A viscosity equation for minimizers of a class of very degenerate elliptic functionals

Giulio Ciraolo*

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Abstract

We consider the functional

$$J(v) = \int_\Omega [f(|\nabla v|) - v]dx,$$

where Ω is a bounded domain and $f : [0, +\infty) \to \mathbb{R}$ is a convex function vanishing for $s \in [0, \sigma]$, with $\sigma > 0$. We prove that a minimizer u of J satisfies an equation of the form

$$\min(F(\nabla u, D^2u), |\nabla u| - \sigma) = 0$$

in the viscosity sense.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega$ of class $C^{2,\alpha}$. We consider the variational problem

$$\inf\{J(v) : \ v \in W_0^{1,\infty}(\Omega)\}, \quad \text{where} \quad J(v) = \int\limits_\Omega [f(|\nabla v|) - v]dx; \tag{1}$$

here, the function $f : [0, +\infty) \to \mathbb{R}$ is convex, monotone, nondecreasing and we assume that there exists $\sigma > 0$ such that

$$f \in C^1([0, +\infty)) \cap C^3((\sigma, +\infty)); \tag{2a}$$

$$f(0) = 0 \text{ and } \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty; \tag{2b}$$

$$f'(s) = 0 \text{ for every } 0 \leq s \leq \sigma; \tag{2c}$$

$$f''(s) > 0 \text{ for } s > \sigma. \tag{2d}$$

Functionals of this kind occur in the study of complex-valued solutions of the *eikonal* equation (see [13]–[16]), as well as in the study of problems linked to traffic congestion (see [2]) and in variational problems which are relaxations of

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*Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Italy, (`g.ciraolo@math.unipa.it`).

non-convex ones (see [4]). We have in mind the following two main examples of a function f:

$$f(s) = \begin{cases} 0, & 0 \le s \le 1 \\ \frac{1}{2}[s\sqrt{s^2-1} - \log(s + \sqrt{s^2-1})], & s > 1, \end{cases} \tag{3}$$

which arises from the study of complex-valued solutions of the eikonal equation, and

$$f(s) = \begin{cases} \frac{1}{q}(s-1)^q, & s > 1, \\ 0, & 0 \le s \le 1, \end{cases} \tag{4}$$

$q > 1$, which is linked to traffic congestion problems.

Since f vanishes in the interval $[0, \sigma]$, problem (1) is strongly degenerate and, as far as we know, few studies have been done. Besides the papers cited before, we mention [1] and [17] where regularity issues were tackled.

In this paper, we shall prove that the minimizer u of (1) satisfies an equation of the form

$$\min\Big(F(\nabla u, D^2 u), |\nabla u| - \sigma\Big) = 0 \tag{5}$$

in the viscosity sense (see Theorems 3.2 and 3.3 for the meaning of F).

Our strategy is to approximate J by a sequence of less degenerating functionals so that the minimizers of the corresponding variational problems converge uniformly to u; this is done in Section 2. Then, the machinery of viscosity equations applies and, in Section 3, we prove that u satisfies (5). To prove Theorems 3.2 and 3.3, which are our main result, we make use of techniques which have been used in the context of the ∞-Laplace operator (see for instance [3],[10],[11]).

2 Preliminary results

We start by recalling some well-known facts. Since Ω is bounded and $\partial\Omega$ is of class $C^{2,\alpha}$, then the following *uniform exterior sphere condition* holds: there exists $\rho > 0$ such that for every $x_0 \in \partial\Omega$ there exists a ball $B_\rho(y)$ of radius ρ centered at $y = y(x_0) \in \mathbb{R}^N \setminus \overline{\Omega}$ such that $\overline{B_\rho(y)} \cap \overline{\Omega} = \overline{B_\rho(y)} \cap \partial\Omega$ and $x_0 \in \overline{B_\rho(y)}$.

Notice that, since f satisfies (2a)-(2d), the functional J is differentiable and a critical point u of J satisfies the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{f'(|\nabla u|)}{|\nabla u|}\nabla u\right) = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

in the weak sense, i.e.

$$\int\limits_\Omega \frac{f'(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\phi dx = \int\limits_\Omega \phi dx, \quad \text{for every } \phi \in C_0^1(\Omega). \tag{7}$$

It will be useful in the sequel to have at hand the solution of (6) when Ω is the ball of given radius R (centered at the origin): it is given by

$$u_R(x) = \int\limits_{|x|}^{R} g'\Big(\frac{s}{N}\Big) ds, \tag{8}$$

where

$$g(t) = \sup\{st - f(s) : \ s \geq 0\}$$

is the Fenchel conjugate of f.

