Non-verbal arithmetic

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Abstract

A teacher can choose to impart his or her knowledge of arithmetic verbally or non-verbally. During the process of learning arithmetic a student may choose between verbalizing and not verbalizing his or her knowledge. That is a matter of decision and may be part of the teaching strategy.

But all above these there is an unconscious element in the teaching-learning process as well. In the development of a child this unconscious learning may play a crucial role.

We give examples of both verbal and non-verbal learning. In the first place we will try to focus on some elements of unconscious learning of arithmetic and algebra.

Introduction

One of the biggest disadvantages of teaching mathematics the verbal way is that the structure of it becomes rigid; the borders are hard to come through.

Non-verbal, (unconscious, imagistic) learning plays a great role in mathematics education. Even at college level the visualization of our teaching can count a lot if we would like to achieve a deep understanding of the mathematical concepts. There are many studies on non-verbal learning. For now the terms "number sense", "symbol sense", "image schemata" became well known in mathematics education. (Arcavi (1994), Bergsten (1999), Dörfler (1991)). These terms refer to informal, non-verbal – basically unconscious – understanding and knowledge. Dörfler writes: "… the subjective meaning of mathematical terms has a non-verbal, non-propositional and geometric-objective component. The individual understanding of a mathematical topic possibly is best grasped as a kind of interplay between the propositional expressions and corresponding image schemata." (Dörfler 1991)

According to Sfard (1994) the role of non-propositional, imagistic thinking is furthermore prevalent at all levels of mathematical thinking. (Although the younger the children are the more we build on non-verbal experiences in the process of developing concepts.)

A non-verbal element can be part of the conscientiously planned teaching/learning process:

- when we create a new concept through examples,
- when we illustrate a concept with pictures, manipulatives,
- when we use models……

These are well known and play very important role in mathematics education. But it is not so well known and probably even more important that non-verbal learning is frequently present in the whole process without being planned or even not noticed. We are convinced that this unplanned, unnoticed learning has a great influence on the achievements of pupils. We strongly believe that this is one of the resources of the outstanding success and also of the hopeless failures of the children.

Famous researchers of mathematics education of the last century – Dienes, Pólya, Tamás Varga, Skemp, … – made a lot of work on this topic. Their results – at least in Hungary – were planted into Hungarian teaching practice only partially. We can also put it another way: at least parts of their results were planted into Hungarian teaching practice.

Here we are going to present a selection of these results.

All the following examples are present in the practice of some Hungarian teachers, but none of them are commonly used. You may as well know some of them but we still hope that we can show something new to all of you.

Examples and activities

I. Equation sign

Example 1. When a child in the school is given the problem: $5+7=\_\_\_\_+3=\_\_\_\_$.

The reaction $5+7=12+3=15$ is understandable.

What is behind that – quite common – problem?

Verbalism is clearly lacking. The teacher missed to interpret the problem properly – based on the knowledge of the children.

The cure is that we make it clear for the children what is written there – verbally and non-verbally. We have to teach them how it all has to be read and understood.

Example 2. Compare the previous problem with the following: $95+27+45=$

Let us see a quite common way of writing the "solution" of it: 95 + 27 = 122 + 45 = 167.

In the above two problems there is an interesting analogy. While the children have to solve a bunch of problems during practicing to carry out operations the role of the equation sign is not symmetric. Typically "the expression on the right hand side has to be simpler than the one on the left hand side". There are many studies which show that the teaching of arithmetic is focusing on the results of calculation rather than on its relational structural aspects. (Kieran 1990, 1992; Malara 1999) That unconscious mistake in the teaching has far reaching consequences in the further understanding of arithmetic and algebra. These routines are nonverbally fixed into their thinking and it is almost impossible to correct them verbally. These are mental pictures which have to be corrected by repainting them.

Let us see some other examples of the possible bad consequences of that kind of badly fixed "sense of symbols".

It is quite common to see children understanding that $2(a+b)=2a+2b$. In spite of it we will surly see children who have difficulties applying it the other way round: $2a+2b=2(a+b)$.

Often you have to find a whole square expression of a quadratic formula like this:

$x^2+4x+3=x^2+4x+4-4+3=x^2+4x+4-1=(x+2)^2-1$.

There is a strong psychic block in many children to make the original expression "more complicated" or longer in order to get the required form. The cure is clearly to start giving arithmetic problems of both types that is not just simplifying an expression but also complicating it. These are essential methods of higher mathematics as well.

How can we put it into practice? Let us list some exercises and activities!

II. Equivalent forms of numbers

Example 3. Every number has many names. Let us choose a number, 52, for example.

$52=4\cdot 13$; $4\cdot 13$ is a name of 52, which tells, that it is a multiple of 13.

$52=100-48$; $100-48$ is a name of 52, one that tells that it is 48 less than 100. Etc.

