

# Discovering functional and dynamic aspects of parametric equations by creating computer animations

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## Abstract

The restriction of lessons in analytical geometry to the treatment of straight lines and planes leads to a dearth of shapes in the lessons as well as to situations where students gain only "static" conceptions of parametric equations and, in particular, fail to grasp the associated functional relationships between parameter values and points. The inclusion of computer visualizations and the creation of simpler animations by students can help incorporate frequently ignored points of view – especially the point set notions, functional relationships as well as dynamic aspects – and "give them life" when tackling parametric equations of straight lines and other geometric objects. For this purpose, by using straight lines as well as circles, spirals, helices and trajectory parabolas conceived as trajectories we can submit suggestions and map out approaches. *You can find sample files, videos and supplementary materials on the website [www.afiller.de/3dcg](http://www.afiller.de/3dcg) for all content described in this article.*

## Introduction, Problem

Parametric equations of straight lines and planes are standard content of lessons in analytical geometry. An introduction to parametric equations is usually very quickly followed by assignments on the transformation of parameters in coordinate form and vice versa, the study of positional relationships, calculations of intersecting points as well as distance and angle calculations. Two important and related aspects in analytical geometry, which could be well understood using parametric equations, have not been shown to adequate advantage:

Students acquire only a rudimentary perception of geometrical objects as point sets.

Students mostly fail to recognize functional relationships between parameter values and associated points. Recognition of this type of relationships naturally requires a view of geometrical objects as point sets but it goes much further than this inasmuch as the dependence of the location of points in space is to be understood from parameters.

To prevent the subject from developing into the concrete and objective aspect of the limited concept of parametric equations, embrace the point set notion and focus on functional relationship, there are two approaches that should merit special attention:

Students construct the points belonging to several parameter values in an equation of the kind  $P = P_0 + t \cdot \vec{a}$  and recognize that these points lie on a straight line. Parametric equations of straight lines can be introduced based on this assumption. Even the parameter-dependent description of various curves is possible in this way. Furthermore, reverse considerations and comparisons of various parameterizations of the same objects make sense in this connection.

The dynamic view of straight lines and other curves as paths can be highlighted whereby students link a concrete meaning to parameters. The interpretation of the parameter as time establishes relationships with the description of motion in physics.

## Using graphic software or CAS for the treatment of parametric equations

The use of computer graphics elements in the lesson creates possibilities where the two above-mentioned approaches to the more detailed treatment of the point set notion and dynamic aspects are implemented. The "construction" of straight lines, curves and planes from points requires only software, which the student can use to make corresponding graphical representations. The

creation of animations (videos) is especially suitable for the development of a dynamic view of parametric equations. This is where positions of objects or the position of the observer is described depending on a time parameter. An argument for the creation of animations to obtain a dynamic view of parametric equations is that most students are generally interested in making videos. Parameter-dependent descriptions are necessary when using suitable software.

To assemble a given straight line or curve as point sets using parametric equations and to be able to create parameter-dependent animations, computer algebraic systems as well as the 3D-graphic software, POV-Ray, among others<sup>1</sup>, can be used. These software packages can also be used for visualizations and calculations in other areas of the core field of analytical geometry. You can find examples using POV-Ray below.

## Straight lines and planes as point sets

### *Introduction of parametric equations by considering individual points*

To introduce the parametric equation of straight lines students can be given the following type of assignment:

Given the point  $P(0,5;1;1,5)$  and the vector  $\vec{a} = \begin{pmatrix} -2,5 \\ 1 \\ -1,5 \end{pmatrix}$ .

Represent point  $P$  as well as vector  $\vec{a}$  (as an arrow, starting from  $P$ ).

Represent the points  $P + 0,5 \cdot \vec{a}$ ,  $P + \vec{a}$ ,  $P + 1,5 \cdot \vec{a}$ ,  $P + 2 \cdot \vec{a}$  as well as  $P - 0,5 \cdot \vec{a}$ ,  $P - \vec{a}$ ,  $P - 1,5 \cdot \vec{a}$  and  $P - 2 \cdot \vec{a}$ .

View the representation from various directions.

Figure 1 shows a solution to this assignment using POV-Ray and the "anageo"-extensions for the simple representation of objects of analytical geometry available from the website [4].

