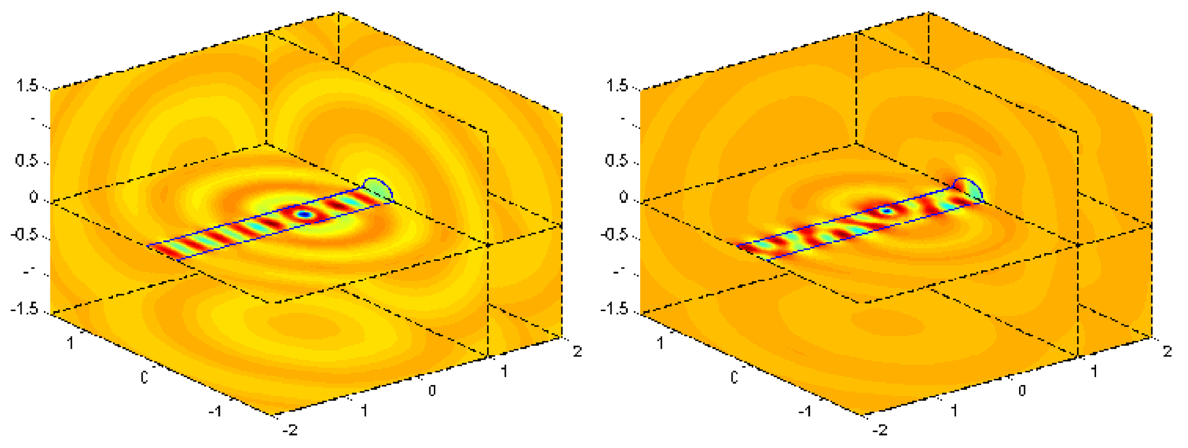

Wave propagation in optical waveguides with imperfections

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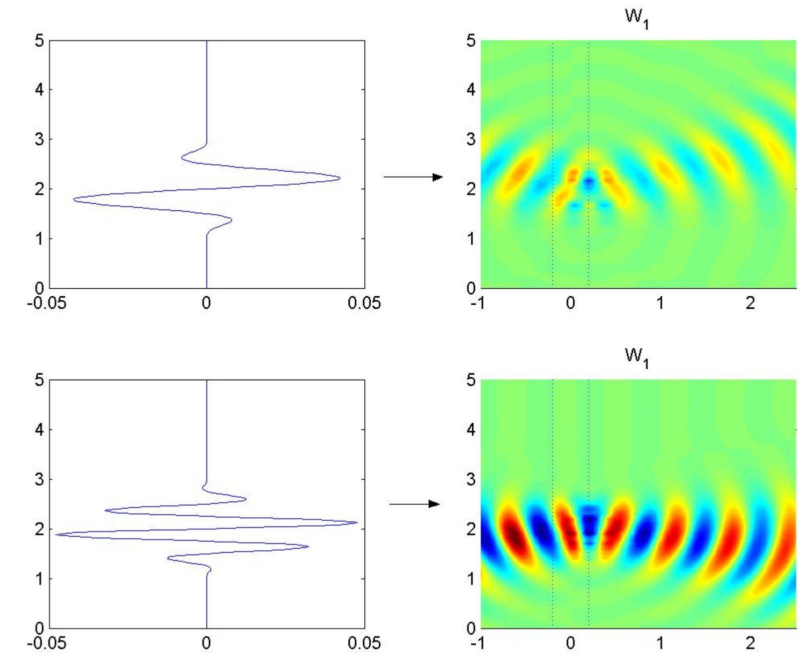
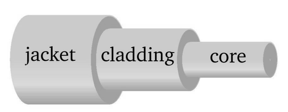


Introduction

We present a mathematical framework for studying the problem of electromagnetic wave propagation in a 2-D or 3-D optical waveguide (optical fiber).

A typical optical fiber is made of silica glass or plastic. Its central region is called *core*, surrounded by a *cladding*, which has a slightly lower index of refraction. The cladding is surrounded by a protective *jacket*.



Most of the electromagnetic radiation propagates without loss as a set of guided modes along the fiber axis. The electromagnetic field intensity of the guided modes in the cladding decays exponentially transversally to the waveguide's axis. That is why the radius of the cladding, which is typically several times larger than the radius of the core, can be considered infinite.

