

# Non-rectilinear optical waveguides: analytical and numerical results based on the Green's function

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# Outline

## 1 Rectilinear waveguides

- Introduction to the problem
- 2-D case: the Green's function
- 3-D case
- Uniqueness

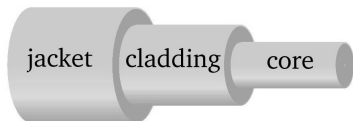
## 2 Non-rectilinear waveguides

- Physical motivations
- Mathematical framework
- Numerical simulations

## Part I

# Rectilinear waveguides

# Optical waveguides



## Notations

- $n$  – index of refraction
- $n_0$  – maximum of  $n$
- $n_{cl}$  – *cladding's* index of refraction
- $k$  – wavenumber ( $2\pi/\lambda$ )

## Kinds of energy

- Guided energy
- Radiation energy
- Evanescent energy

## Mathematical model in 2-D and 3-D

Helmholtz equation

$$\Delta u + k^2 n(\mathbf{x})^2 u = f, \quad \mathbf{x} \in \mathbb{R}^N.$$

2-D

 $n$  even function,

$$n = \begin{cases} n_{co}(x_1) & \text{if } |x_1| \leq h, \\ n_{cl} & \text{otherwise.} \end{cases}$$

3-D

 $n$  axially symmetric with respect of  $x_3$ ,

$$n = \begin{cases} n_{co}(r) & \text{if } 0 \leq r \leq R, \\ n_{cl} & \text{otherwise,} \end{cases}$$

where  $r = \sqrt{x_1^2 + x_2^2}$ .

## 2-D case (Magnanini, Santosa. 2000, SIAM J. Appl. Math.)

$$\Delta u + k^2 n(x_1)^2 u = f \quad \xrightarrow{u=v(x_1,\lambda)e^{ik\beta x_2}} \quad v'' + [\lambda - q(x_1)]v = 0, \text{ in } \mathbb{R}.$$

- $v'(x_1, \lambda) = \frac{\partial v(x_1, \lambda)}{\partial x_1}$
- $n_0 = \max n(x_1)$
- $q(x_1) = k^2[n_0^2 - n(x_1)^2]$
- $d^2 = k^2(n_0^2 - n_{cl}^2)$
- $\lambda = k^2(n_0^2 - \beta^2)$
- $Q = \sqrt{\lambda - d^2}$

## Solutions

$$v_j(x_1, \lambda) = \begin{cases} \phi_j(h, \lambda) \cos Q(x_1 - h) + \frac{\phi_j'(h, \lambda)}{Q} \sin Q(x_1 - h), & x_1 > h, \\ \phi_j(x_1, \lambda), & |x_1| \leq h, \\ \phi_j(-h, \lambda) \cos Q(x_1 + h) + \frac{\phi_j'(-h, \lambda)}{Q} \sin Q(x_1 + h), & x_1 < -h, \end{cases}$$

$j = s, a.$

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# Classification of solutions

## Modes

Modes are solutions of Helmholtz equation of the form

$$v(x_1, \lambda)e^{ik\beta x_2}, \text{ with } \beta k = \sqrt{k^2 n_0^2 - \lambda}.$$

- **guided**: for  $0 < \lambda < d^2$ . Guided modes (and then the energy) propagate mainly inside the core and vanish exponentially outside. They are *finite* in number.
- **radiation**: for  $d^2 < \lambda < n_0^2 k^2$ . Energy doesn't localize inside the core but propagates mainly in the cladding.
- **evanescent**: for  $\lambda > n_0^2 k^2$ . The *constant of propagation*  $\beta$  becomes imaginary. These modes have an exponential decay in one of the directions along the axis  $x_2 \rightarrow \pm\infty$ , while they grow exponentially along the other one.

