Symmetry of minimizers with a level surface parallel to the boundary

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Spherical symmetry of minimizers

Let

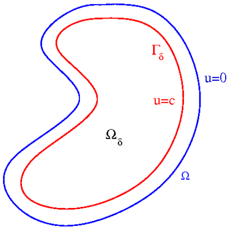
- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded domain with C^2 boundary
- *Distance from the boundary*: $d(x) = \inf_{y \in \partial\Omega} |x - y|$, $\quad x \in \overline{\Omega}$
- $\Omega_\delta = \{x \in \Omega : \; d(x) > \delta\}$
- $\Gamma_\delta = \{x \in \Omega : \; d(x) = \delta\}$. Γ_δ is *parallel* to $\partial\Omega$.

Assume that

- u minimizes $J(v) = \int_\Omega [f(|Dv|) - v] dx, \quad v \in W_0^{1,\infty}(\Omega)$
- $u = c$ on Γ_δ for some $c > 0$.

Problem

Can we conclude that Ω is a ball?



Main result: symmetry for minimizers of J

$$J(v) = \int_\Omega [f(|Dv|) - v]dx, \qquad v \in W_0^{1,\infty}(\Omega).$$

(f1) $f \in C^1([0,+\infty))$ convex, monotone nondecrasing, such that $f(0) = 0$ and $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$;

(f2) $\exists \alpha \geq 0$ s.t. $f \in C^3(\alpha, +\infty)$ and

- $0 \leq s \leq \alpha : \quad f'(s) = 0$
- $s > \alpha \; : \quad f'(s) > 0$ and $f''(s) > 0$;

Theorem

Let u be a minimizer of J, with f satisfying (f1) and (f2). If

- *u has a level surface Γ_δ parallel to $\partial\Omega$, with $\delta < R_\Omega$,*
- *u is C^1 in some open neighborhood A_δ of Γ_δ,*

then Ω is a ball.

$$R_\Omega = \min_{y\in\partial\Omega} \{R > 0 : \; B_R \subset \Omega \text{ is the largest ball tangent to } \partial\Omega \text{ in y}\}.$$

Notice that in some cases the second assumption on u can be removed.

Motivations: Crasta[JEMS 06] – *web functions*

Consider

$$J(v) = \int_\Omega [f(|Dv|) - v]dx, \;\; v \in W_0^{1,1}(\Omega).$$

Assumptions on f

- $f : [0, b) \to \mathbb{R}, \;\; b \in (0, +\infty]$, is a convex, differentiable, monotone nondecreasing function;
- if $0 < b < +\infty$, then $\lim_{s \to b^-} f(s) = +\infty$;

  if $b = +\infty$, then $\lim_{s \to +\infty} \frac{f(s)}{s} > \frac{1}{n} \left(\frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}}$;
- $f'_+(0) = 0$.

Theorem

If J admits a minimizer $u \in W_0^{1,1}(\Omega)$ that depends only on d then Ω is a ball.

Serrin problem

Crasta's result is in connection with the Serrin problem

$$\begin{cases} -\operatorname{div} \frac{f'(|Du|)}{|Du|} Du = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |Du| = const & \text{on } \partial\Omega. \end{cases}$$

Crasta's assumptions: weaker on f and stronger on u

- u web function regular at $\partial\Omega \quad \Rightarrow \quad |Du| = const$ on $\partial\Omega$;
- u web function $\quad \Rightarrow \quad u$ constant on Γ_δ for any δ.

Our proof is based on Alexandrov's method of moving planes by using an argument used in Magnanini-Sakaguchi [AIHP10].

Connection with Serrin problem?

- $u = c_n$ on Γ_{δ_n}, $n \in \mathbb{N} \quad \Rightarrow \quad |Du| = const$ on $\partial\Omega$;
- $|Du| = const$ on $\partial\Omega \quad \Rightarrow \quad \max_{\Gamma_\delta} u - \min_{\Gamma_\delta} u = o(\delta)$ as $\delta \to 0$.