In the model we used, we study the following Helmholtz equation

$$L_0 u := \Delta u + k^2 n(\mathbf{x})^2 u = f, \tag{1}$$

in $\mathbb{R}^2$ or $\mathbb{R}^3$, where k is the *wavenumber* and n is a positive function representing the index of refraction.

In this poster, we consider both the case of a rectilinear waveguide and the one of a waveguide presenting imperfections, with applications to phenomenons of physical interest.

2-D rectilinear waveguides

The eigenvalue problem

A rectilinear waveguide can be described by assuming that

$$n(x) = n_0(x_1) = \begin{cases} n_{co}(x_1), & \text{if } |x_1| \le h, \\ n_{cl}, & \text{if } |x_1| > h, \end{cases}$$

where n_{co} is a bounded even function. Thanks to the symmetry of the problem, we can separate the variables and look for solutions of the homogeneous Helmholtz equation of the form $u(x_1, x_2) = v(x_1, \lambda) e^{ik\beta x_2}$, with $\lambda = k^2(n_*^2 - \beta^2)$ and $n_* = \max n$. This leads to consider the following eigenvalue equation

$$v''(x_1,\lambda) + [\lambda - q(x_1)]v(x_1,\lambda) = 0, \quad x_1 \in \mathbb{R}, \tag{2}$$

where $q(x_1) = k^2[n_*^2 - n(x_1)^2]$.

Classification of the solutions

Bounded solutions of (2) are of the form

$$v_j(x_1,\lambda) = \begin{cases} \phi_j(h,\lambda)\cos Q(x_1-h) + \frac{\phi_j'(h,\lambda)}{Q}\sin Q(x_1-h), & x_1 > h, \\ \phi_j(x_1,\lambda), & |x_1| \le h, \\ \phi_j(-h,\lambda)\cos Q(x_1+h) + \frac{\phi_j'(-h,\lambda)}{Q}\sin Q(x_1+h), & x_1 < -h, \end{cases}$$

$j = s, a$, with $Q = \sqrt{\lambda - d^2}$ and $d^2 = k^2(n_*^2 - n_{cl}^2)$. Here, v_j is symmetric or anti-symmetric in x_1 if $j = s$ or $j = a$, respectively. Solutions can be classified as follows:

- *Guided modes.* For $0 < \lambda < d^2$, only a finite number of eigenvalues λ_m^j are supported by (2). The solutions decay exponentially outside the core, which corresponds to solutions of the Helmholtz equation which propagate most of their energy inside the core;
- *Radiation modes.* For $d^2 < \lambda < k^2 n_*^2$, $v_j(x_1,\lambda)$ are bounded and oscillatory.
- *Evanescent modes.* For $\lambda > k^2 n_*^2$, v_j are bounded and oscillatory, but the corresponding solutions of the Helmholtz equation decay exponentially in one direction along the x_2 axis and increase along the other one.

Resolution formula

By using the theory of Titchmarsh for the eigenvalues problems of singular differential operators, it is possible to construct a resolution formula for (1):

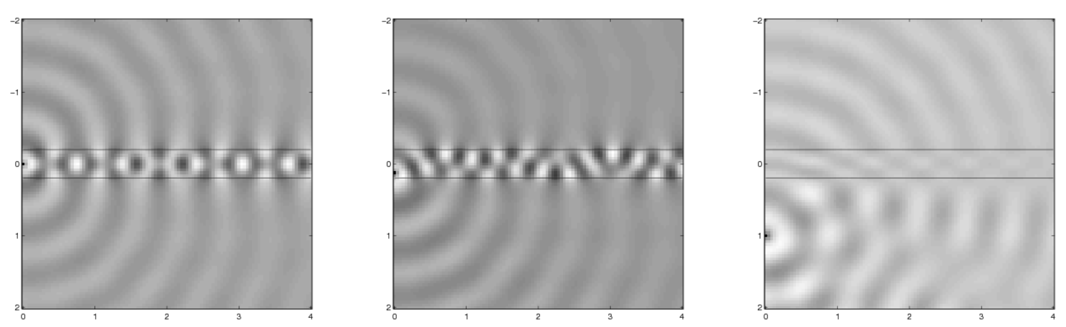
$$u(x) = \int_{\mathbb{R}^2} G(x,y) f(y) dy, \tag{3}$$