It is clear that, when $\sigma = 0$, (1) has a unique solution, since f is strictly convex. When $\sigma > 0$, the uniqueness of a minimizer for (1) is proved in [7].

In this section we shall approximate the functional J by a sequence of strictly convex functionals

$$J_n(v) = \int_\Omega [f_n(|\nabla v|) - v] dx, \tag{9}$$

$n \in \mathbb{N}$, which are less degenerating than J (see Proposition 2.3 for the assumptions on the functions f_n) and prove some uniform bounds for the minimizers u_n of

$$\inf\{J_n(v) : \ v \in W_0^{1,\infty}(\Omega)\}. \tag{10}$$

Notice that, if $f_n \in C^1([0,+\infty)) \cap C^3((0,+\infty))$ satisfies (2b) and it is such that $f_n'(0) = 0$ and $f_n''(s) > 0$ for $s > 0$, then the minimizer u_n of (9) is unique and satisfies

$$\int_\Omega \frac{f_n'(|\nabla u_n|)}{|\nabla u_n|} \nabla u_n \cdot \nabla \phi dx = \int_\Omega \phi dx, \quad \text{for every } \phi \in C_0^1(\Omega). \tag{11}$$

Let u_n and v_n be a subsolution and a supersolutions of (11), respectively. Then, the following *weak comparison principle* holds: if $u_n \leq v_n$ on $\partial\Omega$ then $u_n \leq v_n$ in $\overline{\Omega}$ (see Lemma 3.7 in [8]).

It will be useful to define the following P-function (see [8]):

$$P_n(x) = \Phi_n(|\nabla u_n(x)|) + \frac{2}{N} u_n(x), \quad x \in \overline{\Omega}, \tag{12}$$

where

$$\Phi_n(t) = 2 \int_0^t s f_n''(s) ds. \tag{13}$$

To avoid heavy notations, in Lemmas 2.1 and 2.2 we drop the dependence on n.

Lemma 2.1. *Let $f \in C^1([0,+\infty)) \cap C^3((0,+\infty))$ be such that $f'(0) = 0$ and $f''(s) > 0$ for $s > 0$ and let u be the solution of (1). Then, $|\nabla u|$ attains its maximum on the boundary of Ω and the following estimate holds:*

$$|\nabla u(x)| \leq M, \quad x \in \overline{\Omega}, \tag{14}$$

with

$$M = g'\left(\frac{\rho}{N-1}\left(e^{\frac{(N-1)R^*}{\rho}} - 1\right)\right), \tag{15}$$

where g is the Fenchel conjugate of f, $R^* = \sup\{|x-y| : \ x, y \in \partial\Omega\}$ and ρ is the radius of the uniform exterior sphere.

Furthermore,

$$0 \leq u(x) \leq \min\left(\int_0^{R^*} g'\Big(\frac{s}{N}\Big) ds, \ \frac{N}{2}\Phi(M)\right) \quad x \in \overline{\Omega}. \tag{16}$$

Proof. Since u is a minimizer of J, it is easy to show that $u \geq 0$. Being R^* the diameter of Ω, there exist a ball of radius R^* that contains Ω (we can assume that such ball is centered at the origin). Since $u_{R^*}(x) \geq 0$ for $x \in \partial\Omega$, the weak comparison principle implies that

$$u(x) \leq u_{R^*}(x) \quad \text{for every } x \in \overline{\Omega}. \tag{17}$$

From $u_{R^*}(x) \leq u_{R^*}(0),\ x \in B_{R^*}$ and from (8), we have

$$u(x) \leq \int_0^{R^*} g'\left(\frac{s}{N}\right) ds, \tag{18}$$

for every $x \in \overline{\Omega}$.

Now, we consider the P-function given by (12). As proved in Lemma 3.2 in [8], P attains its maximum on the boundary of Ω and thus

$$P(x) \leq \max_{\partial\Omega} P = \max_{\partial\Omega} \Phi(|\nabla u|), \quad x \in \overline{\Omega}.$$

Since Φ is strictly increasing, then we get

$$\max_{\overline{\Omega}} |\nabla u(x)| = \max_{\partial\Omega} |\nabla u(x)|, \tag{19}$$

i.e. $|\nabla u|$ attains its maximum on the boundary of Ω.