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## Resolution formula

M-S find a solution (in the sense of distributions) of the Helmholtz equation as superposition of guided, radiation and evanescent modes:

$$u(x) = \int_{\mathbb{R}^2} G(x, y) f(y) dy,$$

where

$$G(x, y) = \sum_{j \in \{s, a\}} \int_0^{\infty} \frac{e^{i|x_2 - y_2| \sqrt{k^2 n_0^2 - \lambda}}}{2i \sqrt{k^2 n_0^2 - \lambda}} v_j(x_1, \lambda) v_j(y_1, \lambda) d\rho_j(\lambda),$$

with

$$(d\rho_j, \eta) = \sum_{m=1}^{M_j} r_j^m \eta(\lambda_j^m) + \frac{1}{2\pi} \int_{d^2}^{\infty} \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2) \phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2} \eta(\lambda) d\lambda, \quad \forall \eta \in C_0^\infty([0, +\infty)).$$

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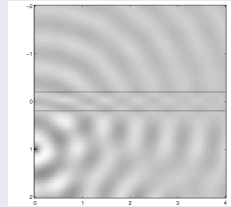
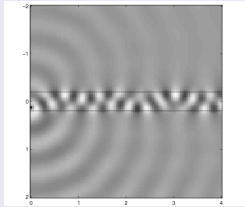
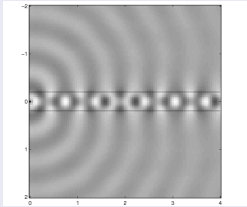
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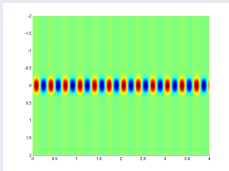
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# Examples

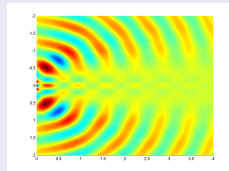
## Green's function



## A guided mode



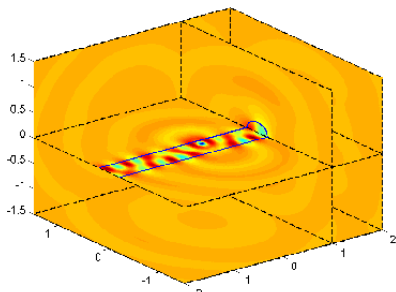
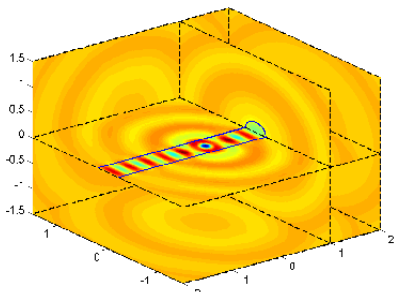
## The radiating energy



## 3-D optical fibers

By using a different technique, **analogous results** for the 3-D case were obtained in

- O. Alexandrov – G. Ciruolo, *Wave propagation in a 3-D optical waveguide*, M3AS, 14 (2004), no.6, 819–852.
- O. Alexandrov – G. Ciruolo, *Wave propagation in a 3-D optical waveguide II. Numerical results*, to appear in 5<sup>th</sup> ISAAC Congress Proceedings.



## 2-D vs 3-D

By using the cylindrical symmetry with respect to the  $x_3$ -axis (the fiber's axis), we find:

$$\Delta u + k^2 n(r)^2 u = f \quad \xrightarrow{u = e^{i\beta k x_3} e^{im\vartheta} \frac{w(r)}{\sqrt{r}}} \quad w'' + \left[ \lambda - q(r) - \frac{m^2 - 1/4}{r^2} \right] w = 0, \quad r > 0.$$

## Differences from the 2-D case

- The term  $\frac{m^2 - 1/4}{r^2}$ , makes the equation singular in  $r = 0$ , besides in  $r = \infty$ .
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in  $r = 0$ .
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

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# Uniqueness

We have a *good solution*:

$$G = G^g + G^r + G^e = \sum \dots + \int_{d^2}^{k^2 n_0^2} \dots + \int_{k^2 n_0^2}^{\infty} \dots$$

but have not a unique solution!

