WAVE PROPAGATION IN A 3-D OPTICAL WAVEGUIDE II. NUMERICAL EXAMPLES.

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Abstract

In this paper we apply the results we obtained in [2] to deduce explicit analytic formulas for the Green's function of the electromagnetic field propagating inside an optical fiber for the particular cases of a step-index fiber and a coaxial waveguide. We use the obtained expressions to calculate numerically the electromagnetic fields and represent them graphically.

1 Introduction

In [2] we defined a transform which enabled us to study the wave propagation in a cylindrical optical fiber. We considered the whole $\mathbb{R}^3$ space and, as model equation, we used the *Helmholtz equation*

$$\Delta u + k^2 n(r)^2 u = f, \tag{1}$$

where r is the radial coordinate in a cylindrical coordinate system (r, ϑ, z), in which z runs along the fiber axis. The positive number k is called the *wavenumber*, and the function f represents a source of energy. It is assumed that the optical fiber has a core, outside which the index of refraction $n(r)$ is constant. Thus, for some $R > 0$ and $n_{cl} \geq 1$, $n(r) = n_{cl}$ for $r \geq R$. For $0 \leq r < R$, $n(r)$ is required to be a positive, bounded, and integrable function.

The main result of [2] was the construction of a representation formula (see (21)) for a solution u of (1) satisfying suitable radiation conditions. Our results generalized a similar formula obtained by Magnanini and Santosa [5] in the two-dimensional case. For the reader's convenience, in Section 2 we summarize the formulas obtained in [2] which are used in our calculations.

In this paper we study two particular cases of (1) in which the Green's function of (1) can be calculated explicitly. We consider a *step-index fiber*, which is characterized by the index of refraction

$$n(r) = \begin{cases} n_{co}, & 0 < r < R, \\ n_{cl}, & r \geq R, \end{cases} \tag{2}$$

and the so-called *coaxial dielectric cable*, where

$$n(r) = \begin{cases} n_{cl}, & 0 < r < a, \\ n_{co}, & a \leq r < R, \\ n_{cl}, & r \geq R, \end{cases} \tag{3}$$

with $n_{co} > n_{cl}$.

In these two cases it is relatively easy to derive from (21) explicit formulas. We present these calculations in Sections 3 (step-index fiber) and 4 (coaxial cable). We are then able to calculate numerically the obtained fields and represent them graphically.

2 A summary of the results

In cylindrical coordinates Eq. (1) becomes

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + k^2 n(r)^2 u = f. \tag{4}$$

Consider the homogeneous version of (4), that is, with $f = 0$ on the right-hand side. We look for a solution of this equation in separated variables

$$u(r, \vartheta, z) = e^{i\beta k z} e^{im\vartheta} \sqrt{r} w(r),$$

with $\beta \in \mathbb{C}$ and $m \in \mathbb{Z}$. Then $w(r)$ must satisfy the differential equation

$$w'' + \left\{ l - q(r) - \frac{m^2 - 1/4}{r^2} \right\} w = 0, \quad r \in (0, \infty), \tag{5}$$

where

$$d^2 = k^2(n_0^2 - n_{cl}^2), \quad l = k^2(n_0^2 - \beta^2), \quad q(r) = k^2[n_0^2 - n(r)^2], \tag{6}$$

with n_0 being the maximum of $n(r)$. The function $q(r)$ is non-negative, with $q(r) = d^2$ for $r \geq R$. We will view (5) as an eigenvalue problem in $l \in \mathbb{C}$ and will call it the *associated eigenvalue problem* to (4).

The next lemma gives us a solution of Eq. (5) which is "well behaved" as $r \to 0$.

Lemma 2.1 *There exists a solution $j_m(r, l)$ ($r > 0$, $l \in \mathbb{C}$, $m \in \mathbb{Z}$) of (5) such that*

$$\lim_{r \to 0} \frac{j_m(r, l)}{r^{|m|+1/2}} = 1, \quad \lim_{r \to 0} \frac{j'_m(r, l)}{(|m| + 1/2)\, r^{|m|-1/2}} = 1. \tag{7}$$

For a non-decreasing function $\chi : \mathbb{R} \to \mathbb{R}$ we denote by $L^2(\chi)$ the space of all functions $G : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |G(\lambda)|^2 \, d\chi(\lambda) < \infty.$$

The next theorem gives us a transform in terms of the eigenfunctions $j_m(r, l)$ of the eigenvalue problem (5).

Theorem 2.2 *For every $m \in \mathbb{Z}$ there exists a non-decreasing function $\chi_m : \mathbb{R} \to \mathbb{R}$ such that if $g \in L^2(0, \infty)$, then the integral*

$$G_m(\lambda) = \int_0^\infty j_m(r, \lambda) g(r) \, dr \tag{8}$$

is convergent in $L^2(\chi_m)$, in the sense that there exists $G_m \in L^2(\chi_m)$ such that $G_m^{cd} \to G_m$ in $L^2(\chi_m)$ as $c \to 0$ and $d \to \infty$, where

$$G_m^{cd}(\lambda) = \int_c^d j_m(r, \lambda) g(r) \, dr, \quad 0 < c < d < \infty.$$

The equality

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r, \lambda) G_m(\lambda) \, d\chi_m(\lambda) \tag{9}$$

holds, in the sense that $g^{\sigma\tau} \to g$ in $L^2(0, \infty)$ as $\tau \to -\infty$, $\sigma \to \infty$, where

$$g^{\tau\sigma}(r) = \frac{1}{\pi} \int_\tau^\sigma j_m(r, \lambda) G_m(\lambda) \, d\chi_m(\lambda), \quad -\infty < \sigma < \tau < \infty.$$

We have the Parseval identity

$$\int_0^\infty |g(r)|^2 \, dr = \frac{1}{\pi} \int_{-\infty}^{\infty} |G_m(\lambda)|^2 \, d\chi_m(\lambda).$$

We will need some definitions and notation to state the next theorems.
Let $m \in \mathbb{Z}$ and J_m, Y_m be the Bessel's functions of the first and second kind respectively. We use the following notations:

$$a_m(r, \lambda) = \sqrt{r} J_m\big(\sqrt{\lambda - d^2}\, r\big), \quad b_m(r, \lambda) = \sqrt{r} Y_m\big(\sqrt{\lambda - d^2}\, r\big), \quad \lambda > d^2. \tag{10}$$