Following [9], we construct a barrier function for u which will give us an upper bound for $|\nabla u|$ on the boundary of Ω. Let $x_0 \in \partial\Omega$ be fixed and let $B_\rho(y(x_0))$ the ball in the exterior sphere condition. Set

$$\delta(x) = \operatorname{dist}(x, \partial B_\rho(y(x_0))), x \in \Omega,$$

and let $w = \psi(\delta(x))$ be a function depending only on the distance from $\partial B_\rho(y(x_0))$; we have

$$\operatorname{div}\left\{f'(|\nabla w|)\frac{\nabla w}{|\nabla w|}\right\} = \psi''(\delta(x))f''(\psi'(\delta(x))) + f'(\psi'(\delta(x)))\Delta\delta(x). \tag{20}$$

Since

$$|\Delta\delta(x)| = \frac{N-1}{|x-y|} \leq \frac{N-1}{\rho},$$

from (20) we obtain

$$\operatorname{div}\left\{f'(|\nabla w|)\frac{\nabla w}{|\nabla w|}\right\}+1 \leq \psi''(\delta(x))f''(\psi'(\delta(x)))+\frac{N-1}{\rho}f'(\psi'(\delta(x)))+1. \tag{21}$$

By choosing

$$\psi(t) = \int_0^t g'\left(\frac{\rho}{N-1}\left(e^{\frac{N-1}{\rho}(R^*-s)} - 1\right)\right) ds,$$

the right hand side of (21) vanishes and thus w is a supersolution of (7). Since $\psi'(t) > 0$ for $t > 0$, then $w(x) \geq 0$ on $\partial\Omega$ and the weak comparison principle yields $u(x) \leq w(x)$ in $\overline{\Omega}$. Since $x_0 \in \partial\Omega$ is arbitrary, we obtain

$$|\nabla u(x)| \leq g'\left(\frac{\rho}{N-1}\left(e^{\frac{(N-1)R^*}{\rho}} - 1\right)\right),$$

for any $x \in \partial\Omega$. According to (19) the same estimate holds in the whole of Ω and (14) holds.

Notice that from (12)

$$u(x) \leq \frac{N}{2} P(x), \quad x \in \overline{\Omega};$$

since P attains its maximum on the boundary of Ω and from (14) we have that

$$u(x) \leq \frac{N}{2} \Phi(M)$$

which, together with (18), implies (16). $\square$

We denote by $H_{\partial\Omega}(x)$ the mean curvature of $\partial\Omega$ at the point $x \in \partial\Omega$ and set

$$H^*_{\partial\Omega} = \min_{x \in \partial\Omega} H_{\partial\Omega}(x).$$

In the following Lemma, we give a further bound for u and $|\nabla u|$ in the case that the mean curvature of $\partial\Omega$ attains a positive minimum. Notice that, when Ω is a ball, (23) is optimal.

Lemma 2.2. *Let f be as in Lemma 2.1 and assume that $H^*_{\partial\Omega} > 0$. Then,*

$$u(x) \leq \frac{N}{2} \Phi\Big(g'\Big(\frac{1}{N H^*_{\partial\Omega}} \Big) \Big) \tag{22}$$

and

$$|\nabla u(x)| \leq g'\bigg(\frac{1}{N H^*_{\partial\Omega}} \bigg) \tag{23}$$

for every $x \in \overline{\Omega}$, where Φ is given by (13) *and g is the Fenchel conjugate of f.*

Proof. Since $|\nabla u| > 0$ on $\partial\Omega$ (see Lemma 2.7 in [7]), equation (7) is nondegenerate in a neighborhood of $\partial\Omega$; from standard elliptic regularity theory (see [18] and [9]), we know that $u \in C^{2,\alpha}(\overline{\Omega} \setminus \{x : \nabla u \neq 0\})$ for some $\alpha \in (0,1)$, and then (7) can be written pointwise on $\partial\Omega$ as

$$f''(|u_\nu(x)|) u_{\nu\nu}(x) - (N-1) f'(u_\nu(x)) H_{\partial\Omega}(x) = -1,$$

where ν denotes the exterior unit normal to $\partial\Omega$. From Lemma 3.3 in [8], we know that

$$N f'(|\nabla u(x)|) H_{\partial\Omega}(x) \leq 1,$$

for every $x \in \partial\Omega$, and then

$$|\nabla u(x)| \leq g'\Big(\frac{1}{N H^*_{\partial\Omega}} \Big),$$

for every $x \in \partial\Omega$. Since P (given by (12)) attains its maximum on $\partial\Omega$ and from $u = 0$ on $\partial\Omega$, we have that

$$P(x) \leq \Phi\Big(g'\Big(\frac{1}{N H^*_{\partial\Omega}} \Big) \Big) \tag{24}$$

for every $x \in \Omega$. From (12) and (24) we conclude. $\square$

Proposition 2.3. *Let* $(f_n)_{n\in\mathbb{N}}$ *be such that:*

(i) $f_n \in C^1([0,+\infty)) \cap C^3((0,+\infty))$*;*

(ii) f_n *converges uniformly to* f *on the compact sets contained in* $[0;+\infty)$*;*