The following commands are entered:

```
#declare a = <-2.5,1,-1.5>;
#declare P = <0.5,1,1.5>;
pluspunkt(P, schwarz)
vektoranpunkt(P, a, silbergrau)
punkt(P-2*a, blau_matt)
punkt(P-1,5*a, blau_matt)
...
punkt(P+1,5*a, blau_matt)
punkt(P+2*a, blau_matt)
```

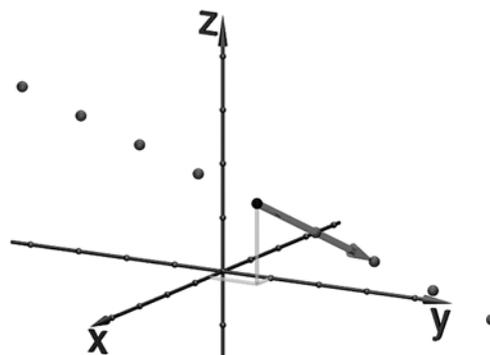


Figure 1: Points of a straight line

It becomes clear in Fig. 1 that all points represented lie on a straight line. After representing larger numbers of points by reducing parameter distances it is shown that all points of the straight line going through  $P$  whose direction is given by  $\vec{a}$  can be represented with  $t \in \mathbf{R}$  in the form  $P + t \cdot \vec{a}$ .

### *Using loops to represent large numbers of points*

To generate larger numbers of points and thereby actually make it obvious that the objects can be "fully assembled" by inserting any parameters (or parameter pairs) in the parameter equations of

<sup>1</sup> For more about the use of 3D computer graphics elements and POV-Ray, please see [1], [2] and [3]. POV-Ray is freely available from [www.povray.org](http://www.povray.org). You can find a short guide to this software's application especially for lessons in analytical geometry as well as associated files on the website [4].

straight lines, curves and planes, elementary programming constructs are used. Loops, among others, make it possible to generate such a large number of points so that the result can no longer be visibly differentiated, say, from straight lines or pathways. Students can thereby get a "plastic impression" of the point set character of geometric objects.

The graphics represented in Fig. 2 and 3 can be created in POV-Ray using loops:

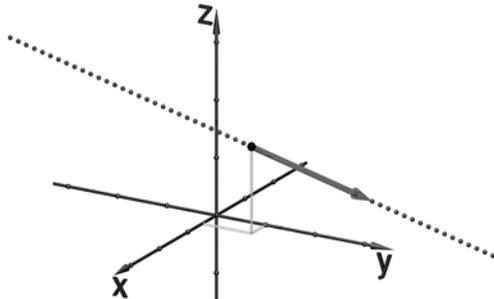


Figure 2: 60 points of a straight line

```
#declare i=-200;
#while (i <= 200)
  punkt(P+i*a/100 blau_matt)
#declare i=i+1;
#end
```

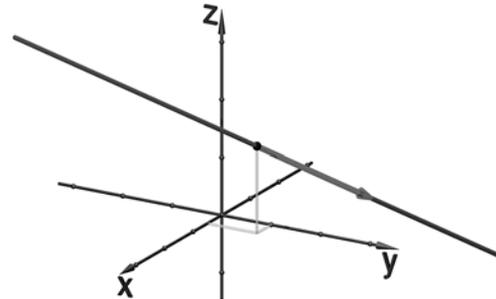


Figure 3: 400 points of a straight line

Provided the condition  $i \leq 200$  is met, this command creates a small sphere with the center point  $P + (i/100) \cdot \vec{a}$  and the value of the loop variables  $i$  is raised by 1. Any number of "points" can be represented by changing the values 200 and 100.

### Time as parameter – generating simple videos

As was already mentioned, creating videos is a very motivating task for many students. Using suitable software coordinates or other features, which are described by numbers, can be expressed based on a time parameter (in POV-Ray `clock`). It is possible, for example, to generate a video with straight-line uniform motion of a sphere by<sup>2</sup>

```
#declare a = <-2.5,1,-1.5>;
#declare P = <0.5,1,1.5>;
punkt(P+2*clock*a blau_matt)
```

The motion track becomes clearer when the "track" of the moving object is displayed simultaneously as pathway between the starting point and the respective position reached. This is possible in POV-Ray using the already described illustration of straight lines or pathways as point sets.

