Wave propagation in optical waveguides. Numerical results.

Giulio Ciraolo

Dipartimento di Matematica "Ulisse Dini"
Università di Firenze

5^{th} ISAAC Congress, Catania, 2005

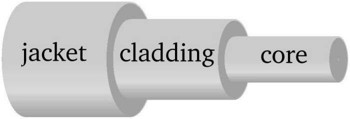
Outline

1. Rectilinear waveguides
   - Introduction to the problem
   - 2-D case
   - 3-D case
   - Numerical examples
2. Non-rectilinear waveguides
   - Physical motivations
   - Mathematical framework
   - Numerical simulations

Part I

Rectilinear waveguides

Optical waveguides



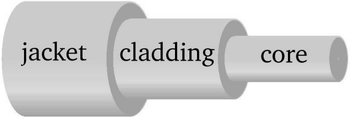
Notations

- n – index of refraction
- n_0 – maximum of n
- n_{cl} – *cladding*'s index of refraction
- k – wavenumber $(2\pi/\lambda)$

Kinds of energy

- Guided energy
- Radiation energy
- Evanescent energy

Optical waveguides



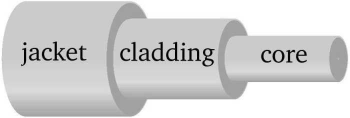
Notations

- n – index of refraction
- n_0 – maximum of n
- n_{cl} – *cladding*'s index of refraction
- k – wavenumber $(2\pi/\lambda)$

Kinds of energy

- Guided energy
- Radiation energy
- Evanescent energy

Optical waveguides



Notations

- n – index of refraction
- n_0 – maximum of n
- n_{cl} – *cladding*'s index of refraction
- k – wavenumber $(2\pi/\lambda)$

Kinds of energy

- Guided energy
- Radiation energy
- Evanescent energy

Mathematical model in 2-D and 3-D

Helmholtz equation

$$\Delta u + k^2 n(\mathbf{x})^2 u = f, \quad \mathbf{x} \in \mathbb{R}^N.$$

2-D

n even function,

$$n = \begin{cases} n_{co}(x_1) & \text{if } |x_1| \le h, \\ n_{cl} & \text{otherwise.} \end{cases}$$

3-D

n axially symmetric with respect of x_3,

$$n = \begin{cases} n_{co}(r) & \text{if } 0 \le r \le R, \\ n_{cl} & \text{otherwise,} \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

Mathematical model in 2-D and 3-D

Helmholtz equation

$$\Delta u + k^2 n(\mathbf{x})^2 u = f, \quad \mathbf{x} \in \mathbb{R}^N.$$

2-D

n even function,

$$n = \begin{cases} n_{co}(x_1) & \text{if } |x_1| \leq h, \\ n_{cl} & \text{otherwise.} \end{cases}$$

3-D

n axially symmetric with respect of x_3,

$$n = \begin{cases} n_{co}(r) & \text{if } 0 \leq r \leq R, \\ n_{cl} & \text{otherwise,} \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

2-D case (Magnanini, Santosa. 2000, SIAM J. Appl. Math.)

$$\Delta u+k^2 n(x_1)^2 u = f \quad \xrightarrow{u=v(x_1,\lambda)e^{ik\beta x_2}} \quad v''+[\lambda-q(x_1)]v = 0, \text{ in } \mathbb{R}.$$

- $v'(x_1,\lambda) = \frac{\partial v(x_1,\lambda)}{\partial x_1}$
- $n_0 = \max n(x_1)$
- $q(x_1) = k^2[n_0^2 - n(x_1)^2]$
- $d^2 = k^2(n_0^2 - n_{cl}^2)$
- $\lambda = k^2(n_0^2 - \beta^2)$
- $Q = \sqrt{\lambda - d^2}$

Solutions

$$v_j(x_1,\lambda) = \begin{cases} \phi_j(h,\lambda)\cos Q(x_1-h) + \frac{\phi_j'(h,\lambda)}{Q}\sin Q(x_1-h), & x_1 > h, \\ \phi_j(x_1,\lambda), & |x_1| \le h, \\ \phi_j(-h,\lambda)\cos Q(x_1+h) + \frac{\phi_j'(-h,\lambda)}{Q}\sin Q(x_1+h), & x_1 < -h, \end{cases}$$

$j = s, a.$

2-D case (Magnanini, Santosa. 2000, SIAM J. Appl. Math.)

$$\Delta u + k^2 n(x_1)^2 u = f \quad \xrightarrow{u=v(x_1,\lambda)e^{ik\beta x_2}} \quad v'' + [\lambda - q(x_1)]v = 0, \text{ in } \mathbb{R}.$$

- $v'(x_1,\lambda) = \frac{\partial v(x_1,\lambda)}{\partial x_1}$
- $n_0 = \max n(x_1)$
- $q(x_1) = k^2[n_0^2 - n(x_1)^2]$
- $d^2 = k^2(n_0^2 - n_{cl}^2)$
- $\lambda = k^2(n_0^2 - \beta^2)$
- $Q = \sqrt{\lambda - d^2}$

Solutions

$$v_j(x_1,\lambda) = \begin{cases} \phi_j(h,\lambda)\cos Q(x_1-h) + \frac{\phi_j'(h,\lambda)}{Q}\sin Q(x_1-h), & x_1 > h, \\ \phi_j(x_1,\lambda), & |x_1| \le h, \\ \phi_j(-h,\lambda)\cos Q(x_1+h) + \frac{\phi_j'(-h,\lambda)}{Q}\sin Q(x_1+h), & x_1 < -h, \end{cases}$$

$j = s, a.$

Classification of solutions

Modes

Modes are solutions of Helmholtz equation of the form $v(x_1,\lambda)e^{ik\beta x_2}$, with $\beta k = \sqrt{k^2 n_0^2 - \lambda}$.