Example 4. Every child knows – even the laziest ones – that a number multiplied (or divided) by 1 equals the original number. $a\cdot 1=a$.

But they usually are not aware that it works in the opposite way as well – a number equals the same number multiplied (or divided) by 1. $a=a\cdot 1$.

That might sound strange, but it can come in very handy in several cases. For example when they have already learnt the method of multiplying two fractions we can ask:

What do you think of that? $\frac{7}{5}=\frac{7}{5}\cdot 1=\frac{7}{5}\cdot\frac{3}{3}=\frac{21}{15}$. So they can realize that any expansion of a fraction can be viewed as a multiplication with an "invisible 1".

Another very useful kind of exercise is when instead of "multiplying" two fractions one fraction is "multiplied apart" into two fractions. $\frac{28}{9} = \boxed{\frac{\quad}{\quad}} \cdot \boxed{\frac{\quad}{\quad}}$.

The task is to fill in the empty places with fractions which make the equality true.
We can start solving it in a whole class activity. (For example we can ask: what do you think, how many solutions are there for that problem? Make guesses. We will see whose guess is closest.)
Usually there are several phases in this class work. First they take apart the numerator and the denominator into two factors. $\frac{2}{3} \cdot \frac{14}{3} = \frac{4}{3} \cdot \frac{7}{3} = \ldots$
The next step is when they realize that 1 can be used either in the numerator either in the denominator. Then usually comes $\frac{1}{3} \cdot \frac{28}{3} = \frac{7}{9} \cdot \frac{4}{1} = \ldots$
And after a while someone hopefully may come to the idea that we can expand the original fraction – or multiply it with 1 – and this way we can get any number of solutions.
$\frac{28}{9} = \frac{28}{9} \cdot \frac{5}{5} = \frac{5}{9} \cdot \frac{28}{5} = \frac{5}{3} \cdot \frac{28}{15} = \ldots$
In weaker classes where this idea does not come, we can give them 1 as a starting number to multiply apart. Here the idea of expansion comes naturally. $1 = \frac{3}{3} = \frac{8}{8} = \frac{15}{15} = \frac{3}{5} \cdot \frac{5}{3} = \frac{1}{15} \cdot \frac{15}{1} \ldots$
This way the idea of reciprocal fraction comes quite naturally as well.
From now on the children may work individually or in groups and they can try to find as many different solutions to these kinds of problems as they can. (We can organize it in a way of competition – the winner is who, in a given time, finds the biggest number of good solutions different from everyone else's.)
After that division becomes easy: $\frac{7}{9} : \frac{3}{2} = \left(\frac{7}{9} \cdot 1\right) : \frac{3}{2} = \left(\frac{7}{9} \cdot \frac{2}{3}\right) \cdot \left(\frac{3}{2} : \frac{3}{2}\right) = \frac{7}{9} \cdot \frac{2}{3}$.
These activities are very simple to children who already knew how to multiply fractions. At the same time they give a different "dimension" to their knowledge. Using only this very simple operation, they can solve many other tasks easily, – like the division with fraction, the expansion and simplification of a fraction – if you can not only multiply fractions, but multiply them apart as well.
At some cases the experience of some children clash with a new concept and this prevents him or her in understanding it. A typical problem is subtraction of negative numbers. It is hard to take that subtracting something you can get more. That is why we need to develop this concept for a longer time and interpret it on several models. There are strong analogies here with the previous examples.
If the children are used to write numbers in different forms, add zero to them that helps a lot in understanding these models as well.

III. Order of operations

Example 5. Let us consider the previous exercise in order to discuss another type of problem.
$\boxed{95} + \boxed{27} + \boxed{45} + \boxed{9} =$ Can we change the order of the terms in an addition? Yes, we can!
$\boxed{95} + \boxed{27} - \boxed{45} + \boxed{9} =$ Can we change the order of the numbers in this case? No, we can not!
But can we change the order of the operations? The answer is not so easy. In such cases it is common to carry out the operations from left to the right. Several textbooks suggest this

procedure. At the same time, for example: $\boxed{95}\boxed{+27}\boxed{-45}\boxed{+9} = \boxed{95}\boxed{-45}\boxed{+27}\boxed{+9}$, etc. Thus the order of the operations is changed yet the result is the same.

In the original operation we had to add four terms. How can it be done and how is it done by the children? The children are taught to consider addition and subtraction, multiplication and division as operations on two operands. This is mathematically correct. But the fact is that in many cases they handle these operations unconsciously as operations on one operand. (Add something to…, subtract something from…, multiply something by…, divide something by... a given number.) Very much so in the beginning. And even later it is strongly but implicitly present in transforming algebraic expressions as well, e.g.: $5xy-3+2xy+16=7xy+13$.

Clearly, there is a contradiction here, which creates confusion in the head of many children, especially when they get to learning algebra.