with

$$G(x,y) = \sum_{j\in\{s,a\}} \int_0^\infty \frac{e^{i|x_2-y_2|\sqrt{k^2n_*^2-\lambda}}}{2i\sqrt{k^2n_*^2-\lambda}} v_j(x_1,\lambda) v_j(y_1,\lambda) d\rho_j(\lambda),$$

where, for every $\eta \in C_0^\infty([0,+\infty))$, it holds that

$$\langle d\rho_j, \eta\rangle = \sum_{m=1}^{M_j} r_j^m \eta(\lambda_j^m) + \frac{1}{2\pi}\int_{d^2}^\infty \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h,\lambda)^2 + \phi_j'(h,\lambda)^2}\eta(\lambda) d\lambda,$$



3-D rectilinear waveguides

We study cylindrically symmetric optical fibers, i.e. when

$$n = n(r) = \begin{cases} n_{co}(r), & \text{if } 0 < r \le R, \\ n_{cl}, & \text{if } r > R, \end{cases}$$

where r is the distance from the fiber's axis. In this case, we separate the variables by using cylindrical coordinates (r, ϑ, x_3) and looking for solutions of the homogeneous Helmholtz equation of the form $u = e^{i\beta k x_3} e^{im\vartheta} w(r) r^{-\frac{1}{2}}$. Hence, the associated eigenvalue problem is

$$w'' + \left[\lambda - q(r) - \frac{m^2 - 1/4}{r^2}\right] w = 0, \ r > 0.$$

The classification of the solutions is analogous to the one obtained in the 2-D case.

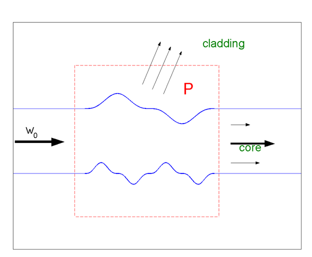
In this case, we can still apply the theory of Titchmarsh in all its power (see [AC1]). Notice that in this case, due to the term $\frac{m^2-1/4}{r^2}$, the equation has a singularity at $r = 0$ besides the one at $r = +\infty$; this adds further technical difficulties.

Numerical results in the 3-D case are shown at the top left corner of this poster.

2-D waveguides with imperfections

Real-life waveguides are never perfect, since they might contain imperfections due to inhomogeneities or changes in the core's width and shape.

When a pure guided mode is excited inside a guide with imperfections, a sort of resonation takes place and the other modes supported by the fiber are excited. This effect causes a signal distortion, since every guided mode propagates at its own characteristic velocity, and a loss in the signal power, due to the transfer of power to radiation, evanescent and the other guided modes.



These effects are not always to be avoided. It is possible to make optical devices which can "propagate the energy as desired". We will show two examples in the last part of the poster.

The mathematical framework

From the mathematical point of view, we consider the Helmholtz equation

$$L_\varepsilon u := \Delta u + k^2 n_\varepsilon(x_1,x_2)^2 u = f, \quad \text{in } \mathbb{R}^2, \tag{4}$$

where the index of refraction n_ε is supposed to be a small perturbation of n_0.

We formally represent L_ε and $u := u_\varepsilon$ in terms of their Neumann series and find

$$L_0 u_0 = f, \ L_0 u_1 = -L_1 u_0, \ldots, L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \ \ldots$$

Each step of the above iterative method can be solved by using the resolution formula (3).

Existence of a solution

We prove the existence of a solution by writing the equation $L_\varepsilon u = f$ as $L_0 u = f + (L_0 - L_\varepsilon)u$ and then as

$$u = L_0^{-1} f + \varepsilon L_0^{-1}\left(\frac{L_0 - L_\varepsilon}{\varepsilon}\right) u.$$

Consider a weight function $\mu(x) = \frac{16}{(4+|x|^2)^2}$. By using estimates on the solution (3), we are able to prove that the linear operators

$$L_0^{-1} : L^2(\mathbb{R}^2, \mu^{-1}) \to H^2(\mathbb{R}^2, \mu) \quad \text{and} \quad \frac{L_0 - L_\varepsilon}{\varepsilon} : H^2(\mathbb{R}^2, \mu) \to L^2(\mathbb{R}^2, \mu^{-1})$$

are continuous. Hence, by choosing ε small enough and using the contraction mapping theorem, we prove the existence of a solution of $L_\varepsilon u = f$.