- **guided**: for $0 < \lambda < d^2$. Guided modes (and then the energy) propagate mainly inside the core and vanish exponentially outside. They are *finite* in number.
- **radiation**: for $d^2 < \lambda < n_0^2 k^2$. Energy doesn't localize inside the core but propagates mainly in the cladding.
- **evanescent**: for $\lambda > n_0^2 k^2$. The *constant of propagation* β becomes imaginary. These modes have an exponential decay in one of the directions along the axis $x_2 \to \pm\infty$, while they grow exponentially along the other one.

Classification of solutions

Modes

Modes are solutions of Helmholtz equation of the form $v(x_1, \lambda)e^{ik\beta x_2}$, with $\beta k = \sqrt{k^2 n_0^2 - \lambda}$.

- **guided**: for $0 < \lambda < d^2$. Guided modes (and then the energy) propagate mainly inside the core and vanish exponentially outside. They are *finite* in number.
- **radiation**: for $d^2 < \lambda < n_0^2 k^2$. Energy doesn't localize inside the core but propagates mainly in the cladding.
- **evanescent**: for $\lambda > n_0^2 k^2$. The *constant of propagation* β becomes imaginary. These modes have an exponential decay in one of the directions along the axis $x_2 \to \pm\infty$, while they grow exponentially along the other one.

Classification of solutions

Modes

Modes are solutions of Helmholtz equation of the form $v(x_1, \lambda)e^{ik\beta x_2}$, with $\beta k = \sqrt{k^2 n_0^2 - \lambda}$.

- **guided**: for $0 < \lambda < d^2$. Guided modes (and then the energy) propagate mainly inside the core and vanish exponentially outside. They are *finite* in number.
- **radiation**: for $d^2 < \lambda < n_0^2 k^2$. Energy doesn't localize inside the core but propagates mainly in the cladding.
- **evanescent**: for $\lambda > n_0^2 k^2$. The *constant of propagation* β becomes imaginary. These modes have an exponential decay in one of the directions along the axis $x_2 \to \pm\infty$, while they grow exponentially along the other one.

Classification of solutions

Modes

Modes are solutions of Helmholtz equation of the form $v(x_1, \lambda)e^{ik\beta x_2}$, with $\beta k = \sqrt{k^2 n_0^2 - \lambda}$.

- **guided**: for $0 < \lambda < d^2$. Guided modes (and then the energy) propagate mainly inside the core and vanish exponentially outside. They are *finite* in number.
- **radiation**: for $d^2 < \lambda < n_0^2 k^2$. Energy doesn't localize inside the core but propagates mainly in the cladding.
- **evanescent**: for $\lambda > n_0^2 k^2$. The *constant of propagation* β becomes imaginary. These modes have an exponential decay in one of the directions along the axis $x_2 \rightarrow \pm\infty$, while they grow exponentially along the other one.

Resolution formula

M-S find a solution (in the sense of distributions) of the Helmholtz equation as superposition of guided, radiation and evanescent modes:

$$u(x) = \int_{\mathbb{R}^2} G(x,y) f(y) dy,$$

where

$$G(x,y) = \sum_{j\in\{s,a\}} \int_0^\infty \frac{e^{i|x_2-y_2|\sqrt{k^2 n_0^2-\lambda}}}{2i\sqrt{k^2 n_0^2-\lambda}} v_j(x_1,\lambda) v_j(y_1,\lambda) d\rho_j(\lambda),$$

with

$$\langle d\rho_j, \eta\rangle = \sum_{m=1}^{M_j} r_j^m \eta(\lambda_j^m) + \frac{1}{2\pi}\int_{d^2}^\infty \frac{\sqrt{\lambda-d^2}}{(\lambda-d^2)\phi_j(h,\lambda)^2+\phi_j'(h,\lambda)^2} \eta(\lambda) d\lambda, \quad \forall \eta \in C_0^\infty([0,+\infty)).$$

Resolution formula

M-S find a solution (in the sense of distributions) of the Helmholtz equation as superposition of guided, radiation and evanescent modes:

$$u(x) = \int_{\mathbb{R}^2} G(x,y) f(y) dy,$$

where

$$G(x,y)=\sum_{j\in\{s,a\}}\int_0^\infty \frac{e^{i|x_2-y_2|\sqrt{k^2n_0^2-\lambda}}}{2i\sqrt{k^2n_0^2-\lambda}} v_j(x_1,\lambda)v_j(y_1,\lambda)d\rho_j(\lambda),$$

with

$$\langle d\rho_j,\eta\rangle=\sum_{m=1}^{M_j} r_j^m\eta(\lambda_j^m)+\frac{1}{2\pi}\int_{d^2}^\infty \frac{\sqrt{\lambda-d^2}}{(\lambda-d^2)\phi_j(h,\lambda)^2+\phi_j'(h,\lambda)^2}\eta(\lambda)d\lambda, \quad \forall\eta\in C_0^\infty([0,+\infty)).$$

3-D case. (Alexandrov, Ciraolo. 2004, M3AS, 14, no.6, 819–852)