- $n$  is not constant outside any compact set.
- Pointwise spectrum  $\rightarrow$  difficult to give a radiation condition analogous to the one of Sommerfeld (cfr. Rellich, Miranker, Jäger & Saitō, Eidus, Zhang, Perthame & Vega...)
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## Notations and assumptions

Assumptions on  $u$ 

$$\lim_{|x_1| \rightarrow +\infty} \frac{u(x_1, x_2)}{e^{|x_1| \sqrt{d^2 - \lambda_*}}} = 0, \quad \lim_{|x_1| \rightarrow +\infty} \frac{u_{x_1}(x_1, x_2)}{e^{|x_1| \sqrt{d^2 - \lambda_*}}} = 0.$$

$$\Delta u + k^2 n(x)^2 u = f$$

Notations:

$$u = u_0 + \sum_{j \in \{s, a\}} \sum_{m=1}^{M_j} u_m^j,$$

- $u_m^j(x_1, x_2) = e_j(x_1, \lambda_m^j) U^j(x_2, \lambda_m^j)$
- $U^j(x_2, \lambda_m^j) = \int_{-\infty}^{\infty} u(\xi, x_2) e_j(\xi, \lambda_m^j) d\xi,$
- $e_j(x_1, \lambda_m^j) = \frac{v_j(x_1, \lambda_m^j)}{\|v_j(\cdot, \lambda_m^j)\|_2}$
- $\lambda_* = \max_{j, m} \lambda_m^j$

# Uniqueness Theorem

## Theorem

There exists at most one weak solution of  $\Delta u + k^2 n(x_1)^2 u = f$  such that

$$\int_0^\infty \int_{\partial B_\rho} \left| \frac{\partial u_0}{\partial r} - i k n_{cl} u_0 \right|^2 d\sigma d\rho +$$

$$\sum_{j \in \{s, a\}} \sum_{m=1}^{M_j} \int_0^\infty \int_{\partial Q_\rho} \left| \frac{\partial u_m^j}{\partial \nu} - i \beta_m^j u_m^j \right|^2 d\sigma d\rho \leq C^2, \quad (\text{RC})$$

## Remarks

- If no guided modes appear, (RC) is

$$\int_{\mathbb{R}^2} \left| \frac{\partial u}{\partial r} - i k n_{cl} u \right|^2 dx < +\infty,$$

already obtained by Rellich (with different assumptions on  $n$ ).

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# Uniqueness Theorem

## Idea of the proof

- $w$  solution of

$$\Delta w + k^2 n(x_1)^2 w = 0; \quad (1)$$

- $\text{Im} \int_{\partial\Omega} \bar{w} \frac{\partial w}{\partial \nu} d\sigma = 0;$

- $w_0$  and  $w_m^j$  satisfy (1);
- (RC) implies  $w_0, w_m^j \in L^2(\mathbb{R}^2);$

- $u \equiv 0$  is the only solution of the Helmholtz equation in  $L^2(\mathbb{R}^2).$

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## Part II

# Non-rectilinear waveguides

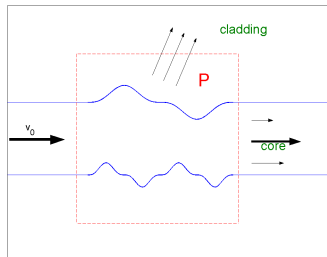
# Motivations

Small imperfections can affect wave propagation:

- Signal distortion
- Loss in the signal power



- Mode coupling
- More radiation and evanescent energy



...not always to be avoided...

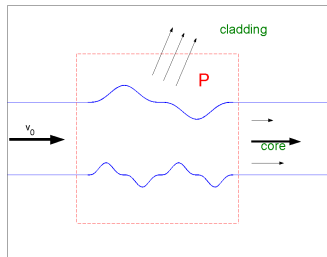
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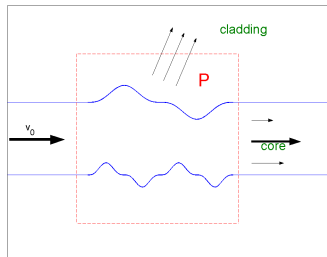
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Mathematical Framework (in  $\mathbb{R}^2$ )

$$L_0 = \Delta + k^2 n_0^2(x_1), \quad L_\varepsilon = \Delta + k^2 n_\varepsilon^2(x_1, x_2).$$

Non-rectilinear waveguides  $\longrightarrow L_\varepsilon u = f$ .