It can be shown that

$$j_m(r, \lambda) = c_m(\lambda) a_m(r, \lambda) + d_m(\lambda) b_m(r, \lambda), \quad r \geq R, \tag{11}$$

where

$$c_m(\lambda) = \frac{\pi}{2} \{ b_m'(R, \lambda) j_m(R, \lambda) - j_m'(R, \lambda) b_m(R, \lambda) \}, \tag{12a}$$

and

$$d_m(\lambda) = -\frac{\pi}{2} \{ a_m'(R, \lambda) j_m(R, \lambda) - j_m'(R, \lambda) a_m(R, \lambda) \}. \tag{12b}$$

Theorem 2.3 *The function χ_m is identically zero for $\lambda \in (-\infty, 0]$, is piecewise constant for $\lambda \in (0, d^2]$ where it has a finite number of discontinuities, and is continuous for $\lambda \in (d^2, \infty)$. Let $0 < \lambda_1^m < \cdots < \lambda_{P_m}^m \le d^2$ $(P_m \ge 0)$ be the points where χ_m is discontinuous and $r_1^m, \ldots, r_{P_m}^m$ be the corresponding jumps. Then*

$$r_k^m = \pi \left\{ \int_0^\infty j_m(r, \lambda_k^m)^2 \, dr \right\}^{-1}, \; k = 1, \ldots, P_m, \tag{13}$$

and

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}, \quad \lambda \in (d^2, \infty). \tag{14}$$

The inverse transform (9) *becomes*

$$g(r) = \frac{1}{\pi} \sum_{k=1}^{P_m} r_k^m j_m(r, \lambda_k^m) G_m(\lambda_k^m) + \frac{1}{2} \int_{d^2}^\infty \frac{j_m(r,\lambda) G_m(\lambda)}{c_m(\lambda)^2 + d_m(\lambda)^2} \, d\lambda.$$

Now we define

$$k_m(r, \lambda) = \sqrt{r} K_m(\sqrt{d^2 - \lambda}\, r), \quad \lambda < d^2, r \ge R, \tag{15}$$

where K_m is the modified Bessel function of the second kind. This function will decay exponentially as $r \to \infty$.

The next theorem will characterize the discontinuity points of the function χ_m.

Theorem 2.4 *A number $\lambda \in (0, d^2]$ is a discontinuity point of χ_m if and only if either $\lambda < d^2$ and*

$$\frac{j_m'(R, \lambda)}{j_m(R, \lambda)} = \frac{k_m'(R, \lambda)}{k_m(R, \lambda)}, \tag{16}$$

which implies

$$j_m(r, \lambda) = \frac{j_m(R, \lambda)}{k_m(R, \lambda)} k_m(r, \lambda), \quad r \ge R, \tag{17}$$

or $\lambda = d^2$, $|m| \ge 2$, and

$$\frac{j_m'(R, \lambda)}{j_m(R, \lambda)} = -\frac{|m| - 1/2}{R},$$

which implies

$$j_m(r, \lambda) = \frac{j_m(R, \lambda)}{R^{1/2 - |m|}} r^{1/2 - |m|}, \quad r \ge R. \tag{18}$$

The next theorem will state that, under certain conditions, the solution of the Helmholtz equation (1), which in the cylindrical coordinate system (r, ϑ, z) is written as (4), is unique. We will find a representation for it in terms of the source $f(r, \vartheta, z)$, the eigenfunctions $j_m(r, \lambda)$ of Eq. (5) satisfying Lemma 2.1, and the measures $d\chi_m(\lambda)$ defined in Theorem 2.2.

We will assume that the source f is continuous and has compact support. Then we give the conditions on the solution u for (4) which guarantee its uniqueness. First, we impose the condition that $u \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Second, we suppose that for all $m \in \mathbb{Z}$, $z \in \mathbb{R}$ the following equality holds:

$$\lim_{r \to \infty} \left[j_m(r, \lambda) \frac{\partial \{\sqrt{r} u_m(r, z)\}}{\partial r} - \frac{\partial j_m(r, \lambda)}{\partial r} \{\sqrt{r} u_m(r, z)\} \right] = 0, \tag{19}$$

with the functions $u_m(r,z)$ being the Fourier coefficients from the Fourier series

$$u(r,\vartheta,z)=\sum_{m\in\mathbb{Z}} e^{im\vartheta}u_m(r,z).$$

We denote by $U_m(\lambda,z)$ the transform of the function $r\to\sqrt{r}u_m(r,z)$ given by (8):

$$U_m(\lambda,z)=\int_0^\infty j_m(\rho,\lambda)\sqrt{\rho}u_m(\rho,z)\,d\rho.$$

The third requirement is the *radiation condition*

$$\begin{cases}\lim\limits_{|z|\to\infty}\left[\dfrac{\partial U_m(\lambda,z)}{\partial|z|}-i\sqrt{k^2n_0^2-\lambda}\,U_m(\lambda,z)\right]=0, & \text{for } \lambda\leqslant k^2n_0^2 \\ & \text{and } d\chi_m(\lambda)\neq 0, \\ \lim\limits_{|z|\to\infty} U_m(\lambda,z)=0, & \text{for } \lambda>k^2n_0^2.\end{cases}\tag{20}$$

These conditions are physically motivated. The condition on f says that the source must be finite in size. Eq. (19) signifies a fast decay of the electromagnetic field intensity as $r\to\infty$. The radiation condition (20) means that as $|z|\to\infty$, the electromagnetic field can be separated in two parts. The first part, for $\lambda\leq k^2n_0^2$, is oscillatory, and then condition (20) is just a version of the Sommerfeld radiation condition, which guarantees that field is *outgoing*, that is, it is being radiated away. The second part implies that the energy of the field decays.

Theorem 2.5 *With the above assumptions, the solution of* (4) *can be represented as*

$$u(r,\vartheta,z)=\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} G(r,\rho;\vartheta,t;z,\zeta)f(\rho,t,\zeta)\,\rho\,dt\,d\rho\,d\zeta,$$

where

$$G(r,\rho;\vartheta,t;z,\zeta)$$
$$=\frac{1}{2\pi^2}\frac{1}{\sqrt{r\rho}}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{+\infty}\frac{e^{i|z-\zeta|\sqrt{k^2n_0^2-\lambda}}}{2i\sqrt{k^2n_0^2-\lambda}}e^{im(\vartheta-t)}\,j_m(\rho,\lambda)\,j_m(r,\lambda)\,d\chi_m(\lambda),$$
$$0<r,\rho;\ 0\leq\vartheta,t\leq 2\pi;\ z,\zeta\in\mathbb{R},\tag{21}$$

and χ_m is the non-decreasing function defined in Theorem 2.2.

In the next two sections we will calculate explicit analytic formulas for the Green's function (21).