By using the cylindrical symmetry with respect to the x_3-axis (the fiber's axis), we find:

$$\Delta u + k^2 n(r)^2 u = f \quad \xrightarrow{u=e^{i\beta k x_3}\, e^{im\vartheta}\, \frac{w(r)}{\sqrt{r}}} \quad w'' + \left[\lambda - q(r) - \frac{m^2-1/4}{r^2}\right] w = 0, \;\; r>0.$$

Differences from the 2-D case

- The term $\frac{m^2-1/4}{r^2}$, makes the equation singular in $r=0$, besides in $r=\infty$.
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in $r=0$.
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

3-D case. (Alexandrov, Ciraolo. 2004, M3AS, 14, no.6, 819–852)

By using the cylindrical symmetry with respect to the x_3-axis (the fiber's axis), we find:

$$\Delta u+k^2 n(r)^2 u=f \xrightarrow{u=e^{i\beta k x_3}\, e^{im\vartheta}\, \frac{w(r)}{\sqrt{r}}} w''+\left[\lambda-q(r)-\frac{m^2-1/4}{r^2}\right]w=0,\ \ r>0.$$

Differences from the 2-D case

- The term $\frac{m^2-1/4}{r^2}$, makes the equation singular in $r=0$, besides in $r=\infty$.
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in $r=0$.
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

3-D case. (Alexandrov, Ciraolo. 2004, M3AS, 14, no.6, 819–852)

By using the cylindrical symmetry with respect to the x_3-axis (the fiber's axis), we find:

$$\Delta u + k^2 n(r)^2 u = f \quad \xrightarrow{u=e^{i\beta k x_3}\, e^{im\vartheta}\, \frac{w(r)}{\sqrt{r}}} \quad w'' + \left[\lambda - q(r) - \frac{m^2-1/4}{r^2}\right] w = 0, \;\; r>0.$$

Differences from the 2-D case

- The term $\frac{m^2-1/4}{r^2}$, makes the equation singular in $r=0$, besides in $r=\infty$.
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in $r=0$.
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

3-D case. (Alexandrov, Ciraolo. 2004, M3AS, 14, no.6, 819–852)

By using the cylindrical symmetry with respect to the x_3-axis (the fiber's axis), we find:

$$\Delta u + k^2 n(r)^2 u = f \quad \xrightarrow{u = e^{i\beta k x_3} e^{im\vartheta} \frac{w(r)}{\sqrt{r}}} \quad w'' + \left[\lambda - q(r) - \frac{m^2 - 1/4}{r^2}\right] w = 0, \;\; r > 0.$$

Differences from the 2-D case

- The term $\frac{m^2-1/4}{r^2}$, makes the equation singular in $r = 0$, besides in $r = \infty$.
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in $r = 0$.
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

3-D case. (Alexandrov, Ciraolo. 2004, M3AS, 14, no.6, 819–852)

By using the cylindrical symmetry with respect to the x_3-axis (the fiber's axis), we find:

$$\Delta u + k^2 n(r)^2 u = f \quad \xrightarrow{u = e^{i\beta k x_3} e^{im\vartheta} \frac{w(r)}{\sqrt{r}}} \quad w'' + \left[\lambda - q(r) - \frac{m^2 - 1/4}{r^2}\right] w = 0, \ r > 0.$$

Differences from the 2-D case

- The term $\frac{m^2-1/4}{r^2}$, makes the equation singular in $r = 0$, besides in $r = \infty$.
- Technical difficulties in adapting the method of M-S.
- We need a study of the behaviour of the solutions in $r = 0$.
- We use the theory of Titchmarsh on the eigenvalues problems for singular differential operators (not trivial to be applied).

Behaviour of the solutions in $r = 0$

Lemma. Let $q \in L^\infty(0,\infty)$. For m fixed, there exists a base of solutions $j_m(r,\lambda)$ and $y_m(r,\lambda)$ of

$$w'' + \left[\lambda - q(r) - \frac{m^2-1/4}{r^2}\right] w = 0, \;\; r>0, \; \lambda \in \mathbb{C}, \; m \in \mathbb{Z} \qquad (1)$$

such that

$$\lim_{r\to 0} \frac{j_m(r,\lambda)}{r^{|m|+1/2}} = 1, \quad \lim_{r\to 0} \frac{j_m'(r,\lambda)}{(|m|+1/2)\, r^{|m|-1/2}} = 1.$$

and

$$\lim_{r\to 0} \frac{y_m(r,\lambda)}{r^{-|m|+1/2}} = 1, \quad \lim_{r\to 0} \frac{y_m'(r,\lambda)}{(-|m|+1/2)\, r^{-|m|-1/2}} = 1, \;\text{ if } |m| \geq 1,$$

or

$$\lim_{r\to 0} \frac{y_m(r,\lambda)}{\sqrt{r}\ln r} = 1, \quad \lim_{r\to 0} \frac{y_m'(r,\lambda)}{\ln r/(2\sqrt{r})} = 1, \;\text{ if } m = 0.$$

Furthermore the functions $j_m(r,\lambda)$ and $j_m'(r,\lambda)$ are analytical in λ for r fixed.

