pairs appear most often.
A tree diagram can be used to represent our experiment and to calculate probabilities using the classical definition. Figure 2 shows the probability tree diagram for the game where there are 6 objects; one of them red and the rest blue.

Figure 2



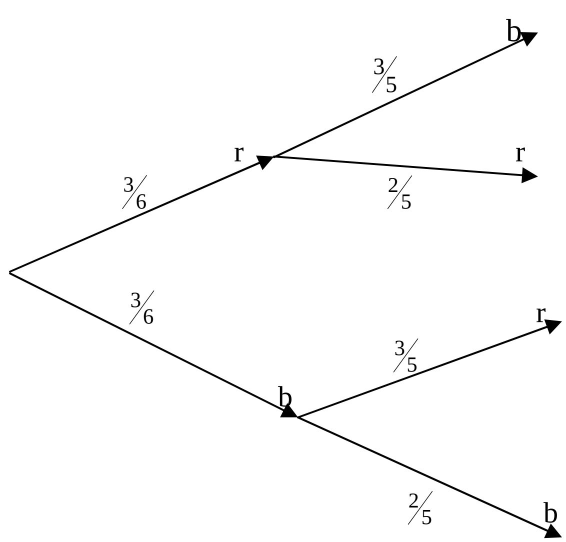
The probability that the first object is blue is $\frac{5}{6}$, the probability that the first object is red is $\frac{1}{6}$.
If the first object was red we do not have a chance to get another red object (there was only one red object in the bag). Therefore, the second object must be blue. The probability that the second object is blue if the first one was red is $\frac{5}{5}=1$. As a result the probability that the first object is red and the second object is blue is equal to $\frac{1}{6} \cdot \frac{5}{5} = \frac{5}{30}$. In short, P(rb) = $\frac{5}{30}$. By the same logic, P(br) = $\frac{5}{6} \cdot \frac{1}{5} = \frac{5}{30}$.
These two events are mutually exclusive so we can add probabilities.
P(a pair of different color) = P(rb) + P(br) = $\frac{5}{30} + \frac{5}{30} = \frac{10}{30}$
So, if there are six objects; one of them red and the rest are blue then the probability to get a pair of the same color is P(bb) = $\frac{5}{6} \cdot \frac{4}{5} = \frac{20}{30}$ which is twice as big as the probability to get a pair of different colors.
A tree diagram for the situation where there are six objects; two of them red and four are blue is shown in Figure 3.

Figure 3



A tree diagram for the situation where there are six objects; three of them red and three are blue is shown in Figure 4.

Figure 4



In order to analyze these results we may combine them in one table (Figure 5 below).

Combinations of colors	Probability of getting a pair of the same color P(bb) + P(rr)	Probability of getting a pair of different color P(rb) + P(br)
0 blue 6 red	1	0
1 blue 5 red	$^{20}/_{30}$	$^{10}/_{30}$
2 blue 4 red	$^{14}/_{30}$	$^{16}/_{30}$
3 blue 3 red	$^{12}/_{30}$	$^{18}/_{30}$
4 blue 2 red	$^{14}/_{30}$	$^{16}/_{30}$
5 blue 1 red	$^{20}/_{30}$	$^{10}/_{30}$
6 blue 0 red	1	0

Looking at this table, students may formulate a couple of hypotheses:

the greater the difference between the numbers of objects of different color the higher the probability to get a pair of the same color

the probability to get a pair of different color is the highest when the numbers of objects of both colors are the same.

It can be helpful to collect more numerical information. For example, students can calculate probabilities for the total number of objects equals nine. The two hypotheses that were formulated for six objects seem to be true for nine objects as well. Would they be true for any number of objects?

An algebraic approach can help us to formulate the generalization. Let's say we have *a* blue and *b* red objects in the bag.

In this case the probability to get a pair of objects of the same color can be expressed as:

$$\frac{a}{a+b}\cdot\frac{(a-1)}{a+b-1}+\frac{b}{a+b}\cdot\frac{(b-1)}{a+b-1} \qquad (1)$$

and the probability to get a pair of objects of different colors can be expressed as:

$$\frac{a}{a+b}\cdot\frac{b}{a+b-1}+\frac{b}{a+b}\cdot\frac{a}{a+b-1} \qquad (2)$$

Taking into consideration two facts: expressions (1) and (2) are positive because a and b are positive and expressions (1) and (2) have the same denominators, we can say that

$$\frac{a}{a+b}\cdot\frac{(a-1)}{a+b-1}+\frac{b}{a+b}\cdot\frac{(b-1)}{a+b-1} < \frac{a}{a+b}\cdot\frac{b}{a+b-1}+\frac{b}{a+b}\cdot\frac{a}{a+b-1}$$

if $a(a-1) + b(b-1) < ab + ab$

or $a^2 - a + b^2 - b < 2\,ab$

$(a-b)^2 < a + b.$

Looking at this inequality, we can be sure that if $a = b$ meaning that the numbers of objects of both colors are the same, the probability of getting a pair of different colors is greater than the probability of getting a pair of the same color. On the other hand the greater the difference between *a* and *b*, meaning that the greater the difference between the numbers of objects of different colors, the greater the probability of getting a pair of the same color.