3 Step-index fibers

For a *step-index fiber*, the index of refraction satisfies (2). Then the function $q(r)$ defined by (6) becomes

$$q(r)=\begin{cases}0, & 0\leqslant r<R,\\ d^2, & r\geqslant R.\end{cases}$$

Let us find a formula for the function $j_m(r,\lambda)$ defined by Lemma 2.1. For $0 \le r < R$, Eq. (5) takes the form

$$w'' + \left\{\lambda - \frac{m^2 - 1/4}{r^2}\right\} w = 0.$$

Two linearly independent solutions of this equation are $\sqrt{r}J_m(r\sqrt{\lambda})$ and $\sqrt{r}Y_m(r\sqrt{\lambda})$. From Eq. (7) we deduce that $j_m(r,\lambda)$ must be a multiple of $\sqrt{r}J_m(r\sqrt{\lambda})$. By using the formulas

$$J_{-m}(z) = (-1)^m J_m(z), \quad m \in \mathbb{Z},\ z \in \mathbb{C}$$

and

$$J_m(z) \sim \frac{1}{m!}\left(\frac{z}{2}\right)^m, \quad m \ge 0,\ z \in \mathbb{C},\ z \to 0$$

(identities (9.1.5) and (9.1.7) from [1]), we find that

$$j_m(r,\lambda) = \alpha_m(\lambda)\sqrt{r}J_m(r\sqrt{\lambda}),\ 0 \leqslant r < R, \tag{22}$$

with

$$\alpha_m(\lambda) = (-1)^{(|m|-m)/2} 2^{|m|} |m|!\, \lambda^{-|m|/2}, \quad m \in \mathbb{Z}.$$

Let us calculate $j_m(r,\lambda)$ for $r > R$ and the measure $d\chi_m(\lambda)$. It will be convenient to represent $j_m(r,\lambda)$ as

$$j_m(r,\lambda) = \alpha_m(\lambda)\widetilde{j}_m(r,\lambda).$$

The measure $d\chi_m$ has a continuous part, for $\lambda > d^2$, and a discrete part, for $\lambda \le d^2$. We will treat these cases separately.

Let $\lambda > d^2$. Denote $Q = \sqrt{\lambda - d^2}$. Define the operator

$$\mathcal{V}_x[f,g](\lambda) = x[\sqrt{\lambda} f(Qx) g'(x\sqrt{\lambda}) - Q f'(Qx) g(x\sqrt{\lambda})].$$

By using (10) - (12) we deduce

$$\widetilde{j}_m(r,\lambda) = \begin{cases} \sqrt{r}\ J_m(r\sqrt{\lambda}), & 0 \leqslant r < R, \\ \\ \frac{\pi}{2}\sqrt{r}\ \big[\beta_m(\lambda) J_m(Qr) + \gamma_m(\lambda) Y_m(Qr)\big], & r \ge R, \end{cases} \tag{23}$$

with

$$\begin{array}{rl} \beta_m(\lambda) = & -\mathcal{V}_R[Y_m, J_m](\lambda), \\ \gamma_m(\lambda) = & \mathcal{V}_R[J_m, J_m](\lambda). \end{array} \tag{24}$$

From (11) and (14) we get

$$d\chi_m(\lambda) = \frac{2}{\pi \alpha_m(\lambda)^2} \frac{d\lambda}{\beta_m(\lambda)^2 + \gamma_m(\lambda)^2}, \quad \lambda \in (d^2, \infty).$$

Let now consider the case $\lambda \le d^2$. Let $\lambda = \lambda_k^m$ in this interval be a discontinuity point of χ_m and r_k^m be the jump of χ_m at λ_k^m. It will be convenient to represent r_k^m as

$$r_k^m = \frac{\pi}{\alpha_m(\lambda)^2}\widetilde{r}_k^m.$$

Also, in this case we will denote

$$\widetilde{j}_m^g(r,\lambda) = \widetilde{j}_m(r,\lambda).$$

The motivation for this is to emphasize that for $\lambda \le d^2$ we are dealing with *guided modes*, as opposed to *radiation modes* for $\lambda > d^2$. We need to consider separately the sub-cases $\lambda < d^2$ and $\lambda = d^2$.

Let $\lambda < d^2$. Denote $Q_0 = \sqrt{d^2-\lambda}$. The condition for discontinuity of χ_m at λ becomes in view of (15), (16) and (22)

$$\frac{\sqrt{\lambda}J_m'(\sqrt{\lambda}R)}{J_m(\sqrt{\lambda}R)} = \frac{Q_0 K_m'(Q_0 R)}{K_m(Q_0 R)}. \tag{25}$$

Using (17) we obtain

$$\widetilde{j}_m^g(r,\lambda) = \begin{cases} \sqrt{r}\ J_m(r\sqrt{\lambda}), & 0 \leqslant r < R, \\ \\ \dfrac{J_m(\sqrt{\lambda}R)}{K_m(Q_0R)}\sqrt{r}\ K_m(Q_0 r), & r \ge R. \end{cases} \tag{26}$$

Now we will calculate the jump of χ_m at λ_k^m. From (13) we have

$$\frac{\pi}{r_k^m} = \int_0^\infty j_m(r,\lambda_k^m)^2\, dr.$$

According to [6], see pp. 87-88, the following indefinite integral formulas hold:

$$\int x Z_m(\alpha x)^2\, dx = \frac{x^2}{2}\left\{\left(1-\frac{m^2}{\alpha^2 x^2}\right) Z_m(\alpha x)^2 + Z_m'(\alpha x)^2\right\}\ ,$$

where $Z_m(x)$ is a solution of the Bessel equation, and

$$\int x W_m(\alpha x)^2\, dx = \frac{x^2}{2}\left\{\left(1+\frac{m^2}{\alpha^2 x^2}\right) W_m(\alpha x)^2 - W_m'(\alpha x)^2\right\}\ ,$$

with $W_m(x)$ a solution of the modified Bessel equation.