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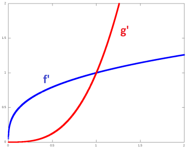
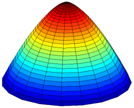
Radial solution and some example

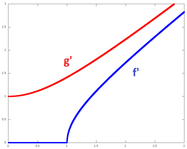
When $\Omega = B_R$, the minimizer is given by $u_R(x) = \displaystyle\int_{|x|}^{R} g'\left(\frac{s}{n}\right) ds$;
$g(t) = \sup\{st - f(s) : \ s \geq 0\}$ is the Fenchel conjugate of f.

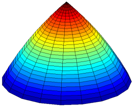
Examples:

Differentiable functionals









Radial solution and some example

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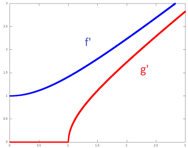
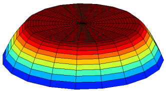
Examples:

Not differentiable functionals





Two cases of tangency:

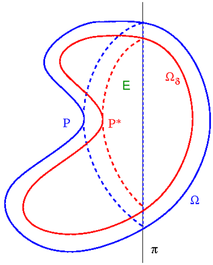


Figure: Case 1

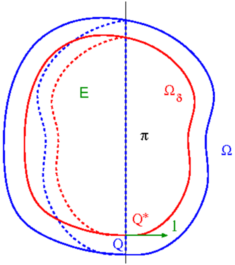


Figure: Case 2

Let u be the minimizer of J and v be its reflection in the plane π.

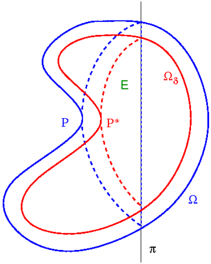
Weak Comparison Principle: $v \leq u$ on $\partial E \;\Rightarrow\; v \leq u$ in $\overline{E}$.

- $\partial\Omega$ is tangent to its reflection at some point $P \notin \pi$;
- Γ_δ is tangent to its reflection at some point $P^* \notin \pi$ which lies in the interior of E;
- $u \equiv c$ on Γ_δ, thus

$$u(P^*) = v(P^*) = c.$$

- Applying the Strong Comparison Principle to u and v in $A_\delta \cap E$ gives $v < u$ in $A_\delta \cap \Omega$

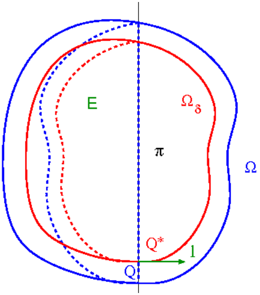
– contradiction –



Case 2

- π is orthogonal to $\partial\Omega$ at some point Q;
- let Q^* be a point on $\Gamma_\delta \cap \pi$; such that ℓ belongs to the tangent hyperplane to Γ_δ at Q^*;
- $u(Q^*) = v(Q^*) = c$ and $\partial_\ell u(Q^*) = \partial_\ell v(Q^*) = 0$.
- Applying a Boundary Point Principle to u and v in $A_\delta \cap E$ gives $\partial_\ell v(Q^*) < \partial_\ell u(Q^*)$

– contradiction –

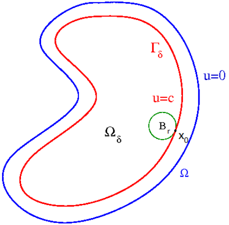


Around Γ_δ

Notice that $u - c$ is a minimizer of

$$J_{\Omega_\delta}(v) = \int_{\Omega_\delta} [f(|Dv|) - v]dx, \;\; v \in W_0^{1,\infty}(\Omega_\delta)$$

$\Rightarrow \;\; u \geq c$ in Ω_δ.