(iii) $f_n'(0) = 0$*, the functions* f_n' *decrease to* f' *in* $[0,+\infty)$ *and* f_n' *converges uniformly to* f' *on the compact sets contained in* $[0,+\infty)$

(iv) $f_n''(t) > 0$ *for* $t > 0$*.*

Let u *(resp.* u_n*) be the solution of* (1) *for* J *(resp. of* (10) *for* J_n*). Then*

(a) u_n *is a minimizing sequence for* J *and* $J_n(u_n) \to J(u)$*;*

(b) u_n *and* ∇u_n *are uniformly bounded and (up to a subsequence)* $(u_n)_{n\in\mathbb{N}}$ *tends to* u *in the sup norm topology and* u *satisfies estimates* (14) *and* (16) *almost everywhere in* Ω*.*

Proof. Since $J_n \to J$ uniformly (a) is standard. Since the sequence $(f_n')_{n\in\mathbb{N}}$ is decreasing in n, then g_n' is increasing in n and converges pointwise to g' (here, we denote by g and g_n the Fenchel conjugates of f and f_n, respectively). Thus, $g_n(t) \le g(t)$ and $g_n'(t) \le g'(t)$ for every $t \in [0,+\infty)$ and (b) follows by Lemma 2.1 and an application of the Ascoli-Arzelà's theorem. ☐

3 Viscosity Euler-Lagrange equation

In this section we prove that the solution u of (1) satisfies an equation of the form (5) in the viscosity sense. Firstly, we do it for $f \in C^2((0,+\infty)) \cup C^3((\sigma,+\infty))$ and then we deal with the case that f is not twice differentiable at $s = \sigma$.

Consider a sequence of approximating functions $\{f_n\}_{n\in\mathbb{N}}$ satisfying $(i)-$ $-(iv)$ in Proposition 2.3. The minimizer u_n for (10) satisfies

$$-\operatorname{div}\frac{f_n'(|\nabla u_n|)}{|\nabla u_n|} = 1,$$

in weak sense. Assume for a moment that u_n is regular enough so that we can differentiate, then u_n satisfies

$$-\frac{|\nabla u_n| f_n''(|\nabla u_n|) - f_n'(|\nabla u_n|)}{|\nabla u_n|^3}\Delta_\infty u_n - \frac{f_n'(|\nabla u_n|)}{|\nabla u_n|}\Delta u_n = 1.$$

This equation is fully nonlinear and it makes sense to define and study its viscosity solutions.

Let $P \in \mathbb{R}^N$ and $X \in \mathcal{S}^N$, where $\mathcal{S}^N$ is the space of real-valued $N \times N$ symmetric matrices. Consider the function

$$F_n(P,X) := \begin{cases} -\dfrac{|P| f_n''(|P|) - f_n'(|P|)}{|P|^3}\langle P, X\cdot P\rangle - \dfrac{f_n'(|P|)}{|P|}\operatorname{tr}(X) - 1, & P \neq 0, \\ -1, & P = 0. \end{cases} \tag{25}$$

Notice that, if

$$\lim_{s\to 0^+} \frac{sf_n''(s) - f_n'(s)}{s^3} = 0, \qquad \text{and} \qquad \lim_{s\to 0^+} \frac{f_n'(s)}{s} = 0, \tag{26}$$

then F_n is continuous. For future use, we shall assume that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is such that

$$\lim_{n\to +\infty} \frac{sf_n''(s) - f_n'(s)}{s^3} = 0, \quad \text{and} \quad \lim_{n\to +\infty} \frac{f_n'(s)}{s} = 0, \tag{27}$$

for any $0 < s < \sigma$.

Definition. An upper semicontinuous function u defined in Ω is a *viscosity subsolution* of

$$F_n(\nabla v, D^2 v) = 0, \tag{28}$$

$x \in \Omega$, if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$ if $x \neq x_0$, then

$$F_n(\nabla\phi(x_0), D^2\phi(x_0)) \leq 0.$$

A lower semicontinuous function u defined in Ω is a *viscosity supersolution* of (28) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$ if $x \neq x_0$, then

$$F_n(\nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

Finally, $u \in C^0(\Omega)$ is a *viscosity solution* of (28) if it is both a viscosity subsolution and a viscosity supersolution of (28).