An even more interesting question to ask is: are these probabilities ever equal? If we played with six objects and ten objects we did not have combinations of colors that would give us equal probabilities for a pair of the same color and a pair of different colors. In the cases with nine objects, the combination of 6 red and 3 blue would give us a probability of getting a pair of the same color equal to the probability of getting a pair of different colors. So what combinations of different colors would have a probability of ½ for both events: getting a pair of the same color and getting a pair of different colors?

In order to answer this question, we have to set expressions (1) and (2) equal to each other.

Taking into consideration two facts: expressions (1) and (2) are positive because *a* and *b* are positive and expressions (1) and (2) have the same denominators, we can say that

$$\frac{a}{a+b}\cdot\frac{(a-1)}{a+b-1}+\frac{b}{a+b}\cdot\frac{(b-1)}{a+b-1} = \frac{a}{a+b}\cdot\frac{b}{a+b-1}+\frac{b}{a+b}\cdot\frac{a}{a+b-1}$$

if $a(a-1) + b(b-1) = ab + ab$

or $a^2 - a + b^2 - b = 2ab$

$(a-b)^2 = a + b$ (3)

$(a-b)^2 = a + b$ means that we would like to find pairs of positive whole numbers such that their difference squared is equal to their sum. Playing with 9 objects, we already noticed that 6 and 3 gives us equal probabilities. Substituting $a = 6$ and $b = 3$ into (3), we will get $(6-3)^2 = 6 + 3$ which is a true expression.

Students can find different pairs of numbers that will satisfy (3) by substituting whole positive numbers for b and then solving the equation for a.

For example, if $b = 1$ $(a-1)^2 = a + 1$

$a^2 - 2a + 1 = a + 1$

$a^2 - 3a = 0$

$a \cdot (a-3) = 0$

$a = 0$ or $a = 3$

By the condition of our game a cannot be equal zero. Therefore, we obtained one pair of numbers (3;1).

if $b = 3$ $(a-3)^2 = a + 3$

$a^2 - 6a + 9 = a + 3$

$a^2 - 7a + 6 = 0$

$(a-6) \cdot (a-1) = 0$

$a = 6$ or $a = 1$

Therefore, we obtained two pairs of numbers (1;3) and (6;3).

This way students can collect as many examples as they need to formulate some hypothesis about pairs a and b that satisfy (3). Usually looking carefully at the pairs (1,3); (3,6); (6,10), some students are ready to suspect that they are dealing with triangular numbers. Here we can use algebra as a tool to confirm our hypothesis.

Is it true that if we choose any two consecutive triangular numbers as the numbers of different colored objects then the probability of getting a pair of the same color will be equal to the probability of getting different colors?

Any triangular numbers can be written as: $1 + 2 + \ldots + n$. Therefore any pair of consecutive triangular numbers can be written as $1 + 2 + \ldots + n$ and $1 + 2 + \ldots + n + (n + 1)$.

Let's say $a = 1 + 2 + \ldots + n + (n + 1)$ and $b = 1 + 2 + \ldots + n$.

Then $(a-b)^2 = (n+1)^2$

and $a + b = (1 + 2 + \ldots + n + n + 1) + (1 + 2 + \ldots + n)$.

In other words, $a + b = 2(1 + 2 + \ldots + n) + n + 1$.

Therefore, $a + b = 2\cdot\frac{(n+1)\cdot n}{2} + n + 1$.

As a result, $a + b = (n+1)\cdot n + (n+1)\cdot 1 = (n+1)\cdot(n+1)$.

So we have: $a + b = (n+1)^2$ and $(a-b)^2 = (n+1)^2$ and we confirmed our hypothesis using algebra.

Students enjoy playing this game. At the same time the game helps students to better understand conditions of the problem at hand and be more successful in solving it. Usually this activity works rather well with the students of different abilities. It allows students to "go as far as they can". They can calculate probabilities in numbers independently and wait for help with the generalizations. They can formulate a hypothesis and try proving it independently.

On the whole, doing this activity with students helps achieve an important goal of showing connections between different topics in mathematics. Also it demonstrates for students that algebra is a powerful tool that can be used for generalizations and confirming hypothesis in different areas including probability.