Since $\delta < R_\Omega$ then Ω_δ has the uniform interior touching sphere property (with radius $r = R_\Omega - \delta$).
Let $x_0 \in \Gamma_\delta$ and let B_r be tangent to Γ_δ at x_0.
The weak comparison principle applied to u and $c + u_r$ in B_r implies that

$$\frac{\partial u}{\partial \nu}(x_0) \geq \frac{\partial u_\rho}{\partial \nu}(x_0) = g'\left(\frac{r}{n}\right) > \alpha.$$

From elliptic regularity theory $u \in C^{2,\beta}(\{|Du| > \alpha\})$ and thus

$u \in C^{2,\beta}$ and $|Du| > \alpha$ in an open neighborhood A_δ of Γ_δ.

Comparison results

Let u and v be local minimizers of J in $A \subseteq \Omega$.

- **Weak Comparison Principle**:
  let assume that $|A \cap (\{|Du| > \alpha\} \cup \{|Dv| > \alpha\})| > 0$.
  If $v \leq u$ on $\partial A \;\Rightarrow\; v \leq u$ in A.
- **Strong Comparison Principle**: assume $u, v \in C^1(\overline{A})$ with $|Du|, |Dv| > \alpha$. If $v \leq u$ in $A \Rightarrow$ either $v \equiv u$ or $v < u$ in A.
- **Boundary Point Principle**: let $u, v \in C^2(\overline{A})$ with $|Du|, |Dv| > \alpha$ and $v \leq u$ in A. Suppose that $v = u$ at some point P on the boundary of A admitting an internally touching tangent sphere. Then, either $v \equiv u$ in A or else $v < u$ in A and $\partial_\ell v < \partial_\ell u$ at P.

Notice that, if the three principles hold, our proof works also when u is a classical or viscosity solution of $F(u, Du, D^2u) = 0$.

Not differentiable functionals

$$J(v) = \int_\Omega [f(|Dv|) - v]dx, \quad v \in W_0^{1,\infty}(\Omega)$$

(f1) $f \in C^1([0,+\infty))$ convex, monotone nondecrasing, such that $f(0) = 0$ and $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$;

(f3) $f'(0) > 0$, $f \in C^3(0,+\infty)$ and $f''(s) > 0$ for every $s > 0$.

The map $s \mapsto f(|s|)$ is not differentiable at the origin and we have

The minimizer u of J is unique and satisfies

$$\left| \int_{\Omega^\sharp} f'(|Du|)\frac{Du}{|Du|} \cdot D\phi \, dx - \int_\Omega \phi dx \right| \le f'(0) \int_{\Omega^0} |D\phi| dx,$$

for any $\phi \in C_0^1(\Omega)$, where

$$\Omega^0 := \{Du = 0\} \quad \text{and} \quad \Omega^\sharp := \Omega \setminus \Omega^0.$$

$\Omega = B_R$

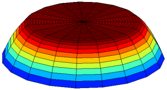
If $\Omega = B_R$, then the solution is given by

$$u_R(x) = \int_{|x|}^{R} g'\left(\frac{s}{n}\right) ds,$$

where $g(t) = \sup\{st - f(s) : \; s \geq 0\}$ is the Fenchel conjugate of f.

Notice that

- if $R \leq nf'(0)$, then $u \equiv 0$;
- if $R > nf'(0)$, then $u > 0$.



Symmetry?

Let

$$h(\Omega) = \inf_{\phi \in C_0^1(\Omega)} \frac{\int_\Omega |D\phi| dx}{\int_\Omega |\phi| dx}.$$

Theorem

Let u be the minimizer of J, with f satisfying (f1) and (f3).

$$u \equiv 0 \;\Leftrightarrow\; f'(0)h(\Omega) \geq 1.$$

Symmetry result

Let u be the minimizer of J, with f satisfying (f1) and (f3). If

$$u \equiv c \text{ on } \Gamma_\delta \quad \text{and} \quad R_\Omega - \delta > nf'(0),$$

then Ω is a ball.