Lemma 3.1. *Let u_n be the minimizer of J_n, where $f_n \in C^1([0,+\infty)) \cup C^3((0,+\infty))$ satisfies* (26) *and is such that $f_n''(s) > 0$ for $s > 0$. Then u_n is a viscosity solution of* (28).

Proof. We present the details for the case of supersolutions. Let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ be such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$ for $x \neq x_0$. We have to show that

$$-\frac{|\nabla\phi(x_0)| f_n''(|\nabla\phi(x_0)|) - f_n'(|\nabla\phi(x_0)|)}{|\nabla\phi(x_0)|^3} \Delta_\infty \phi(x_0) - \frac{f_n'(|\nabla\phi(x_0)|)}{|\nabla\phi(x_0)|} \Delta\phi(x_0) - 1 \geq 0.$$

By contradiction, suppose that this is not the case. By continuity, there exists $r > 0$ small enough such that

$$-\frac{|\nabla\phi(x)| f_n''(|\nabla\phi(x)|) - f_n'(|\nabla\phi(x)|)}{|\nabla\phi(x)|^3} \Delta_\infty \phi(x) - \frac{f_n'(|\nabla\phi(x)|)}{|\nabla\phi(x)|} \Delta\phi(x) < 1,$$

for any $|x - x_0| < r$. Let $m = \inf\{u_n(x) - \phi(x) : \; |x - x_0| = r\}$ and set $\eta = \phi + \frac{1}{2}m$. Since $m > 0$ then $\eta < u$ on $\partial B_r(x_0)$, $\eta(x_0) > u_n(x_0)$ and

$$-\frac{|\nabla\eta(x)| f_n''(|\nabla\eta(x)|) - f_n'(|\nabla\eta(x)|)}{|\nabla\eta(x)|^3} \Delta_\infty \eta(x) - \frac{f_n'(|\nabla\eta(x)|)}{|\nabla\eta(x)|} \Delta\eta(x) < 1,$$

for any $|x - x_0| < r$. By multiplying by $(\eta - u_n)^+$, integrating in $B_r(x_0)$ and using an integration by parts, we have

$$\int\limits_{\{\eta > u_n\}} f_n'(|\nabla \eta|)\frac{\nabla \eta}{|\nabla \eta|} \cdot \nabla(\eta - u_n)dx < \int\limits_{\{\eta > u_n\}} (\eta - u_n)dx. \tag{29}$$

The function $(\eta - u_n)^+$ extended to zero outside $B_r(x_0)$ can be used as a test function in (11):

$$\int\limits_{\{\eta > u_n\}} f_n'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|} \cdot \nabla(\eta - u_n)dx = \int\limits_{\{\eta > u_n\}} (\eta - u_n)dx. \tag{30}$$

Subtracting (30) from (29) we have

$$\int\limits_{\{\eta > u_n\}} \left[f_n'(|\nabla \eta|)\frac{\nabla \eta}{|\nabla \eta|} - f_n'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|} \right] \cdot \nabla(\eta - u_n)dx < 0. \tag{31}$$

Since

$$\begin{aligned} \left[f_n'(|\nabla \eta|)\frac{\nabla \eta}{|\nabla \eta|} - f_n'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|} \right] \cdot \nabla(\eta - u_n) = \\ = f_n'(|\nabla \eta|)|\nabla \eta| + f_n'(|\nabla u_n|)|\nabla u_n| + \\ - f_n'(|\nabla \eta|)\frac{\nabla \eta}{|\nabla \eta|} \cdot \nabla u_n - f_n'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|} \cdot \nabla \eta, \end{aligned} \tag{32}$$