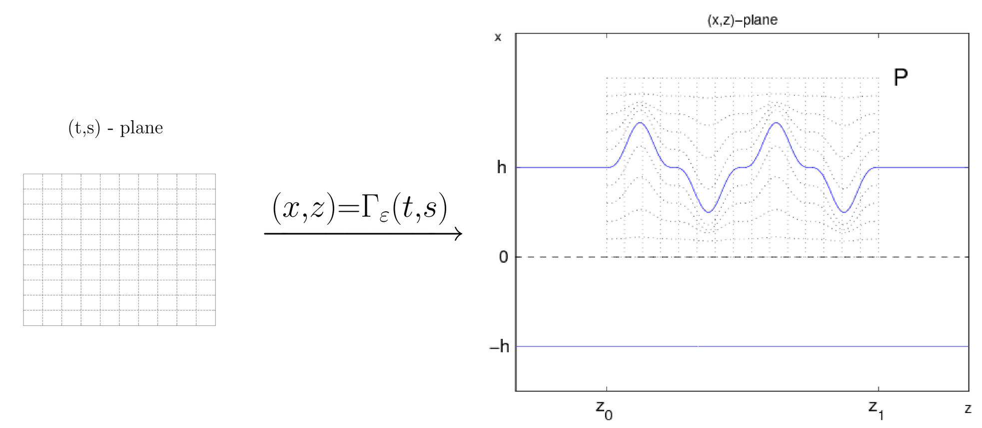
Numerical results

We look for a suitable change of coordinates such that n_ε can be rewritten in a more handy way. The best we can hope is that, after having changed the coordinates, n_ε depends only on the "new" transversal coordinate or it can be represented in Neumann series with the zeroth order term depending on the transversal coordinate. In such cases, the iterative method described above can be applied.

We will change the coordinates by using

$$\Gamma^\varepsilon(t_1,t_2) = \begin{cases} x_1 = t_1 + \varepsilon T(t_1) S(t_2), \\ z_2 = t_2, \end{cases} \tag{5}$$

where $T \in C_0^2(\mathbb{R})$ is a piecewise polynomial function and $S \in C_0^2(\mathbb{R})$ describe the "profile" of the perturbation.



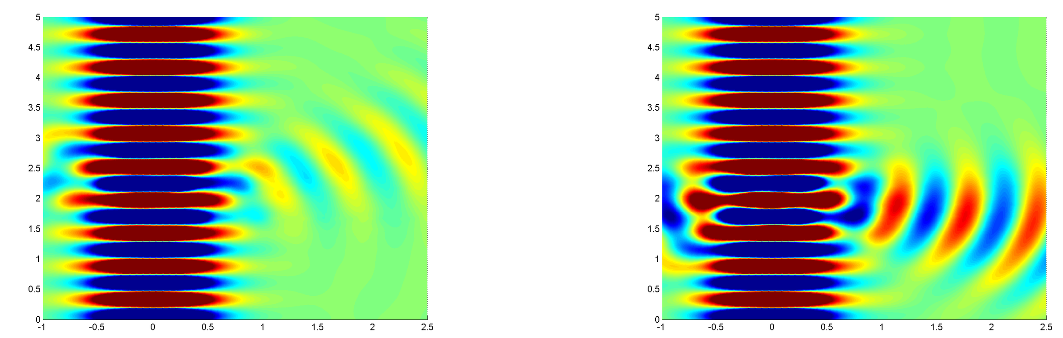
Hence, in the new coordinates, the operator L_ε becomes a second order elliptic operator of the form

$$L^\varepsilon w(t_1,t_2) = \sum_{i,j=1}^2 a_{ij}^\varepsilon w_{x_i x_j} + \sum_{i=1}^2 b_i^\varepsilon w_{x_i} + n_\varepsilon w,$$

where $w(t_1,t_2) = u(x_1,x_2)$. We represent this operator and w in Neumann series, we consider w_0 as a pure guided mode and compute the first order approximation of w. This corresponds to study what happens in the wave propagation when a pure guided mode finds imperfections in the waveguide.