Then we can use (25) and (26) to deduce

$$\widetilde{r}_k^m = \frac{2\lambda_k^m(d^2-\lambda_k^m)}{d^2}\cdot\frac{1}{\lambda_k^m R^2 J_m'(\sqrt{\lambda_k^m}\,R)^2 - m^2\, J_m(\sqrt{\lambda_k^m}\,R)^2}.$$

Lastly, let $\lambda = d^2$. From Theorem 2.4 and formulas (15) and (22) we obtain that the condition for discontinuity of χ_m at λ is

$$\frac{dJ_m'(Rd)}{J_m(Rd)} = -\frac{|m|}{R}. \tag{27}$$

From (18) we get

$$\widetilde{j}_m^g(r,\lambda) = \begin{cases} \sqrt{r}\ J_m(rd), & 0 \leqslant r < R, \\ \\ R^{|m|} J_m(dR)\, r^{1/2-|m|}, & r \ge R. \end{cases} \tag{28}$$

In a similar fashion to the case $\lambda < d^2$ we find that, if $\chi_m(\lambda)$ is discontinuous at $\lambda = d^2$, then $|m| \ge 2$ and

$$\widetilde{r}_k^m = \frac{2\,(|m|-1)}{|m| R^2 J_m(dR)^2}.$$

We can now define a transform in terms of $\widetilde{\jmath}_m(\rho,\lambda)$,

$$\widetilde{G}_m(\lambda) = \int_0^\infty \widetilde{\jmath}_m(\rho,\lambda)\, g(\rho)\, d\rho. \tag{29}$$

For $\lambda = \lambda_k^m \le d^2$ a discontinuity point for χ_m we will denote $\widetilde{G}_m(\lambda)$ by $\widetilde{G}^g_m(\lambda_k^m)$, again with the purpose of emphasizing that we are in the guided mode case.

We have the following corollary of Theorem 2.3.

Corollary 3.1 *Let $\beta_m(\lambda)$ and $\gamma_m(\lambda)$ be defined by (24), and denote by $\{\lambda_k^m\}_{k=1,..,P_m(R,d)} \subset (0,d^2]$ the points of discontinuity of χ_m. The following* inversion formula *holds for all $g \in L^2(0,\infty)$ and $m \in \mathbb{Z}$,*

$$g(r) = \sum_{k=1}^{P_m(R,d)} \widetilde{r}_k^m\, \widetilde{\jmath}^g_m(r,\lambda_k^m)\, \widetilde{G}^g_m(\lambda_k^m) + \frac{2}{\pi^2}\int_{d^2}^{+\infty} \frac{\widetilde{\jmath}_m(r,\lambda)\,\widetilde{G}_m(\lambda)}{\beta_m(\lambda)^2+\gamma_m(\lambda)^2}\, d\lambda,$$

where

$$\widetilde{r}_k^m = \begin{cases} \dfrac{2\lambda_k^m(d^2-\lambda_k^m)}{d^2\left[\lambda_k^m R^2 J'_m(\sqrt{\lambda_k^m}\,R)^2 - m^2\, J_m(\sqrt{\lambda_k^m}\,R)^2\right]}, & \text{if } \lambda_k^m < d^2, \\[2ex] \dfrac{2\,(|m|-1)}{|m| R^2 J_m(dR)^2}, & \text{if } \lambda_k^m = d^2, \end{cases} \tag{30}$$

and $\widetilde{\jmath}_m(r,\lambda)$, $\widetilde{\jmath}^g_m(r,\lambda_k^m)$, $\widetilde{G}_m(\lambda)$, $\widetilde{G}^g_m(\lambda_k^m)$ are defined by (23), (26) together with (28), and (29) respectively.

At this point it is easy to write the Green's function for a step-index fiber.

Corollary 3.2 *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a continuous function with compact support and let $\widetilde{\chi}_m : \mathbb{R} \to \mathbb{R}$ be a function such that*

$$\langle \mathrm{d}\widetilde{\chi}_m, h\rangle = \sum_{k=1}^{P_m(R,d)} \widetilde{r}_k^m\, h(\lambda_k^m) + \frac{2}{\pi^2}\int_{d^2}^{+\infty} \frac{h(\lambda)}{\beta_m(\lambda)^2+\gamma_m(\lambda)^2}\, d\lambda$$

for all $h \in C_0^\infty(\mathbb{R})$, where $\lambda_k^m \in (0,d^2]$, $k = 1 \dots P_m(R,d)$, $m \in \mathbb{Z}$, are the solutions of (25) or (27), the jumps $\widetilde{r}_k^m$ are defined by (30), and $\beta_m(\lambda)$ and $\gamma_m(\lambda)$ are given by (24). Define

$$G(r,\vartheta,z;\rho,\eta,\zeta)$$
$$= \frac{1}{2\pi^2}\frac{1}{\sqrt{r\rho}} \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{i|z-\zeta|\sqrt{k^2n_{co}^2-\lambda}}}{2i\sqrt{k^2 n_{co}^2-\lambda}}\, \widetilde{\jmath}_m(\rho,\lambda)\, \widetilde{\jmath}_m(r,\lambda) e^{im(\vartheta-\eta)}\, \mathrm{d}\widetilde{\chi}_m(\lambda),$$
$$0 < r,\rho;\ 0 \le \vartheta, t \le 2\pi;\ z,\zeta \in \mathbb{R}. \tag{31}$$

If $u \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is a weak solution of the Helmholtz equation which satisfies (19) and (20) of Theorem 2.5, then

$$u(r,\vartheta,z) = \int_0^\infty \int_0^{2\pi} \int_{-\infty}^{\infty} G(r,\vartheta,z,\rho,\eta,\zeta)\ f(\rho,\eta,\zeta)\ \rho\ d\rho\, d\eta\, d\zeta.$$

3.1 Numerical examples.

Using the explicit formulas deduced for the step-index fiber we are able to show some numerical examples.

Figures 1-5 show the real part of Green's function (31). In these examples, the wavenumber is $k = 10$, the indexes of refraction of the core and cladding are respectively $n_{co} = 2$, $n_{cl} = 1$, and the fiber radius is $R = 0.2$.

Under these assumptions, Eq. (25) has three solutions in λ, with m being $-1, 0, 1$. We have $\lambda = 84.87$ for $m = 0$ and $\lambda = 205.65$ for $m = -1, 1$. These form the discrete spectrum, and correspond to the guided modes, represented by the finite sum in the Green's function.