Cauchy-Schwarz inequality and the convexity of f_n yield

$$\begin{aligned} \left[f_n'(|\nabla \eta|)\frac{\nabla \eta}{|\nabla \eta|} - f_n'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|} \right] \cdot \nabla(\eta - u_n) \geq \\ \geq \Big(f_n'(|\nabla \eta|) - f_n'(|\nabla u_n|) \Big) \Big(|\nabla \eta| - |\nabla u_n| \Big) \geq 0, \end{aligned}$$

which gives the desired contradiction on account of (31). □

Theorem 3.2. *Let u be the minimizer of* (1) *and assume that f satisfies* (2) *and $f \in C^2((0, +\infty))$. Then, u is a viscosity solution of*

$$\min\left(-\frac{|\nabla u| f''(|\nabla u|) - f'(|\nabla u|)}{|\nabla u|^3}\Delta_\infty u - \frac{f'(|\nabla u|)}{|\nabla u|}\Delta u - 1,\ |\nabla u| - \sigma \right) = 0. \tag{33}$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be an approximating sequence of the function f satisfying (i)–(iv) in Proposition 2.3, (26), (27) and such that f_n'' converges to f'' uniformly on the compact sets contained in $(0, +\infty)$. From Proposition 2.3, we can assume that u_n converges to u uniformly as n tends to infinity. By using a standard argument from the theory of viscosity solutions (see [5] and [12]), we shall prove that u is a viscosity supersolution and subsolution of (33). The two proofs are not symmetric and we prove firstly that u is a viscosity supersolution and then that it is also a viscosity subsolution.

Assume ϕ is a smooth function touching u from below at $\hat{x} \in \Omega$, i.e., $u(\hat{x}) = \phi(\hat{x})$ and $u(x) > \phi(x)$ for any $x \neq \hat{x}$. Since u_n is a viscosity solution of (28) and u_n converges uniformly to u, there exist $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ such that

(i) for any $x \in \Omega$, $u_n(x_n) - \phi(x_n) \geq u_n(x) - \phi(x)$;

(ii) x_n tends to $\hat{x}$ as n tends to infinity.

Being u_n a viscosity supersolution of (28), we can conclude that

$$F_n(\nabla\phi(x_n), D^2\phi(x_n)) \geq 0.$$

Let assume that $|\nabla\phi(\hat{x})| < \sigma$: by taking the limit as $n \to \infty$ and from (27) we get a contradiction. Thus, we may exclude that $|\nabla\phi(\hat{x})| < \sigma$.

Now assume that $|\nabla\phi(\hat{x})| \geq \sigma$. Again, we take the limit as $n \to \infty$ and get that both

$$-\frac{|\nabla\phi(\hat{x})|f''(|\nabla\phi(\hat{x})|) - f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|^3}\Delta_\infty\phi(\hat{x}) - \frac{f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|}\Delta\phi(\hat{x}) - 1 \geq 0,$$

and

$$|\nabla\phi(\hat{x})| - \sigma \geq 0$$

are satisfied. Hence the claim is proven.

Now, we prove that u is a viscosity subsolution of (33). Assume ϕ is a smooth function such that $u(\hat{x}) = \phi(\hat{x})$ and $u(x) < \phi(x)$ for any $x \neq \hat{x}$. As claimed at the previous case, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

(i) $u_n(x_n) - \phi(x_n) \leq u_n(x) - \phi(x)$;

(ii) x_n tends to $\hat{x}$ as n tends to infinity.

If $|\nabla\phi(x_n)| \leq \sigma$, then obviously

$$\min\Big(-\frac{|\nabla\phi(\hat{x})|f''(|\nabla\phi(\hat{x})|) - f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|^3}\Delta_\infty\phi(\hat{x}) - \frac{f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|}\Delta\phi(\hat{x}) - 1, \\ |\nabla\phi(\hat{x})| - \sigma\Big) \leq 0$$

holds. In case $|\nabla\phi(x_n)| > \sigma$, from the fact that u_n is a viscosity subsolution of (28) we have $F_n(\nabla\phi(x_n), D^2\phi(x_n)) \leq 0$. Since f_n and its first and second derivatives converges uniformly as $n \to +\infty$, by taking the limit leads to

$$-\frac{|\nabla\phi(\hat{x})|f''(|\nabla\phi(\hat{x})|) - f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|^3}\Delta_\infty\phi(\hat{x}) - \frac{f'(|\nabla\phi(\hat{x})|)}{|\nabla\phi(\hat{x})|}\Delta\phi(\hat{x}) - 1 \leq 0,$$

which completes the proof. □

Now, we assume that f satisfies (2) and $f \notin C^2((0,+\infty))$ (i.e. f is not twice differentiable at $s = \sigma$). Since it is not possible to choose f_n such that f_n'' converges uniformly to f'', we can not proceed as in Theorem 3.2.