What are the reasons of this ambiguity?

It seems to be a lot easier to say that you do the operations from left to right than giving a correct verbal "rule" how to change the order of operations in a more complex formula.

At the same time there are efficient non-verbal methods such as games and activities through which this can be taught efficiently.

What are the advantages of each of the two schemes?

When we talk about the commutative or the associative law we consider the basic operations as two variable operations. On the other hand to understand the transforming of algebraic expressions it is vital to be able to consider them as one variable operations. Considering multiplication and division as one variable operations can be a big help for example when carrying out operations on fractions, e.g.: $\frac{7}{3}\cdot 3 = \left(7\boxed{:3}\right)\boxed{\cdot 3} = 7\left(\boxed{\cdot 3}\boxed{:3}\right) = 7$. As a consequence, frequently, problems in algebra can be derived from not understanding these two schemes. As we claimed in the beginning, one of the disadvantages of verbal teaching is that the mathematical structure becomes unnecessarily rigid.

Here we give a counter-example to shows how meaningless verbal teaching can be: The so called FOIL method works only if you know what you are doing. Imagine if you accidentally remembered another word instead of foil. Imagine if you suddenly forgot what F, O, I, L stood for.

IV. Constant, parameter, variable

We think that a major problem in understanding algebra is that the functionalities of the letters in algebraic expressions are not clear for most of the children. That is something again we do not teach explicitly because it is very complicated to explain them verbally. So the method in many cases is that the pupils are supposed to get used to these different functionalities of the letters slowly and implicitly, through exercises.

For the most skilled children that method works, they understand the rules and make mistakes only by chance. (And even when they make a mistake they learn from it.) For them school algebra becomes a good tool to solve more complex questions.

Some of the diligent children do not understand algebra clearly but learn to use the procedures successfully. Sometimes they mix up these procedures and in more complex cases they become uncertain.

But many other children – the less skilled and/or less diligent ones – get hopelessly lost.

A simple teaching material – sets of letter-number cards - may help a lot to clarify these concepts non-verbally and very efficently for every child.

Letter-number cards have a number on one side and a letter on the other side. We can start using them already at primary school and they help the children get used to the idea, that the numbers may be substituted by letters and the other way round, letters may hide numbers behind them. Let us see two basic versions of using these tools.
We can use the letter as a parameter. Let us have two cards for example 65 on one card with *a* on the back and 35 on the other one with *b* on the back.
Without showing the numbers – we show only the back of the two cards – we tell the children that there is a number on the other side of each card and the sum of these is 100. We can put the cards on the blackboard $\boxed{a}+\boxed{b}=100$.
Then we ask the children to tell the sum if we add 5 to the number behind *a* and take away 15 from the number behind *b*. After they found out the solution we can translate it into the form: $(\boxed{a}+5)+(\boxed{b}-15)=90$.
Of course, we can do the same with other operations. For example we say that the product of numbers *c* and *d* is 3600: $\boxed{c}\cdot\boxed{d}=3600$. We multiply $\boxed{c}$ by 20 and divide $\boxed{d}$ by 6. Tell the product without knowing what is behind c and d.
These exercises are easy for the children because in the beginning they can solve them by trial and error. On one hand these exercises provide opportunities to use algebra parallel with arithmetic quite early and on the other hand they give a deeper understanding of the operations. In these cases the numbers are hidden behind the letters, but one letter hides only one number. Here we use the letters like parameters.
We can use the letters as variables as well. We need a set of number-letter cards with different numbers but with the same letter on the back, x for example. Give a card to each child and put an open sentence on the board with an empty frame for a number card. Divide the rest of the board into two and write TRUE on one side and FALSE on the other side. The children put in their number cards to the empty frame and decide whether the statement is true or false. According to it they put the number card to the side they think is appropriate.
This exercise is easy as well. It can be played from the primary school up to the secondary school as well. It depends on the open sentence and the number set we use depending on the age group and teaching purposes. Starting with the open sentence $2\cdot\boxed{}+22>50$ and giving integers from 0 to 20 on number cards we have created a problem for a group of 7-8 years of age. The open sentence $5\cdot\log p>\frac{1}{100}$ can be used for secondary level purposes. The number set can contain negative numbers, fractions, "small" powers of 10 etc.
In these kinds of activities the letters are variables.
This is another topic which is impossible to explain verbally. But well planned activities can help a great deal in understanding it. These activities make understanding easy and at the same time are consistent and mathematically correct. They do not have to be revised and corrected even in higher mathematics education. They can create a mathematically strong background of understanding.

4. Conclusion

All these examples clearly show how important the non-verbal way of learning is.
It results in a great flexibility of thinking. It provides strong connections between the elements in the structure of the student's mathematical knowledge. As we could see in some of the examples in the learning process some erroneous connections might be adopted and it is our responsibility to prevent, discover and correct these mistakes.

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