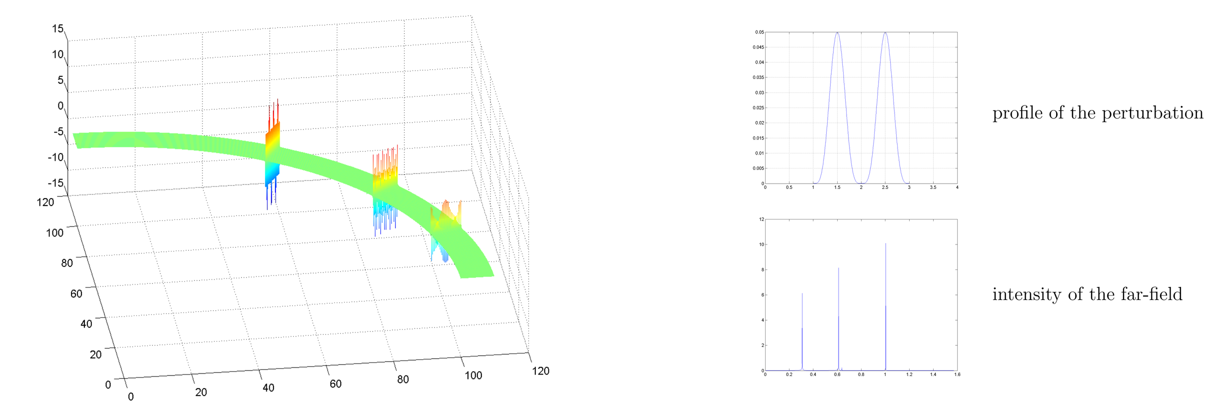
Near-field

In the two figures below, the real part of $w_0 + \varepsilon w_1$ in proximity of the waveguide is represented. They correspond to the two different kind of perturbations of the waveguide represented in the figures in the top right corner of the poster, where only the real part of w_1 is shown.



Grating-couplers

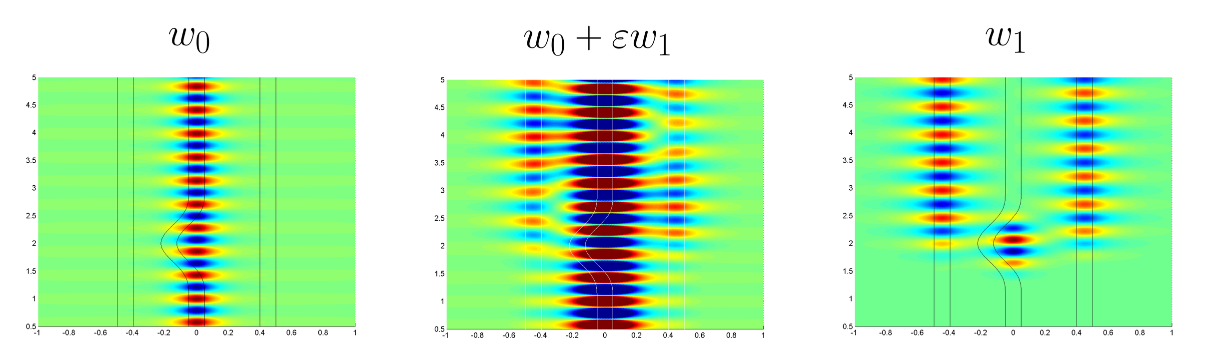
Grating couplers are optical devices where the radiation energy is directed along precise directions. The figure below represents the far-field of w_1 due to a perturbation in the waveguide as in the figure on the right. The last figure shows the intensity of the far-field.



The Mach-Zehnder coupler

In this kind of optical devices, the perturbation in a waveguide excites the guided modes of other waveguides which are close to the perturbed one.

The figures below show how a pure guided mode propagating in the central waveguide excites the guided modes of two waveguides nearby. The third figure shows the details of w_1.



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[AC1] O. Alexandrov – G. Ciraolo, *Wave propagation in a 3-D optical waveguide*, Math. Models Methods Appl. Sci. 14 (2004), no. 6, 819–852.

[AC2] O. Alexandrov – G. Ciraolo, *Wave propagation in a 3-D optical waveguide II. Numerical Results* (preprint).

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