Behaviour of the solutions as $r \longrightarrow \infty$

If $r \in [R, +\infty)$, equation (1) is

$$w'' + \left\{ \lambda - d^2 - \frac{m^2 - 1/4}{r^2} \right\} w = 0.$$

Solutions are linear combinations of

$$\sqrt{r} H_m^{(1)}(r\sqrt{\lambda - d^2}), \quad \sqrt{r} H_m^{(2)}(r\sqrt{\lambda - d^2}),$$

where $H^{(1)}$ and $H^{(2)}$ are Hankel's functions.

We look for C^1 and bounded solutions

- $\lambda < d^2$: $w(r, \lambda)/\sqrt{r}$ vanishes exponentially;
- $\lambda > d^2$: $w(r, \lambda)/\sqrt{r}$ oscillates and decays slowly.

Behaviour of the solutions as $r \longrightarrow \infty$

If $r \in [R, +\infty)$, equation (1) is

$$w'' + \left\{ \lambda - d^2 - \frac{m^2 - 1/4}{r^2} \right\} w = 0.$$

Solutions are linear combinations of

$$\sqrt{r} H_m^{(1)}(r\sqrt{\lambda - d^2}), \quad \sqrt{r} H_m^{(2)}(r\sqrt{\lambda - d^2}),$$

where $H^{(1)}$ and $H^{(2)}$ are Hankel's functions.

We look for C^1 and bounded solutions

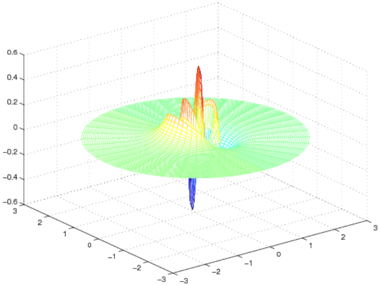
- $\lambda < d^2$: $w(r, \lambda)/\sqrt{r}$ vanishes exponentially;
- $\lambda > d^2$: $w(r, \lambda)/\sqrt{r}$ oscillates and decays slowly.

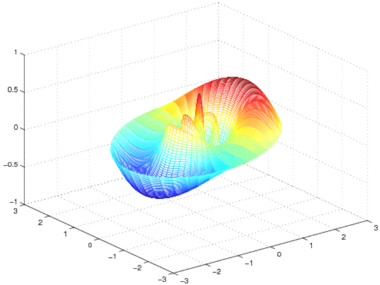
Classification of solutions

The modes corresponds to the bounded solutions of (1)

As in the 2-D case we find the following classification:

- **guided**: for $0 < \lambda \leq d^2$
- **radiation**: for $d^2 < \lambda < n_0^2 k^2$.
- **evanescent**: for $\lambda > n_0^2 k^2$.





Eigenvalues problems for singular differential operators. Titchmarsh 1946.

Let $\varphi_1(r,l)$ and $\varphi_2(r,l)$ be the solution of

$$-(p(x)y')' + Q(x)y = ly, \quad l \in \mathbb{C},$$

with p absolutely continuous and Q locally integrable, which verifies the conditions:

$$\begin{aligned} \varphi_1(R,l) &= 0, & \varphi_1'(R,l) &= -1, \\ \varphi_2(R,l) &= 1, & \varphi_2'(R,l) &= 0. \end{aligned}$$

Let $g \in L^2(0,\infty)$ and

$$\Gamma_1(\lambda) = \int_0^\infty \varphi_1(r,\lambda) g(r)\, dr, \quad \Gamma_2(\lambda) = \int_0^\infty \varphi_2(r,\lambda) g(r)\, dr.$$

There exsist $\xi(\lambda), \eta(\lambda)$ and $\zeta(\lambda)$, $\lambda \in \mathbb{R}$, such that the following formulas hold:

Inversion transform formula

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \{\varphi_1(r,\lambda)\Gamma_1(\lambda)\, d\xi(\lambda) + \varphi_1(r,\lambda)\Gamma_2(\lambda)\, d\eta(\lambda) + \varphi_2(r,\lambda)\Gamma_1(\lambda)\, d\eta(\lambda) + \varphi_2(r,\lambda)\Gamma_2(\lambda)\, d\zeta(\lambda)\}$$

Parseval indentity

$$\int_0^{\infty} g(r)^2 dr = \frac{1}{\pi} \int_{-\infty}^{\infty} \{\Gamma_1(\lambda)^2 d\xi(\lambda) + 2\Gamma_1(\lambda)\Gamma_2(\lambda) d\eta(\lambda) + \Gamma_2(\lambda)^2 d\zeta(\lambda)\}.$$

Titchmarsh theory in our case

In the case of (1), it is possible to find a positive measure $\chi_m(\lambda)$ which simplifies the previous formulas. In particular, if we define

$$G_m(\lambda) = \int_0^{\infty} j_m(r,\lambda) g(r)\, dr,$$

then $\forall g \in L^2(0,\infty)$ we have the following *inverse transform formula*

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r,\lambda) G_m(\lambda)\, d\chi_m(\lambda)$$

and *Parseval identity*

$$\int_0^{\infty} g(x)^2 dx = \frac{1}{\pi} \int_{-\infty}^{\infty} G_m(\lambda)^2 d\chi_m(\lambda).$$

Titchmarsh theory in our case

In the case of (1), it is possible to find a positive measure $\chi_m(\lambda)$ which simplifies the previous formulas. In particular, if we define

$$G_m(\lambda) = \int_0^\infty j_m(r,\lambda) g(r)\, dr,$$

then $\forall g \in L^2(0,\infty)$ we have the following *inverse transform formula*