Parametric equations in animations acquire an aspect that does not affect the geometric shape of these objects: the velocity of motion. For example the two parametric equations below:

$$(1) \quad X(t) = P + t \cdot \vec{a} \quad \text{and}$$

$$(2) \quad X(t) = P + t^2 \cdot \vec{a}$$

(with  $t \in \mathbf{R}^+$ ) describe the same half-line.

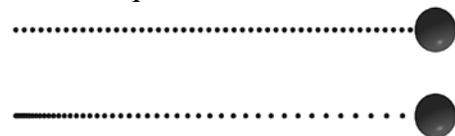


Figure 4

<sup>2</sup> For technical aspects of the creation of videos using POV-Ray, we refer you to the help section of the program or a brief guide available on the website [4] (Downloads).

If these parametric equations are used to generate animations, then (1) yields a uniform and (2) a constantly accelerated motion on these half-lines. This can be recognized in Figure 4 through the distances of the points; the same amount of time elapses between two adjacent points.

### Vector parametric equations – the oblique projection

If vector descriptions and procedures come to the fore in the lessons, one can delve deeper into the above-mentioned aspect of the velocity of motions. Animations allow you to establish links to the physics lessons, work out functional aspects by considering various functions  $f(t)$ , which replace the time parameter as well as create simple simulations. An example is the oblique projection. This can be interpreted as motion assembled from uniform motion and constantly accelerated motion. By adding a linear component in  $t$  and the acceleration vector multiplied with  $t^2$  to the equation

$$\vec{x} = \vec{x}_0 + \vec{v} \cdot t + \frac{1}{2} \vec{g} \cdot t^2$$

of the oblique projection, we get the trajectory parabola. A corresponding animation can be generated in POV-Ray by means of the following commands (see Figure 5).

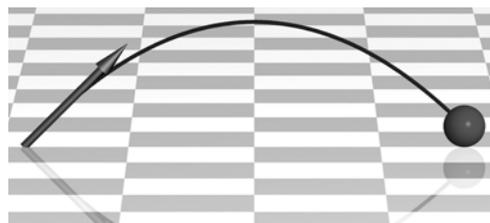


Figure 5: Trajectory parabola

```
#declare x0 = <-2.5,0,0>;
#declare v0 = <5,5,0>;
#declare g = <0,-10,0>;
sphere { x0 + v0*clock + g/2*clock*clock 0.25 }
```

### Parametric equations of circles and several other curves

The representation of geometric objects given by parametric equations by means of point sets as well as the creation of animations through the time-dependent description of coordinates is not limited to linear objects (straight lines and planes). On the contrary, considerations and representations that are significantly more interesting and more appealing in terms of shape are possible using curves (as well as surfaces). Circles are well suited as starting points for the description of curves through parametric equations.

#### Parametric equations of circles

Students from secondary level I are already familiar with the most important foundation of parametric equations of circles in the plane – the customary sinus and cosine function on the unit circle with the designations used in Figure 6 as

$$\sin a = y_\alpha ; \quad \cos a = x_\alpha .$$

A generalization on circles in center location with any radius  $r$  is easily possible from which the parametric equation

$$x(\alpha) = r \cdot \cos \alpha ; \quad y(\alpha) = r \cdot \sin \alpha \quad \text{with } \alpha \in [0; 2\pi)$$

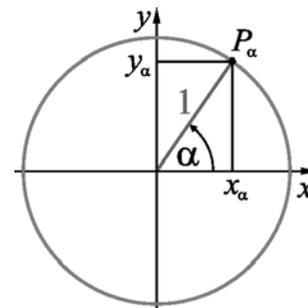


Figure 6

can be derived of a circle of the plane whose center point lies in the origin of coordinates.<sup>3</sup> If different quantities have to be animated, it then makes sense in terms of clarity to standardize the interval of the animation parameter (time) and to write the above parametric equation in the form

<sup>3</sup> By adding center point coordinates this parametric equation can be generalized on any circles in the plane:

$x(\alpha) = r \cdot \cos \alpha + x_M$  ,  $y(\alpha) = r \cdot \sin \alpha + y_M$  .

$$x(t) = r \cdot \cos(2\pi \cdot t) ; \quad y(t) = r \cdot \sin(2\pi \cdot t) \quad \text{with } t \in [0;1].$$

Parametric equations of circles which lie in space on coordinate planes or planes parallel to them arise by representing one of three coordinates as constants, for example,  $z(t) = h$ . With these considerations, students can create animations of circular motions. In POV-Ray, for example, the command

```
sphere { < 10*cos(2*pi*clock), 0, 10*sin(2*pi*clock) > 1 }
```

creates the animation of a sphere on a circular trajectory (see Figure 7). Since the representation of motion trajectories used often makes sense to obtain an overview of the process of animation, the "track" of the sphere can also be depicted so that when looking at the animation it is obvious which path the object has covered. The procedure here corresponds to the generation of a multitude of small spheres already described for straight lines (see Figure 2).