The continuous spectrum, $\lambda > d^2$, is divided into two parts. The interval $d^2 < \lambda < k^2 n_{co}^2$ corresponds to G^r, the radiating part of Green's function, while for $\lambda > k^2 n_{co}^2$ we have G^e, the evanescent part of Green's function. To simplify the computations it has been useful to rewrite G^r and G^e. By substituting in (31) $\beta_r = \sqrt{k^2 n_{co}^2 - \lambda}/k$ in G^r and $\beta_e = \sqrt{\lambda - k^2 n_{co}^2}/k$ in G^e, we find:

$$G^r(r, \vartheta, z, \rho, \eta, \zeta) = \\ -\frac{ik}{2\pi\sqrt{r\rho}} \sum_{m\in\mathbb{Z}} e^{im(\vartheta-\eta)} \int_0^{n_{cl}} e^{i|z-\zeta|k\beta_r} \Big[\widetilde{j}_m(\rho,\lambda)\widetilde{j}_m(r,\lambda)\widetilde{\chi}_m(\lambda)\Big]_{\lambda=k^2(n_{co}^2-\beta_r^2)} d\beta_r, \quad (32)$$

$$G^e(r, \vartheta, z, \rho, \eta, \zeta) = \\ -\frac{k}{2\pi\sqrt{r\rho}} \sum_{m\in\mathbb{Z}} e^{im(\vartheta-\eta)} \int_0^{\infty} e^{-|z-\zeta|k\beta_e} \Big[\widetilde{j}_m(\rho,\lambda)\widetilde{j}_m(r,\lambda)\widetilde{\chi}_m(\lambda)\Big]_{\lambda=k^2(n_{co}^2+\beta_e^2)} d\beta_e. \quad (33)$$

In the sum over $\mathbb{Z}$ only the terms with $|m| \leq 10$ have been retained. To compute the integrals we use the trapezoidal rule, with the sampling rates $\Delta\beta_r = 0.025$ and $\Delta\beta_e = 0.05$. The integral giving the evanescent part has been truncated at $\beta_e = 15$.

Figures 1 and 2 are volume visualizations of the real part of Green's function in which is possible to see how the wave propagates inside and outside the fiber. Because the source in positioned inside the waveguide, most of the energy remains in the fiber in both cases. In Fig. 1 the source is in on the waveguide axis, exciting only the guided mode corresponding to $m = 0$ and generating a cylindrically symmetric wave. In Fig. 2 the source is half the radius off axis, exciting all three guided modes.

Figures 3, 4 and 5 emphasize the different behavior of the wave amplitude inside and outside the fiber. These figures are a section of the global result, that is to say, they represent the real part of Green's function in the Cartesian plane $y = 0$. Fig. 3 and 4 are the equivalent of Fig. 1 and 2 respectively. In Fig. 5 the source is outside the fiber. Note that in this case the guided modes are weakly excited and most of the energy propagates outside the fiber.

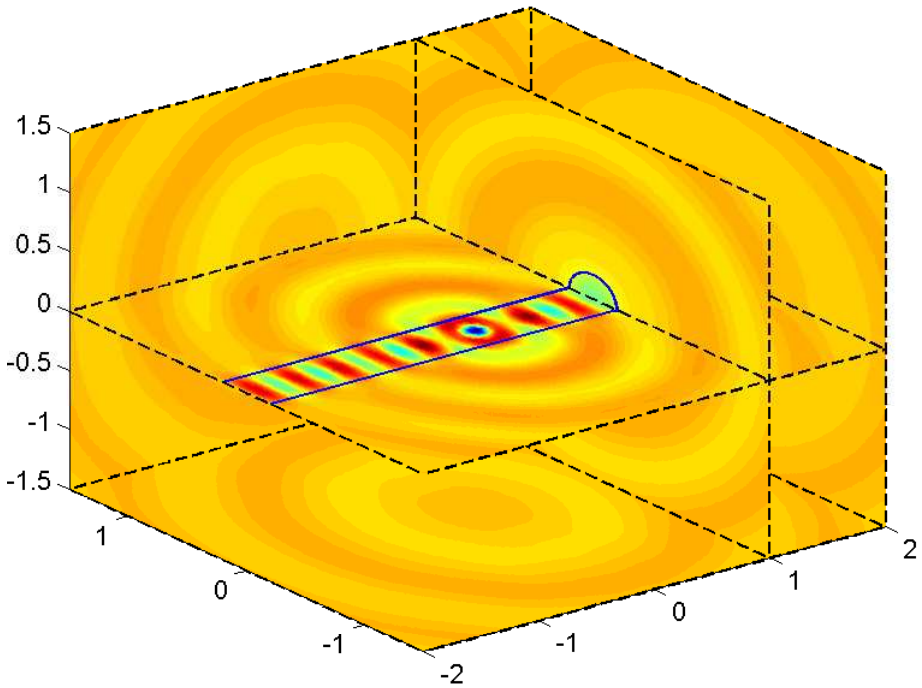

Figure 1: Real part of Green's function with the source at the origin. The wave is cylindrically symmetric and most of the energy propagates inside the fiber. The continuous lines represent the border of the guide.

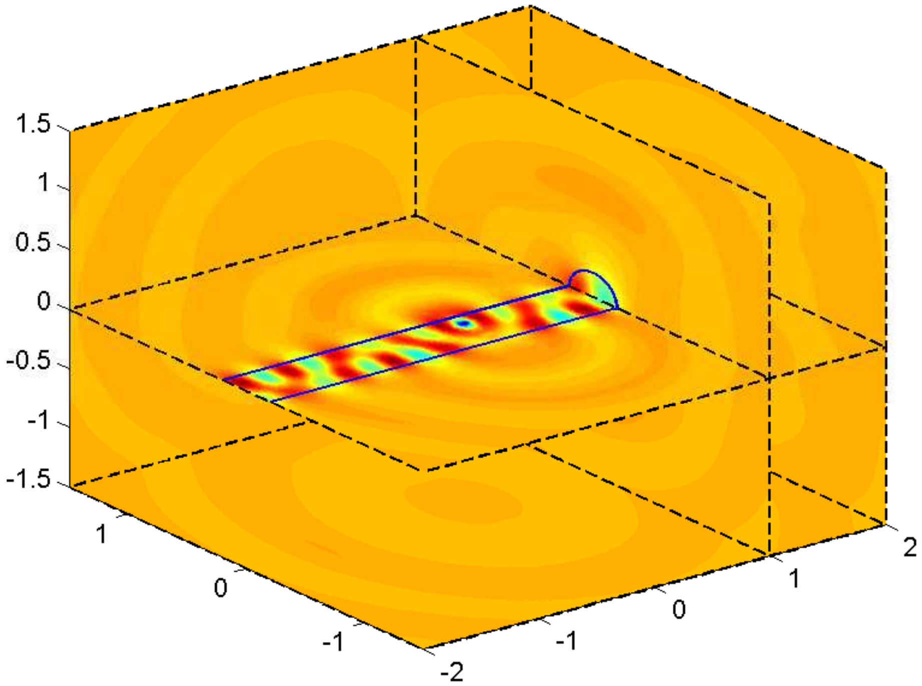

Figure 2: Real part of Green's function with the source in $\rho = 0.1,\ \eta = 0,\ \zeta = 0$. All the guided modes are excited.