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r,\lambda) G_m(\lambda)\, d\chi_m(\lambda)$$

and *Parseval identity*

$$\int_0^\infty g(x)^2 dx = \frac{1}{\pi} \int_{-\infty}^{\infty} G_m(\lambda)^2 d\chi_m(\lambda).$$

Properties of $\chi_m(\lambda)$

- $\chi_m(\lambda) = 0$ for $\lambda \leq 0$.
- For $\lambda \in (0, d^2]$, χ_m is piecewise constant, with discontinuities at exactly those λ_m^k for which $j_m(\cdot, \lambda_m^k) \in L^2(0,\infty) \cap C^1(0,\infty)$. The jumps:

$$\chi_m(\lambda_m^k + 0) - \chi_m(\lambda_m^k - 0) = \frac{\pi}{||j(\cdot, \lambda_m^k)||_{L^2}^2}.$$

- For $\lambda > d^2$, let $c_m(\lambda)$ and $d_m(\lambda)$ be such that for $r \geq R$

$$j_m(r,\lambda) = c_m(\lambda)\sqrt{r} J_m(\sqrt{\lambda - d^2}\, r) + d_m(\lambda)\sqrt{r} Y_m(\sqrt{\lambda - d^2}\, r)$$

(J_m and Y_m are Bessel's functions). Then:

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}.$$

Properties of $\chi_m(\lambda)$

- $\chi_m(\lambda) = 0$ for $\lambda \leq 0$.
- For $\lambda \in (0, d^2]$, χ_m is piecewise constant, with discontinuities at exactly those λ_m^k for which $j_m(\cdot, \lambda_m^k) \in L^2(0, \infty) \cap C^1(0, \infty)$. The jumps:

$$\chi_m(\lambda_m^k + 0) - \chi_m(\lambda_m^k - 0) = \frac{\pi}{||j(\cdot, \lambda_m^k)||_{L^2}^2}.$$

- For $\lambda > d^2$, let $c_m(\lambda)$ and $d_m(\lambda)$ be such that for $r \geq R$

$$j_m(r, \lambda) = c_m(\lambda)\sqrt{r} J_m\left(\sqrt{\lambda - d^2}\, r\right) + d_m(\lambda)\sqrt{r}\, Y_m\left(\sqrt{\lambda - d^2}\, r\right)$$

(J_m and Y_m are Bessel's functions). Then:

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}.$$

Resolution formula

$$u(r,\vartheta,z)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\pi}^{\pi}G(r,\vartheta,z;\rho,\varphi,\zeta)f(\rho,\varphi,\zeta)\rho d\varphi d\rho d\zeta,$$

where

$$G(r,\rho;\vartheta,t;z,\zeta)=$$

$$\frac{1}{2\pi^2}\frac{1}{\sqrt{r\rho}}\sum_{m\in\mathbb{Z}}\int_{0}^{+\infty}\frac{e^{i|z-\zeta|\sqrt{k^2n_0^2-\lambda}}}{2i\sqrt{k^2n_0^2-\lambda}}e^{im(\vartheta-t)}\,j_m(\rho,\lambda)\,j_m(r,\lambda)\,d\chi_m(\lambda),$$

$$r,\rho>0;\ -\pi\le\vartheta,t\le\pi;\ z,\zeta\in\mathbb{R}.$$

Conclusion

We found a solution of the Helmholtz equation as a superposition of guided ($0<\lambda\le d^2$), radiation ($d^2<\lambda\le k^2n_0^2$) and evanescent ($\lambda>k^2n_0^2$) modes.

Uniqueness

$$\Delta u + k^2 n(x)^2 u = f, \quad \text{in } \mathbb{R}^2 \text{ or } \mathbb{R}^3.$$

We have *a good solution*:

$$G = G^g + G^r + G^e = \sum \ldots + \int_{d^2}^{k^2 n_0^2} \ldots + \int_{k^2 n_0^2}^{\infty} \ldots$$

but have not a unique solution!

- n is not constant outside any compact set.
- Pointwise spectrum → difficult to give a radiation condition analogous to the one of Sommerfeld (cfr. Rellich, Miranker, Jäger & Saitō, Costabel & Dauge, Perthame & Vega..)
- Ciraolo-Magnanini in 2-D and Alexandrov in 3-D proved that the Green's function is *outgoing*, i.e. it creates waves propagating towards infinity.

Uniqueness

$$\Delta u + k^2 n(x)^2 u = f, \quad \text{in } \mathbb{R}^2 \text{ or } \mathbb{R}^3.$$

We have *a good solution*:

$$G = G^g + G^r + G^e = \sum \dots + \int_{d^2}^{k^2 n_0^2} \dots + \int_{k^2 n_0^2}^{\infty} \dots$$

but have not a unique solution!

- n is not constant outside any compact set.
- Pointwise spectrum → difficult to give a radiation condition analogous to the one of Sommerfeld (cfr. Rellich, Miranker, Jäger & Saitō, Costabel & Dauge, Perthame & Vega..)
- Ciraolo-Magnanini in 2-D and Alexandrov in 3-D proved that the Green's function is *outgoing*, i.e. it creates waves propagating towards infinity.