We formally represent  $u$  and  $L_\varepsilon$  by their Neumann series:

$$u = u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots; \quad L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \dots,$$

we have

$$L_0 u_0 = f, \quad L_0 u_1 = -L_1 u_0, \dots, \quad L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \dots$$

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## Existence of a solution

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$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left( \frac{L_0 - L_\varepsilon}{\varepsilon} \right) u.$$

## Strategy

- Find  $X_1$  and  $X_2$  such that

$$L_0^{-1} : X_2 \rightarrow X_1,$$

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are continuous.

- $\varepsilon > 0$  small enough  $\rightarrow$  contraction mapping theorem.

Spaces:  $\begin{cases} X_1 = H^2(\mathbb{R}^2, \mu), \\ X_2 = L^2(\mathbb{R}^2, \mu^{-1}), \end{cases}$

where  $\mu(x) = \frac{16}{(4 + |x|^2)^2}$ .

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# Remarks on the proof

- **Uniform asymptotic expansion** of the eigenfunctions and their first derivatives *inside the core*. If  $q \in L^1_{loc}(\mathbb{R})$ ,  $x \in [-h, h]$ :

$$\phi_s(x, \lambda) = \cos(\sqrt{\lambda}x) \{1 + \mathcal{O}(\lambda^{-1})\},$$

$$\phi_a(x, \lambda) = \sin(\sqrt{\lambda}x) \{1 + \mathcal{O}(\lambda^{-1})\},$$

as  $\lambda \rightarrow +\infty$ .

- $G(x_1, x_2; y_1, y_2) \in L^2(\mathbb{R}^4, \mu \times \mu)$ . In particular,  $G^s$  and  $G^r$  are bounded.
- Global estimates on first and second derivatives:

$$\begin{cases} u \in L^2(\mathbb{R}^2, \mu), \\ f \in L^2(\mathbb{R}^2, \mu^{-1}), \end{cases} \xrightarrow{\text{standard techniques}} u \in H^2(\mathbb{R}^2, \mu).$$

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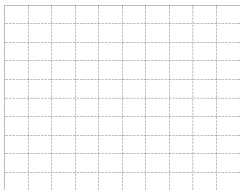
## Numerical simulations

$$\Gamma_\varepsilon(t, s) = \begin{cases} x = t + \varepsilon T(t)S(s), \\ z = s, \end{cases}$$

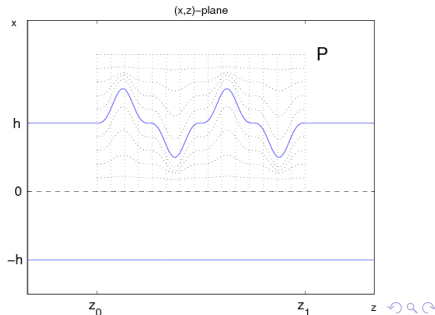
$$\Gamma_\varepsilon \in C^2(\mathbb{R}^2), \Gamma_\varepsilon \equiv Id_{\mathbb{R}^2} \text{ in } \mathbb{R}^2 \setminus P.$$

$$\tilde{L}_\varepsilon v = g$$

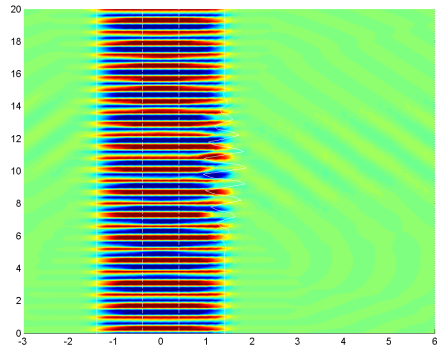
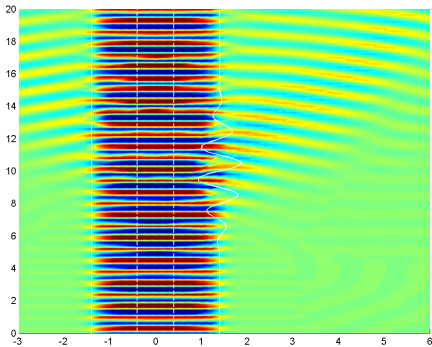
$$\Delta u + k^2 n(x, z)^2 u = f$$



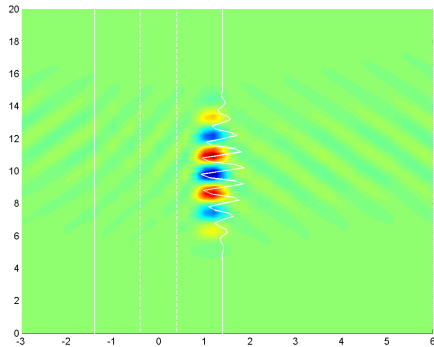
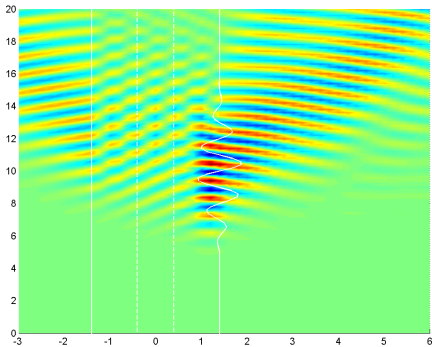
$$(x, z) = \Gamma_\varepsilon(t, s)$$



## Near Field

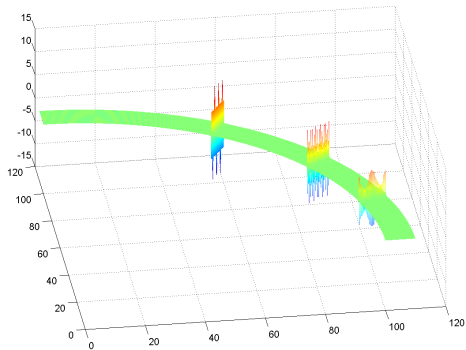
Real part of  $u_0 + \varepsilon u_1$ :

## Near Field

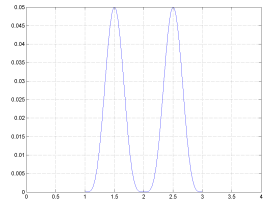
Real part of  $u_1$ :

## Far Field

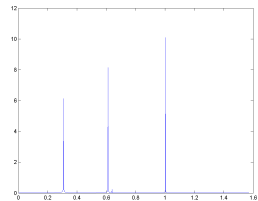
## Real part of the far-field



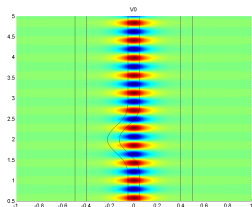
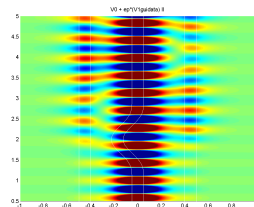
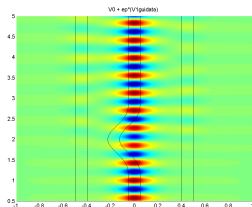
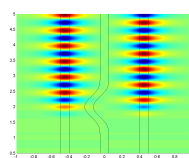
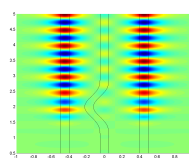
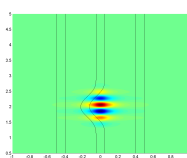
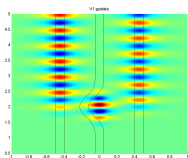
## Profile of the perturbation



## Intensity of the far-field



## Coupling between guided modes

 $u_0$  $u_0 + \varepsilon u_1$ Details of the guided part of  $u_1$ :

## Behaviour of the solutions in $r = 0$

**Lemma.** Let  $q \in L^\infty(0, \infty)$ . For  $m$  fixed, there exists a base of solutions  $j_m(r, \lambda)$  and  $y_m(r, \lambda)$  of

$$w'' + \left[ \lambda - q(r) - \frac{m^2 - 1/4}{r^2} \right] w = 0, \quad r > 0, \quad \lambda \in \mathbb{C}, \quad m \in \mathbb{Z} \quad (2)$$

such that

$$\lim_{r \rightarrow 0} \frac{j_m(r, \lambda)}{r^{|m|+1/2}} = 1, \quad \lim_{r \rightarrow 0} \frac{j'_m(r, \lambda)}{(|m|+1/2) r^{|m|-1/2}} = 1.$$

and

$$\lim_{r \rightarrow 0} \frac{y_m(r, \lambda)}{r^{-|m|+1/2}} = 1, \quad \lim_{r \rightarrow 0} \frac{y'_m(r, \lambda)}{(-|m|+1/2) r^{-|m|-1/2}} = 1, \quad \text{if } |m| \geq 1,$$

or

$$\lim_{r \rightarrow 0} \frac{y_m(r, \lambda)}{\sqrt{r} \ln r} = 1, \quad \lim_{r \rightarrow 0} \frac{y'_m(r, \lambda)}{\ln r / (2\sqrt{r})} = 1, \quad \text{if } m = 0.$$

Furthermore the functions  $j_m(r, \lambda)$  and  $j'_m(r, \lambda)$  are analytical in  $\lambda$  for  $r$  fixed.

## Behaviour of the solutions as $r \rightarrow \infty$

If  $r \in [R, +\infty)$ , equation (2) is

$$w'' + \left\{ \lambda - d^2 - \frac{m^2 - 1/4}{r^2} \right\} w = 0.$$

Solutions are linear combinations of

$$\sqrt{r} H_m^{(1)}(r\sqrt{\lambda - d^2}), \quad \sqrt{r} H_m^{(2)}(r\sqrt{\lambda - d^2}),$$

where  $H^{(1)}$  and  $H^{(2)}$  are Hankel's functions.

We look for  $C^1$  and bounded solutions

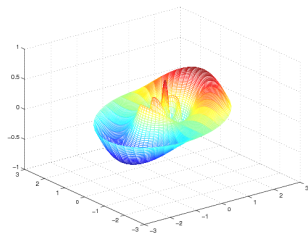
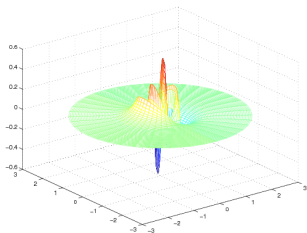
- $\lambda < d^2$ :  $w(r, \lambda)/\sqrt{r}$  vanishes exponentially;
- $\lambda > d^2$ :  $w(r, \lambda)/\sqrt{r}$  oscillates and decays slowly.

## Classification of solutions

The modes corresponds to the bounded solutions of (2)

As in the 2-D case we find the following classification:

- **guided**: for  $0 < \lambda \leq d^2$
- **radiation**: for  $d^2 < \lambda < n_0^2 k^2$ .
- **evanescent**: for  $\lambda > n_0^2 k^2$ .



# Eigenvalues problems for singular differential operators. Titchmarsh 1946.

Let  $\varphi_1(r, l)$  and  $\varphi_2(r, l)$  be the solution of

$$-(p(x)y')' + Q(x)y = ly, \quad l \in \mathbb{C},$$

with  $p$  absolutely continuous and  $Q$  locally integrable, which verifies the conditions:

$$\begin{aligned} \varphi_1(R, l) &= 0, & \varphi_1'(R, l) &= -1, \\ \varphi_2(R, l) &= 1, & \varphi_2'(R, l) &= 0. \end{aligned}$$

Let  $g \in L^2(0, \infty)$  and

$$\Gamma_1(\lambda) = \int_0^{\infty} \varphi_1(r, \lambda)g(r) dr, \quad \Gamma_2(\lambda) = \int_0^{\infty} \varphi_2(r, \lambda)g(r) dr.$$

There exist  $\xi(\lambda), \eta(\lambda)$  and  $\zeta(\lambda)$ ,  $\lambda \in \mathbb{R}$ , such that the following formulas hold: ▶

### Inversion transform formula

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \{ \varphi_1(r, \lambda) \Gamma_1(\lambda) d\xi(\lambda) + \varphi_1(r, \lambda) \Gamma_2(\lambda) d\eta(\lambda) \\ + \varphi_2(r, \lambda) \Gamma_1(\lambda) d\eta(\lambda) + \varphi_2(r, \lambda) \Gamma_2(\lambda) d\zeta(\lambda) \}$$

### Parseval identity

$$\int_0^{\infty} g(r)^2 dr = \frac{1}{\pi} \int_{-\infty}^{\infty} \{ \Gamma_1(\lambda)^2 d\xi(\lambda) \\ + 2\Gamma_1(\lambda)\Gamma_2(\lambda) d\eta(\lambda) + \Gamma_2(\lambda)^2 d\zeta(\lambda) \}.$$

## Titchmarsh theory in our case

In the case of (2), it is possible to find a positive measure  $\chi_m(\lambda)$  which simplifies the previous formulas. In particular, if we define

$$G_m(\lambda) = \int_0^{\infty} j_m(r, \lambda) g(r) dr,$$

then  $\forall g \in L^2(0, \infty)$  we have the following *inverse transform formula*

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r, \lambda) G_m(\lambda) d\chi_m(\lambda)$$

and *Parseval identity*

$$\int_0^{\infty} g(x)^2 dx = \frac{1}{\pi} \int_{-\infty}^{\infty} G_m(\lambda)^2 d\chi_m(\lambda).$$

## Properties of $\chi_m(\lambda)$

- $\chi_m(\lambda) = 0$  for  $\lambda \leq 0$ .
- For  $\lambda \in (0, d^2]$ ,  $\chi_m$  is piecewise constant, with discontinuities at exactly those  $\lambda_m^k$  for which  $j_m(\cdot, \lambda_m^k) \in L^2(0, \infty) \cap C^1(0, \infty)$ . The jumps:

$$\chi_m(\lambda_m^k + 0) - \chi_m(\lambda_m^k - 0) = \frac{\pi}{\|j(\cdot, \lambda_m^k)\|_{L^2}^2}.$$

- For  $\lambda > d^2$ , let  $c_m(\lambda)$  and  $d_m(\lambda)$  be such that for  $r \geq R$ 

$$j_m(r, \lambda) = c_m(\lambda)\sqrt{r}J_m(\sqrt{\lambda - d^2} r) + d_m(\lambda)\sqrt{r}Y_m(\sqrt{\lambda - d^2} r)$$
 ( $J_m$  and  $Y_m$  are Bessel's functions). Then:

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}.$$

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## Resolution formula

$$u(r, \vartheta, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\pi}^{\pi} G(r, \vartheta, z; \rho, \varphi, \zeta) f(\rho, \varphi, \zeta) \rho d\varphi d\rho d\zeta,$$

where

$$G(r, \rho; \vartheta, t; z, \zeta) =$$

$$\frac{1}{2\pi^2} \frac{1}{\sqrt{r\rho}} \sum_{m \in \mathbb{Z}} \int_0^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_0^2 - \lambda}}}{2i\sqrt{k^2 n_0^2 - \lambda}} e^{im(\vartheta-t)} j_m(\rho, \lambda) j_m(r, \lambda) d\chi_m(\lambda),$$

$$r, \rho > 0; -\pi \leq \vartheta, t \leq \pi; z, \zeta \in \mathbb{R}.$$

### Conclusion

We found a solution of the Helmholtz equation as a superposition of guided ( $0 < \lambda \leq d^2$ ), radiation ( $d^2 < \lambda \leq k^2 n_0^2$ ) and evanescent ( $\lambda > k^2 n_0^2$ ) modes. ▶