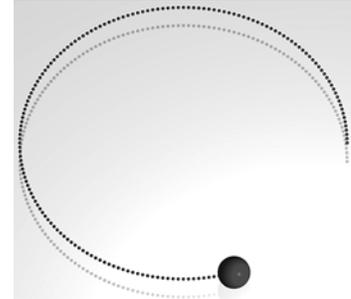


Figure 7

### Camera animations

If instead of a geometric object the position of the "camera" from which the scene is viewed is described as time-dependent, we get an animation where the view of all objects of a scene changes. For example, with

```
camera{ location < r*cos(2*pi*clock), 4, r*sin(2*pi*clock) >
        angle 12 look_at <0,0,0> }
```

a camera flight can be simulated in POV-Ray on a circular path where the camera remains directed at the origin of coordinates.<sup>4</sup> With regard to the necessary mathematical considerations, it does not matter whether students create the animation of an object moving on a trajectory or a camera animation where the view of an entire scene changes. From experience it is known that the latter is more interesting for most students. However, to make motion curves visible, it is recommended to not only create camera animations but also to animate visible objects.

### Variations of circles: spirals and helices

After the treatment of parameter equations of circles it stands to reason to describe "cognate" curves as parameters by making suitable changes to them. The following questions, which students will ask in connection with camera animations, can be tackled:

1. How can the camera circle around an object and simultaneously approach it?
2. In a circular motion, how can the camera simultaneously change its height so that objects can be viewed from different heights?

Of course, both questions can also be formulated so that they pertain to the course of curves. In order to realize the first property students can (at least with assistance) work out that the constant  $r$ , which was used for the radius of the circle previously under consideration, is replaced by a function  $r(t)$  of the temporally changing parameter  $t$ , e. g. by  $r \cdot (1 - t)$ , if the distance to the center point during the course of the animation should decrease from  $r$  to 0 (for  $t \in [0;1]$ ). This notion yields the *parametric description of an Archimedes' spiral*:

<sup>4</sup> You can find several examples of camera animations on the website [4].

$$\begin{aligned}
 x(t) &= r \cdot (1 - t) \cdot \cos(2\pi \cdot t) \\
 y(t) &= r \cdot (1 - t) \cdot \sin(2\pi \cdot t) \quad t \in [0;1] \\
 z(t) &= h .
 \end{aligned}$$

On this basis, objects or the camera can move along an Archimedes' spiral (for which the commands described above should be varied). The shape of the spiral is better visible when two revolutions are run and this can be done by replacing  $(2\pi \cdot t)$  with  $(4\pi \cdot t)$  in the trigonometric terms of the parametric equation (see Fig. 8).

In the discussion of question 2 above, students may easily recognize that for the time-dependent alteration of the "height" the previously constantly held third coordinate has to be replaced with a function of the parameter. If a linear function is selected (in the simplest case  $y = t$  or  $z = t$  – depending on which coordinate was held constant in the description of circles), then the circle equation yields the equation of a helix, e. g:

$$\begin{aligned}
 x(t) &= r \cdot \cos(4\pi \cdot t) \\
 y(t) &= t \quad t \in [0;1] \\
 z(t) &= r \cdot \sin(4\pi \cdot t) .
 \end{aligned}$$

By combining the two considerations which led from the circle to the spiral or to the helix (parameter-dependent descriptions of the radius and the "height" in the original parametric equation of a circle) we get using linear functions in  $t$  a conical spiral with parametric equation of the form

$$\begin{aligned}
 x(t) &= r \cdot (1 - t) \cdot \cos(4\pi \cdot t) \\
 y(t) &= t \quad t \in [0;1] \\
 z(t) &= r \cdot (1 - t) \cdot \sin(4\pi \cdot t) .
 \end{aligned}$$