Step-index fiber

The index of the core is constant ⟶ *explicit formulas*

Jumps

- We cannot apply the abstract formula to find the jumps at λ_m^k in a concrete form.
- We use integration in the complex plane and the Residues Theorem instead.

$$r_m^k = \frac{2\lambda_m^k(d^2 - \lambda_m^k)}{d^2\left[\lambda_m^k R^2 J_m'(\sqrt{\lambda_m^k}\,R)^2 - m^2\,J_m(\sqrt{\lambda_m^k}\,R)^2\right]}$$

Step-index fiber

The index of the core is constant ⟶ *explicit formulas*

The continuous part of χ_m $(\lambda > d^2)$

Easy to compute from the abstract formula:

$$d\chi_m(\lambda) = \frac{2}{\pi} \frac{d\lambda}{U_m(\lambda)^2 + W_m(\lambda)^2},$$

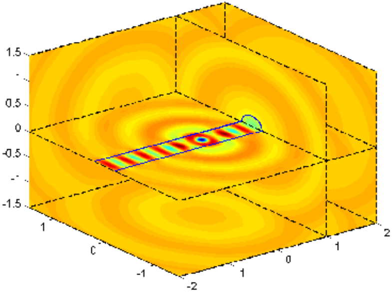
with

$$U_m(\lambda) = QR\, J_m(R\sqrt{\lambda}) Y_m'(QR) - R\sqrt{\lambda}\, J_m'(R\sqrt{\lambda}) Y_m(QR),$$
$$W_m(\lambda) = R\sqrt{\lambda} J_m'(R\sqrt{\lambda}) J_m(QR) - QR\, J_m'(QR) J_m(R\sqrt{\lambda}),$$

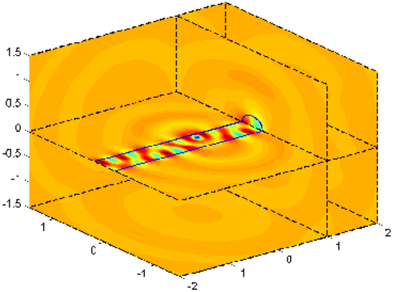
where R is the fiber radius and $Q = \sqrt{\lambda - d^2}$.

Step-index fiber. Figures.

Real part of the Green's function with source on the fiber's axis.



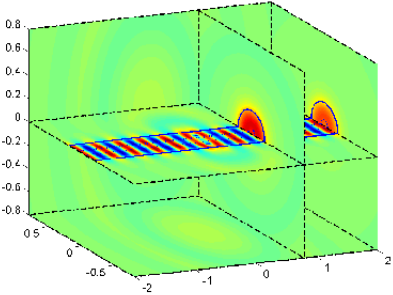
Real part of the Green's function with source not on the fiber's axis.



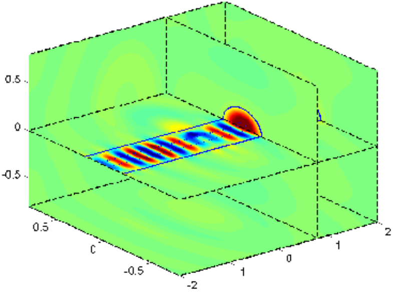
Coaxial cable

$$n(r) = \begin{cases} n_{cl}, & 0 < r < a, \\ n_{co}, & a \leq r < R, \\ n_{cl}, & r \geq R. \end{cases}$$

Source on the fiber's axis.



Source not on the fiber's axis.



Part II

Non-rectilinear waveguides (Ciraolo, Magnanini)

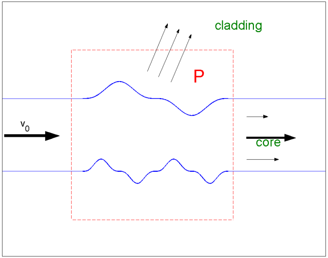
Motivations

Small imperfections can affect wave propagation:

- Signal distortion
- Loss in the signal power

⇓

- Mode coupling
- More radiation and evanescent energy



...not always to be avoided...

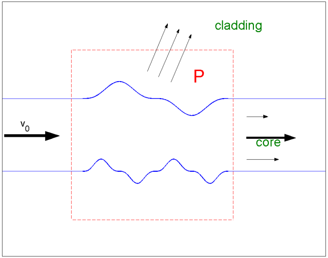
Motivations

Small imperfections can affect wave propagation:

- Signal distortion
- Loss in the signal power

⇓

- Mode coupling
- More radiation and evanescent energy



...not always to be avoided...

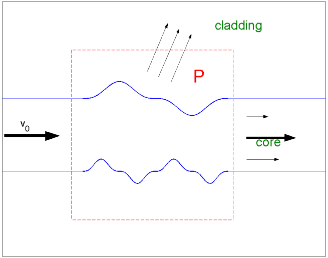
Motivations

Small imperfections can affect wave propagation:

- Signal distortion
- Loss in the signal power

$\Downarrow$

- Mode coupling
- More radiation and evanescent energy



...not always to be avoided...

Mathematical Framework (in $\mathbb{R}^2$)

$$L_0 = \Delta + k^2 n_0^2(x_1), \quad L_\varepsilon = \Delta + k^2 n_\varepsilon^2(x_1, x_2).$$

Non-rectilinear waveguides $\longrightarrow L_\varepsilon u = f$.

We formally represent u and L_ε by their Neumann series:

$$u = u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots; \quad L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \dots,$$

we have

$$L_0 u_0 = f, \; L_0 u_1 = -L_1 u_0, \dots, L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \; \dots$$

We solve each step by using L_0^{-1}.