In the following, we shall assume that $\lim\limits_{s\to\sigma^+} f''(s)$ and $\lim\limits_{s\to+\infty} f''(s)$ exist; since f is not twice differentiable at $s = \sigma$ then $\lim\limits_{s\to\sigma^+} f''(s) \neq 0$. Let

$$a(s) = \begin{cases} \dfrac{f'(s)}{sf''(s)}, & s > \sigma, \\ 0, & 0 \leq s \leq \sigma, \end{cases} \tag{34}$$

and

$$b(s) = \begin{cases} \dfrac{s^2}{f''(s)}, & s > \sigma, \\ \sigma^2 \displaystyle\lim_{s\to\sigma^+} \dfrac{1}{f''(s)}, & s \leq \sigma. \end{cases} \tag{35}$$

Notice that $a, b \in C^0([0, +\infty))$.

Theorem 3.3. *Let u be the minimizer of* (1)*. Then u is a viscosity solution of*

$$\min\big(- [1 - a(|\nabla u|)]\Delta_\infty u - |\nabla u|^2 a(|\nabla u|)\Delta u - b(|\nabla u|), |\nabla u(x)| - \sigma\big) = 0, \tag{36}$$

where a and b are defined by (34) *and* (35)*, respectively.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence satisfying (i)–(iv) in Proposition 2.3, (26) and (27). Let

$$a_n(s) = \frac{f_n'(s)}{s f_n''(s)},$$

$s > 0$. We consider f_n such that a_n converges uniformly to a in the compact sets contained in $[\sigma, +\infty)$. Furthermore, the approximating sequence $\{f_n\}_{n\in\mathbb{N}}$ will be such that the functions $s^2(f_n''(s))^{-1}$ converge uniformly to $b(s)$ in the compact sets contained in $(\sigma, +\infty)$.

The proof splits in two parts. First we prove that u is a viscosity supersolution, then that it is also a subsolution. The earlier is slightly more involved and we deal with it first.

The function u is a viscosity supersolution of (36).
Assume ϕ is a smooth function touching u from below at $\hat{x} \in \Omega$, i.e., $u(\hat{x}) = \phi(\hat{x})$ and $u(x) > \phi(x)$ for any $x \neq \hat{x}$. Recall that u_n is a viscosity solution of (28) and that, from Proposition 2.3, we can assume that u_n converges uniformly to u as n tends to infinity. Thus, there exist $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ such that for any $x \in \Omega$, $u_n(x_n) - \phi(x_n) \geq u_n(x) - \phi(x)$ and x_n tends to $\hat{x}$ as n tends to infinity.

Since u_n is a viscosity supersolution of (28), we can conclude that

$$F_n(\nabla\phi(x_n), D^2\phi(x_n)) \geq 0. \tag{37}$$

Let assume that $|\nabla\phi(\hat{x})| < \sigma$: by taking the limit in (37) as $n \to \infty$ and thanks to (27) we get a contradiction. Thus, we may exclude that $|\nabla\phi(\hat{x})| < \sigma$.

Now assume that $|\nabla\phi(\hat{x})| \geq \sigma$. Hence, we may assume that $\nabla\phi(x_n) \neq 0$ (at least for n large). By multiplying both sides of (37) by

$$\frac{|\nabla\phi(x_n)|^2}{f_n''(|\nabla\phi(x_n)|)},$$

we have

$$\begin{aligned} -\Big[1 - \frac{f_n'(|\nabla\phi(x_n)|)}{|\nabla\phi(x_n)| f_n''(|\nabla\phi(x_n)|)}\Big]\Delta_\infty\phi(x_n) + \\ - \frac{|\nabla\phi(x_n)| f_n'(|\nabla\phi(x_n)|)}{f_n''(|\nabla\phi(x_n)|)}\Delta\phi(x_n) - \frac{|\nabla\phi(x_n)|^2}{f_n''(|\nabla\phi(x_n)|)} \geq 0. \end{aligned}$$

By taking the limit as $n \to \infty$ we get that both

$$-[1 - a(|\nabla\phi(\hat{x})|)]\Delta_\infty\phi(\hat{x}) - |\nabla\phi(\hat{x})|^2 a(|\nabla\phi(\hat{x})|)\Delta\phi(\hat{x}) - b(|\nabla\phi(\hat{x})|) \geq 0,$$

and

$$|\nabla\phi(\hat{x})| - \sigma \geq 0$$

are satisfied. Hence the claim is proven.

The function u is a viscosity subsolution of (36).
Assume ϕ is a smooth function such that $u(\hat{x}) = \phi(\hat{x})$ and $u(x) < \phi(x)$ for any $x \neq \hat{x}$. Thus, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $u_n(x_n) - \phi(x_n) \leq u_n(x) - \phi(x)$ and x_n tends to $\hat{x}$ as n tends to infinity.