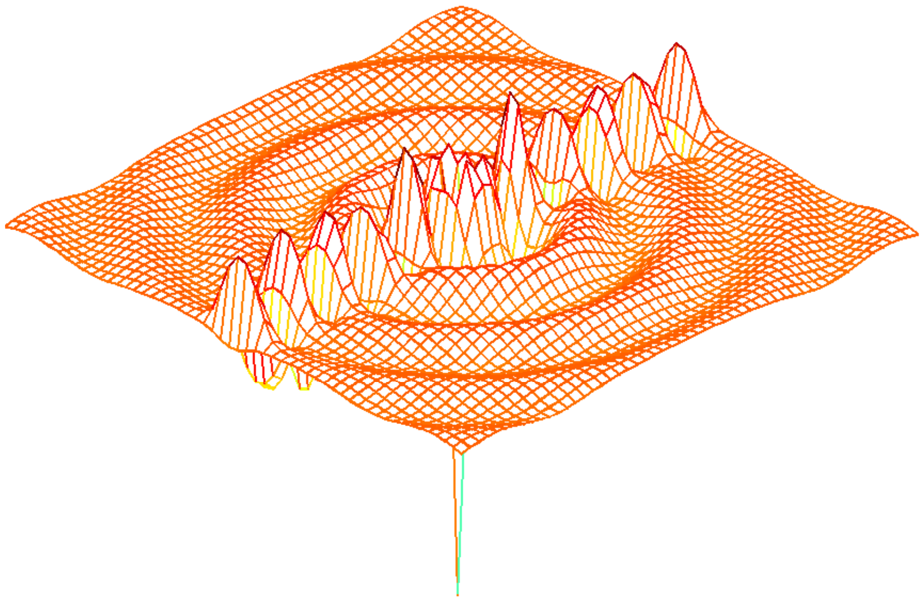

Figure 3: Real part of Green's function in the $x - z$ plane, with the source at the origin. Only the $m = 0$ guided mode is present.

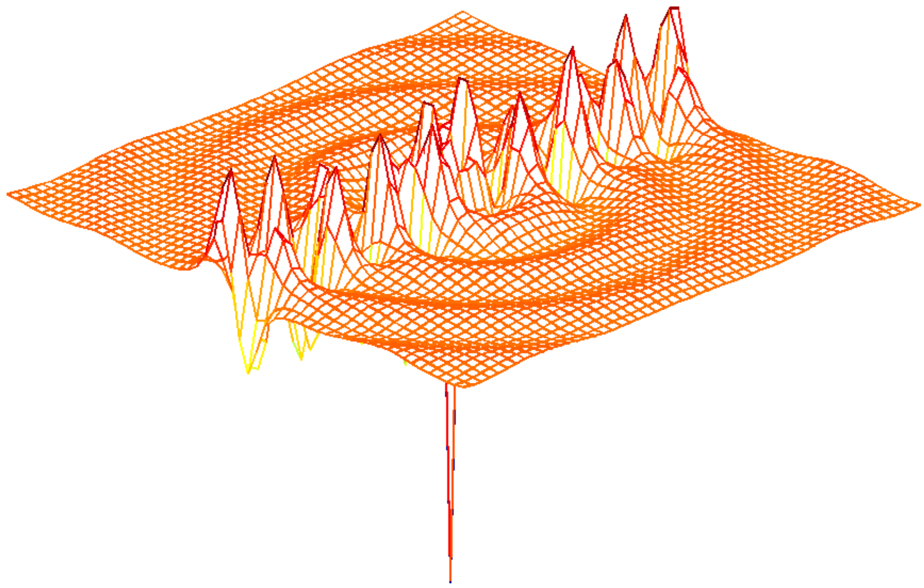

Figure 4: Real part of Green's function in the $x - z$ plane with the source at $\rho = 0.1,\ \eta = 0,\ \zeta = 0$. Three guided modes are excited.

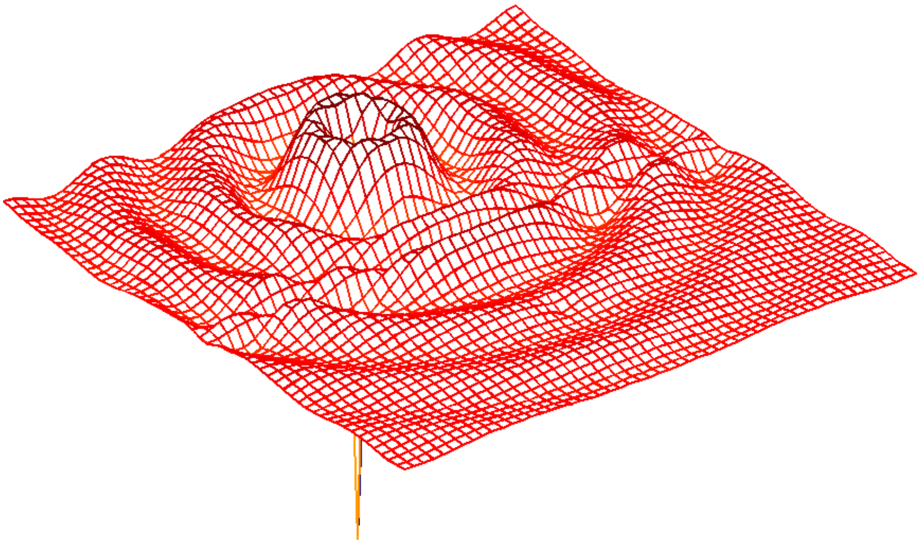

Figure 5: Real part of Green's function in the $x-z$ plane with the source at $\rho=1,\ \eta=0,\ \zeta=0$. The guided modes are weakly excited.

4 The coaxial cable

For a coaxial cable the index of refraction satisfies (3). Then the function $q(r)$, defined by (6), becomes

$$q(r)=\begin{cases} d^2, & 0\le r<a,\\ 0, & a\leqslant r<R,\\ d^2, & r\geqslant R.\end{cases}$$

The way to calculate the function $j_m(r,\lambda)$, the measure $d\chi_m(\lambda)$ and the corresponding Green's function is analogous to the one followed in the case of a step-index fiber. We will list the main results, omitting the details.