Figure 10: Conical spiral

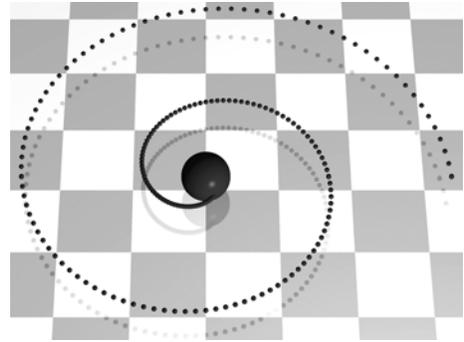


Figure 8: Archimedes' spiral

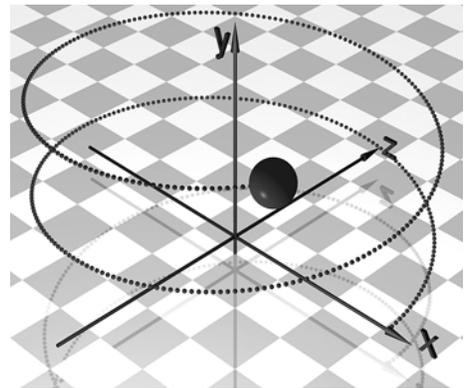
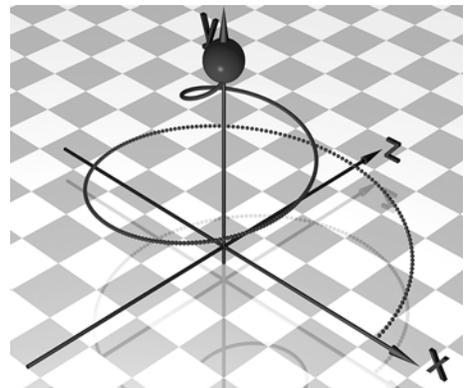


Figure 9: Helix



Other variations of the previously considered curves emerge by using nonlinear function terms in  $t$  for the height or the radius. One can then formulate the task of changing the parametric equation of the helix so that its points depart at first very slowly and then more quickly from those of the originally considered circle. Variations of the Archimedes' spiral are likewise possible. Multiplying the trigonometric terms in the parametric equations with different factors also allows you to create elliptical orbits as well as orbits on "elliptical spirals" and "elliptical helices".

### Summary

The above examples show that with simple elementary mathematical tools which tie up with the secondary level I lessons it is possible to make models of interesting curves and create animations based on them. However, the time needed to do this must not be underestimated.

Even for seemingly simple functional considerations which, say, lead from circles to spirals or helices, students in my seminars needed quite a lot of time. As they found these considerations interesting, they fell into the joy of experimentation and were willing to spend a relatively large amount of free time on them. This could also apply to many school students since the creation of interesting videos, as already mentioned several times, is a motivating endeavor for youth everywhere.

Through variations on parametric equations of curves and the resulting description of "new" curves, we get rich possibilities for functional considerations where students – based on qualitative descriptions of desired curve progressions – ponder the function terms through which these can arise and check their considerations using software.

Vector calculus is not needed to describe geometric spirals as well as spirals that are especially interesting with regard to the creation of camera animations. It makes more sense here to work with coordinate descriptions. However, interesting considerations of motion trajectories are also possible based on vector line equations. From the standpoint of teaching mathematics, I consider the above-mentioned subject matter especially worthy of mention because the activity triggers sophisticated considerations about functional relationships which often inadequately appear in the current dominant treatment of parametric equations in lessons.

### **Literature**

- [1] Filler, A.: Einbeziehung von Elementen der 3D-Computergrafik in den Mathematikunterricht der Sekundarstufe II (Habilitationsschrift). e-doc Dokumenten- und Publikationsserver der Humboldt-Universität zu Berlin: <http://dochost.rz.hu-berlin.de> (2007).
- [2] Filler, A.; Rieper, F.: 3D Computer Graphics and Analytic Geometry in Mathematics Education in Grammar Schools. In: Rogerson, A. (Editor): The Mathematics Education into the 21st Century Project, Proceedings of the International Conference "The Future of Mathematics Education", Ciechocinek, Poland, 2004, p. 39-44.
- [3] Krumpel, N.: Creating Three-Dimensional Scenes. In: Mathematics Teacher 98 (2005), 6, p. 394-398.

### **Website with materials about this article**

- [4] 3D computer graphics and the mathematics behind it: <http://www.afiller.de/3dgc>