If $|\nabla\phi(x_n)| \leq \sigma$, then obviously

$$\min\Big(- [1 - a(|\nabla\phi(\hat{x})|)]\Delta_\infty\phi(\hat{x}) - |\nabla\phi(\hat{x})|^2 a(|\nabla\phi(\hat{x})|)\Delta\phi(\hat{x}) - b(|\nabla\phi(\hat{x})|), \\ |\nabla\phi(\hat{x})| - \sigma\Big) \leq 0$$

holds. In case $|\nabla\phi(x_n)| > \sigma$, from the fact that u_n is a viscosity subsolution of (28), we can conclude (carrying out the same algebraic manipulation showed at the previous step)

$$- \Big[1 - \frac{f_n'(|\nabla\phi(x_n)|)}{|\nabla\phi(x_n)|f_n''(|\nabla\phi(x_n)|)}\Big]\Delta_\infty\phi(x_n) + \\ - \frac{|\nabla\phi(x_n)|f_n'(|\nabla\phi(x_n)|)}{f_n''(|\nabla\phi(x_n)|)}\Delta\phi(x_n) - \frac{|\nabla\phi(x_n)|^2}{f_n''(|\nabla\phi(x_n)|)} \leq 0.$$

Taking the limit leads to the desired conclusion. □

Remark 3.4. Sequences of $(f_n)_{n\in\mathbb{N}}$ that satisfy the assumptions required in the proofs of Theorems 3.2 and 3.3 can be constructed in various fashions. A convenient way is to modify f' only in the interval $(0, \sigma + \varepsilon)$, with $\varepsilon > 0$ small enough. By assuming $\sigma = 1$, the following is an example:

$$f_\varepsilon'(s) = \begin{cases} f'(1+\varepsilon)\Big[2\Big(\dfrac{s}{1+\varepsilon}\Big)^{p_\varepsilon} - \Big(\dfrac{s}{1+\varepsilon}\Big)^{q_\varepsilon}\Big], & 0 \leq s \leq 1+\varepsilon, \\ f'(s), & s > 1+\varepsilon, \end{cases}$$

where

$$p_\varepsilon = \frac{(1+\varepsilon)f''(1+\varepsilon)}{f'(1+\varepsilon)}\left[\frac{3}{2} + \frac{1}{2}\sqrt{2 - \frac{f'''(1+\varepsilon)f'(1+\varepsilon)}{f''(1+\varepsilon)^2} - 2\frac{f'(1+\varepsilon)}{(1+\varepsilon)f''(1+\varepsilon)}}\right],$$

and

$$q_\varepsilon = \frac{(1+\varepsilon)f''(1+\varepsilon)}{f'(1+\varepsilon)}\left[1 + \sqrt{2 - \frac{f'''(1+\varepsilon)f'(1+\varepsilon)}{f''(1+\varepsilon)^2} - 2\frac{f'(1+\varepsilon)}{(1+\varepsilon)f''(1+\varepsilon)}}\right].$$

Example 3.5. *Let f be given by* (3). *Then*

$$f'(s) = \sqrt{(s^2-1)^+}$$

and a and b in (34) *and* (35) *read as*

$$a(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1 - \dfrac{1}{s^2}, & s > 1, \end{cases}$$

and

$$b(s) = s\sqrt{(s^2-1)^+}.$$

We notice that, working as in the proof of Theorem 3.3, we can prove that u satisfies other equations in the viscosity sense which are of the same form as (5). For instance, let $a^* > 0$ be such that $a(s) < a^*$ for any $s \geq 0$; then it can be shown that u satisfies

$$\min\Big\{ -\Big[1+\frac{1-a^*}{a^*-a(|\nabla u|)}\Big]\Delta_\infty u - \frac{|\nabla u|^2 a(|\nabla u|)}{a^*-a(|\nabla u|)}\Delta u - \frac{b(|\nabla u|)}{a^*-a(|\nabla u|)}, \\ |\nabla u(x)| - \sigma\Big\} = 0, \tag{38}$$

in the viscosity sense. If f is given by (3), we can choose $a^* = 1$ and (38) reads as

$$\min\big(-\Delta_\infty u - |\nabla u|^2(|\nabla u|^2-1)^+\Delta u - |\nabla u|^3\sqrt{(|\nabla u|^2-1)^+}, |\nabla u(x)| - 1\big) = 0.$$

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