Consider first the case $\lambda>d^2$. Denote $Q=\sqrt{\lambda-d^2}$. Define the operator

$$\mathcal{V}_x[f,g](\lambda)=x[\sqrt{\lambda}f(Qx)g'(x\sqrt{\lambda})-Qf'(Qx)g(x\sqrt{\lambda})].$$

By using the fact that $j_m(r,\lambda)$ is a continuously differentiable function, we obtain that

$$j_m(r,\lambda)=\begin{cases}\alpha_m(\lambda)\sqrt{r}J_m(Qr), & 0<r<a,\\ \frac{\pi}{2}\sqrt{r}[\beta_m(\lambda)J_m(r\sqrt{\lambda})+\gamma_m(\lambda)Y_m(r\sqrt{\lambda})], & a<r<R,\\ \left(\frac{\pi}{2}\right)^2\sqrt{r}[\delta_m(\lambda)J_m(Qr)+\epsilon_m(\lambda)Y_m(Qr)], & r>R,\end{cases} \tag{34}$$

where

$$\begin{aligned}
&\alpha_m(\lambda)=(-1)^{\frac{|m|-m}{2}}|m|!2^{|m|}Q^{-|m|},\\
&\beta_m(\lambda)=\alpha_m(\lambda)\mathcal{V}_a[J_m,Y_m](\lambda),\\
&\gamma_m(\lambda)=-\alpha_m(\lambda)\mathcal{V}_a[J_m,J_m](\lambda),\\
&\delta_m(\lambda)=-\{\beta_m(\lambda)\mathcal{V}_R[Y_m,J_m](\lambda)+\gamma_m(\lambda)\mathcal{V}_R[Y_m,Y_m](\lambda)\},\\
&\epsilon_m(\lambda)=\beta_m(\lambda)\mathcal{V}_R[J_m,J_m](\lambda)+\gamma_m(\lambda)\mathcal{V}_R[J_m,Y_m](\lambda).
\end{aligned}$$

Use (11) and (14) to deduce

$$d\chi_m(\lambda) = \left(\frac{2}{\pi}\right)^3 \frac{d\lambda}{\delta_m(\lambda)^2 + \epsilon_m(\lambda)^2}. \tag{35}$$

If $0 < \lambda < d^2$, we introduce

$$\mathcal{V}_x^g[f,g](\lambda) = x[\sqrt{\lambda} f(Q_0 x) g'(x\sqrt{\lambda}) - Q_0 f'(Q_0 x) g(x\sqrt{\lambda})],$$

where

$$Q_0 = \sqrt{d^2 - \lambda}.$$

The eigenvalues λ_k^m (that is, the points of discontinuity of $\chi_m(\lambda)$) verify

$$\mathcal{V}_a^g[I_m, Y_m](\lambda)\mathcal{V}_R^g[K_m, J_m](\lambda) - \mathcal{V}_a^g[I_m, J_m](\lambda)\mathcal{V}_R^g[K_m, Y_m](\lambda) = 0.$$

We have

$$j_m^g(r, \lambda_k^m) = \begin{cases} \alpha_m^g(\lambda_k^m)\sqrt{r} I_m(Q_0 r), & 0 < r < a, \\ \frac{\pi}{2}\sqrt{r}[\beta_m^g(\lambda_k^m) J_m(r\sqrt{\lambda_k^m}) \\ \qquad + \gamma_m^g(\lambda_k^m) Y_m(r\sqrt{\lambda_k^m})], & a < r < R, \\ \sqrt{r}\delta_m^g(\lambda_k^m) K_m(Q_0 r), & r > R, \end{cases} \tag{36}$$

where

$$\begin{aligned} \alpha_m^g(\lambda_k^m) &= |m|! 2^{|m|} Q_0^{-|m|}, \\ \beta_m^g(\lambda_k^m) &= \alpha_m^g(\lambda_k^m)\mathcal{V}_a^g[I_m, Y_m](\lambda_k^m), \\ \gamma_m^g(\lambda_k^m) &= -\alpha_m^g(\lambda_k^m)\mathcal{V}_a^g[I_m, J_m](\lambda_k^m), \\ \delta_m^g(\lambda_k^m) &= \frac{\pi}{2} \cdot \frac{\beta_m^g(\lambda_k^m) J_m(R\sqrt{\lambda_k^m}) + \gamma_m^g Y_m(R\sqrt{\lambda_k^m})}{K_m(Q_0 R)}. \end{aligned}$$

The formula for the jumps is

$$r_k^m = \frac{1}{\alpha_m^g(\lambda_k^m)^2} \cdot \frac{2\pi \lambda_k^m d^{-2}(d^2 - \lambda_k^m)}{\{T[I_m](Q_0 a)\}^2 - \delta_m^g(\lambda_k^m)^2 \{T[K_m](Q_0 R)\}^2}, \tag{37}$$

where T is an operator defined by

$$T[f_m](x) = m^2 f_m(x)^2 - x^2 f_m'(x)^2.$$

Consider the case $\lambda = d^2$. If we assume that this λ is a discontinuity point for $\chi_m(\lambda)$, we can introduce the operator

$$\mathcal{V}_x^c[f,g](\lambda) = x[\sqrt{\lambda} f(x) g'(x\sqrt{\lambda}) - f'(x) g(x\sqrt{\lambda})].$$

We denote $j_m(r,\lambda)$ by $j_m^c(r,\lambda)$ to emphasize that it is a *cutoff* mode, standing intermediately between guided modes ($\lambda < d^2$) decaying exponentially, and radiation modes ($\lambda > d^2$) which are oscillatory. We have

$$j_m^c(r,\lambda) = \begin{cases} P_m(r), & 0 < r < a, \\ \frac{\pi}{2}\sqrt{r}[\beta_m^c(\lambda) J_m(r\sqrt{\lambda}) + \gamma_m^c(\lambda) Y_m(r\sqrt{\lambda})], & a < r < R, \\ \sqrt{r}\delta_m^c(\lambda) S_m(r), & r > R, \end{cases} \tag{38}$$

where

$$
\begin{aligned}
P_m(r) &= r^{m+1/2}, \\
S_m(r) &= r^{1/2-m}, \\
\beta^c_m(\lambda) &= \mathcal{V}^c_a[P_m, Y_m](\lambda), \\
\gamma^c_m(\lambda) &= -\mathcal{V}^c_a[P_m, J_m](\lambda), \\
\delta^c_m(\lambda) &= \frac{\pi}{2} \cdot \frac{\beta^c_m(\lambda) J_m(R\sqrt{\lambda}) + \gamma^c_m Y_m(R\sqrt{\lambda})}{S_m(R)}.
\end{aligned}
$$

The condition for discontinuity of $\chi_m(\lambda)$ at $\lambda = d^2$ becomes

$$
\frac{\beta^c_m(\lambda) J'_m(R\sqrt{\lambda}) + \gamma^c_m Y'_m(R\sqrt{\lambda})}{\beta^c_m(\lambda) J_m(R\sqrt{\lambda}) + \gamma^c_m Y_m(R\sqrt{\lambda})} = -\frac{|m|}{R}.
$$

Denote

$$
\varepsilon^c_m(\lambda) = \frac{\pi}{2}\left[\beta^c_m(\lambda) J_m(R\sqrt{\lambda}) + \gamma^c_m Y_m(R\sqrt{\lambda})\right].
$$

The formula for the jump at $\lambda = d^2$ is

$$
r^m_k = \frac{m^2-1}{|m|} \cdot \frac{2\pi}{\varepsilon^c_m(\lambda)^2 R^2(|m|+1) - a^2(|m|-1)}. \tag{39}
$$

As done in the previous section, the radiating and the evanescent parts of the Green's function (21) can be obtained by using formulas (34) and (35), while the guided part follows from (36) – (39).