Mathematical Framework (in $\mathbb{R}^2$)

$$L_0 = \Delta + k^2 n_0^2(x_1), \quad L_\varepsilon = \Delta + k^2 n_\varepsilon^2(x_1, x_2).$$

Non-rectilinear waveguides $\longrightarrow$ $L_\varepsilon u = f$.

We formally represent u and L_ε by their Neumann series:

$$u = u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots; \quad L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \ldots,$$

we have

$$L_0 u_0 = f, \; L_0 u_1 = -L_1 u_0, \ldots, L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \; \ldots$$

We solve each step by using L_0^{-1}.

Mathematical Framework (in $\mathbb{R}^2$)

$$L_0 = \Delta + k^2 n_0^2(x_1), \quad L_\varepsilon = \Delta + k^2 n_\varepsilon^2(x_1, x_2).$$

Non-rectilinear waveguides $\longrightarrow$ $L_\varepsilon u = f$.

We formally represent u and L_ε by their Neumann series:

$$u = u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots; \quad L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \ldots,$$

we have

$$L_0 u_0 = f,\ L_0 u_1 = -L_1 u_0, \ldots, L_0 u_j = -\sum_{r=0}^{j-1} L_{j-r} u_r, \ \ldots$$

We solve each step by using L_0^{-1}.

Existence of a solution

Problem

$$L_\varepsilon u = f$$
$$\downarrow$$
$$L_0 u = f + (L_0 - L_\varepsilon)u$$
$$\downarrow$$
$$u = L_0^{-1} f + \varepsilon L_0^{-1}\left(\frac{L_0 - L_\varepsilon}{\varepsilon}\right) u.$$

Strategy

- Find X_1 and X_2 such that
$$L_0^{-1} : X_2 \to X_1,$$
$$\frac{L_0 - L_\varepsilon}{\varepsilon} : X_1 \to X_2$$
are continuous.
- $\varepsilon > 0$ small enough $\to$ contraction mapping theorem.

Spaces: $\begin{cases} X_1 = H^2(\mathbb{R}^2, \mu), \\ X_2 = L^2(\mathbb{R}^2, \mu^{-1}), \end{cases}$ where $\mu(x) = \dfrac{16}{(4 + |x|^2)^2}$.

Existence of a solution

Problem

$$L_\varepsilon u = f$$
$$\downarrow$$
$$L_0 u = f + (L_0 - L_\varepsilon) u$$
$$\downarrow$$
$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left(\frac{L_0 - L_\varepsilon}{\varepsilon} \right) u.$$

Strategy

- Find X_1 and X_2 such that

$$L_0^{-1} : X_2 \to X_1, \qquad \frac{L_0 - L_\varepsilon}{\varepsilon} : X_1 \to X_2$$

  are continuous.
- $\varepsilon > 0$ small enough → contraction mapping theorem.

Spaces: $\begin{cases} X_1 = H^2(\mathbb{R}^2, \mu), \\ X_2 = L^2(\mathbb{R}^2, \mu^{-1}), \end{cases}$ where $\mu(x) = \dfrac{16}{(4+|x|^2)^2}$.

Existence of a solution

Problem

$$L_\varepsilon u = f$$

$$\downarrow$$

$$L_0 u = f + (L_0 - L_\varepsilon) u$$

$$\downarrow$$

$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left(\frac{L_0 - L_\varepsilon}{\varepsilon} \right) u.$$

Strategy

- Find X_1 and X_2 such that

$$L_0^{-1} : X_2 \to X_1, \quad \frac{L_0 - L_\varepsilon}{\varepsilon} : X_1 \to X_2$$

  are continuous.
- $\varepsilon > 0$ small enough → contraction mapping theorem.

Spaces: $$\begin{cases} X_1 = H^2(\mathbb{R}^2, \mu), \\ X_2 = L^2(\mathbb{R}^2, \mu^{-1}), \end{cases} \quad \text{where } \mu(x) = \frac{16}{(4 + |x|^2)^2}.$$

Remarks on the proof

- Uniform asymptotic expansion of the eigenfunctions and their first derivatives *inside the core*. If $q \in L^1_{loc}(\mathbb{R})$, $x \in [-h, h]$:

$$\phi_s(x,\lambda) = \cos(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\},$$
$$\phi_a(x,\lambda) = \sin(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\},$$

  as $\lambda \to +\infty$.

- $G(x_1, x_2; y_1, y_2) \in L^2(\mathbb{R}^2, \mu) \times L^2(\mathbb{R}^2, \mu)$. In particular, G^g and G^r are bounded.
- Global estimates on first and second derivatives:

$$\begin{cases} u \in L^2(\mathbb{R}^2, \mu), \\ f \in L^2(\mathbb{R}^2, \mu^{-1}), \end{cases} \xrightarrow{\text{standard techniques}} u \in H^2(\mathbb{R}^2, \mu).$$

Remarks on the proof

- Uniform asymptotic expansion of the eigenfunctions and their first derivatives *inside the core*. If $q \in L^1_{loc}(\mathbb{R})$, $x \in [-h, h]$:

$$\phi_s(x,\lambda) = \cos(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\},$$
$$\phi_a(x,\lambda) = \sin(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\},$$

  as $\lambda \to +\infty$.

- $G(x_1, x_2; y_1, y_2) \in L^2(\mathbb{R}^2, \mu) \times L^2(\mathbb{R}^2, \mu)$. In particular, G^g and G^r are bounded.