Remark 4.1 *Our method and notation introduced in this section can be used to calculate the Green's function of any other types of coaxial waveguides, for example to the case of an "all-dielectric" coaxial waveguide, very important in recent applications, see [3] and [4].*

4.1 Numerical examples.

As was done in the previous section, Figures 6 and 7 show the real part of the Green's function for the coaxial cable.

The fiber's parameters are the same as the ones used in the step-index case, $k = 10$, $n_{co} = 2$, $n_{cl} = 1$, $R = 0.2$, $d^2 = 300$, but a new region has been introduced inside the core, with radius $a = 0.05$ and index of refraction $n_{cl} = 1$.

Also in this case there are three different guided modes, corresponding to the following values of the parameter λ: $\lambda = 113.04$ for $m = 0$ and $\lambda = 208.66$ for $m = -1, 1$.

We again separate the Green's function in three parts: radiating, evanescent and guided. Using the notation of formulas (32) and (33), we compute the integrals using the trapezoidal rule, with sampling rates $\Delta\beta_r = 0.01$, $\Delta\beta_e = 0.075$, and we truncate the integral in the evanescent part at $\beta_e = 15$. In the sum over $\mathbb{Z}$ only the terms with $|m| \leq 10$ have been retained.

Acknowledgement

We wish to thank our advisers, Fadil Santosa and Rolando Magnanini[1], for proposing this topic to us and for continual feedback while writing this paper.

[1]Fadil Santosa is the adviser of O. A. and Rolando Magnanini is the adviser of G. C.

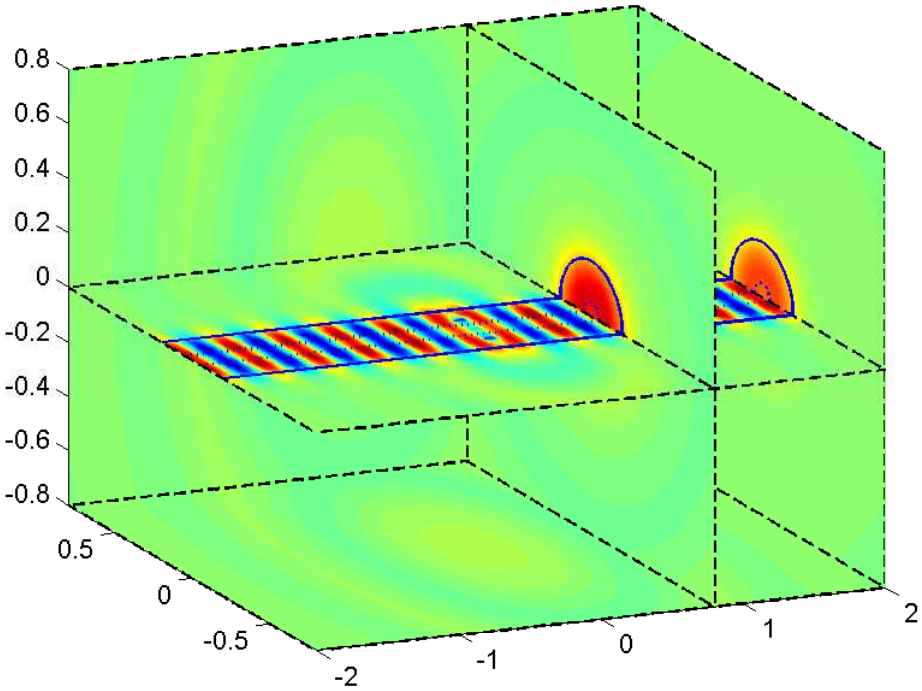

Figure 6: Real part of Green's function with the source at the origin. The wave is cylindrically symmetric and most of the energy propagates inside the fiber. The continuous lines represent the border of the guide while the dotted ones delimitate the zone with $r < a$.

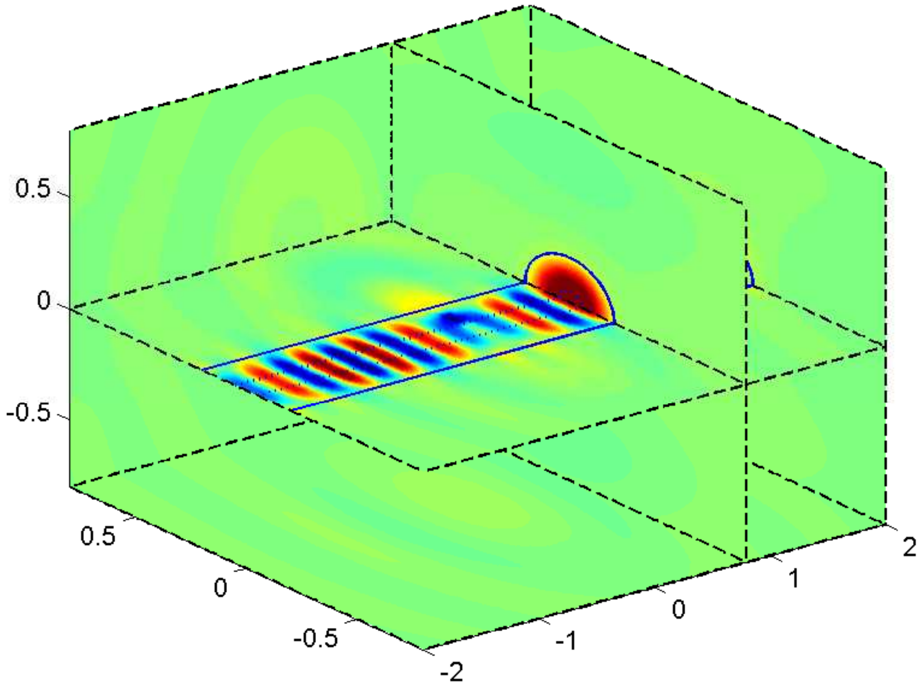

Figure 7: Real part of Green's function with the source in $\rho = 0.1,\ \eta = 0,\ \zeta = 0$. All the guided modes are excited.

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