- Global estimates on first and second derivatives:

$$\begin{cases} u \in L^2(\mathbb{R}^2, \mu), \\ f \in L^2(\mathbb{R}^2, \mu^{-1}), \end{cases} \xrightarrow{\text{standard techniques}} u \in H^2(\mathbb{R}^2, \mu).$$

Remarks on the proof

- Uniform asymptotic expansion of the eigenfunctions and their first derivatives *inside the core*. If $q \in L^1_{loc}(\mathbb{R}),\ x \in [-h, h]$:

$$\begin{aligned} \phi_s(x,\lambda) &= \cos(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\}, \\ \phi_a(x,\lambda) &= \sin(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\lambda^{-1}\right)\right\}, \end{aligned}$$

as $\lambda \to +\infty$.

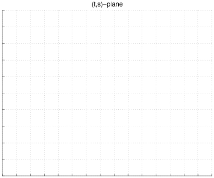
- $G(x_1, x_2; y_1, y_2) \in L^2(\mathbb{R}^2, \mu) \times L^2(\mathbb{R}^2, \mu)$. In particular, G^g and G^r are bounded.
- Global estimates on first and second derivatives:

$$\begin{cases} u \in L^2(\mathbb{R}^2, \mu), \\ f \in L^2(\mathbb{R}^2, \mu^{-1}), \end{cases} \xrightarrow{\text{standard techniques}} u \in H^2(\mathbb{R}^2, \mu).$$

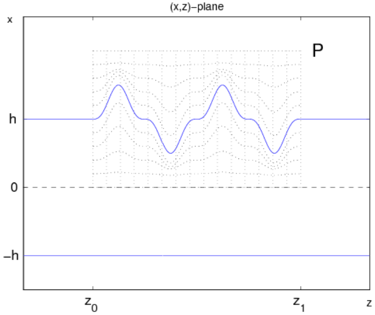
Numerical simulations

$$\Gamma_\varepsilon(t,s) = \begin{cases} x = t + \varepsilon T(t)S(s), \\ z = s, \end{cases} \qquad \Gamma_\varepsilon \in C^2(\mathbb{R}^2), \Gamma_\varepsilon \equiv Id_{\mathbb{R}^2} \text{ in } \mathbb{R}^2 \setminus P.$$

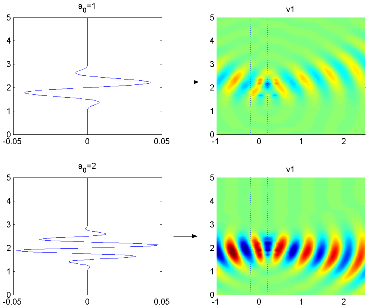
$$\tilde{L}_\varepsilon v = g \qquad \Delta u + k^2 n(x,z)^2 u = f$$

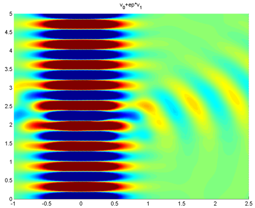


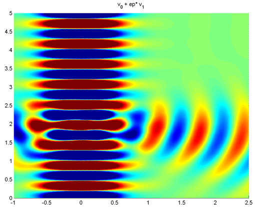
$$\xrightarrow{(x,z)=\Gamma_\varepsilon(t,s)}$$



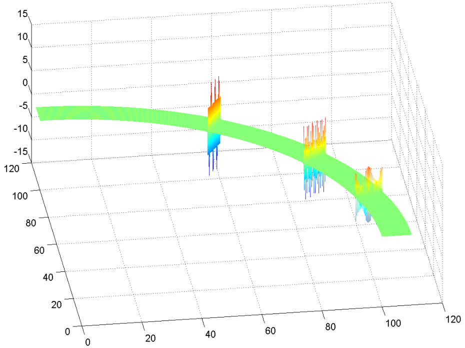
Near Field

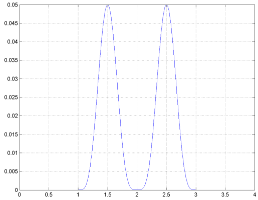

a$_0$=1
5
4
3
2
1
0
-0.05
0
0.05
v1
5
4
3
2
1
0
-1
0
1
2
a$_0$=2
5
4
3
2
1
0
-0.05
0
0.05
v1
5
4
3
2
1
0
-1
0
1
2

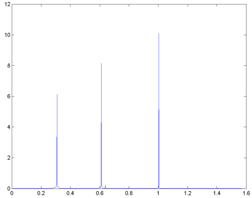

v$_0$+ep*v$_1$
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1
1.5
2
2.5


v$_0$ + ep* v$_1$
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1
1.5
2
2.5

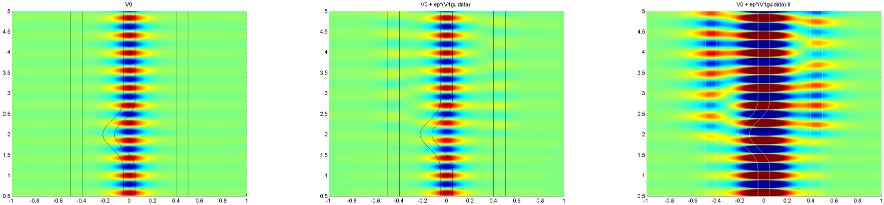
Far Field


15
10
5
0
-5
-10
-15
120
100
80
60
40
20
0
0
20
40
60
80
100
120


0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.5
1
1.5
2
2.5
3
3.5
4


12
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Mode coupling



Details of the guided part of v_1: