INFINITELY MANY SOLUTIONS TO BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Diego Averna 1, Angela Sciammetta 2, Elisabetta Tornatore 3

Abstract

Variational methods and critical point theorems are used to discuss existence of infinitely many solutions to boundary value problem for fractional order differential equations where Riemann-Liouville fractional derivatives and Caputo fractional derivatives are used. An example is given to illustrate our result.

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1. Introduction

The aim of this paper is to establish the existence of infinitely many solutions to the following boundary value problem for fractional order differential equations

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d^\alpha}{dx^\alpha} \left( C_aD_\alpha^\alpha u(x) \right) + u(x) = \lambda f(x, u(x)) \quad \text{in } ]a, b[,
\quad u(a) = u(b) = 0,
\end{array} \right.
\end{aligned}
\]

where \( \frac{d^\alpha}{dx^\alpha} \), \( C_aD_\alpha^\alpha \) are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative of order \( \frac{1}{2} < \alpha \leq 1 \) respectively, \( \lambda \) is a positive real parameter and \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a given continuous function. The fractional differential equations have received a great deal of interest in recent years and we refer the reader to [20], [21], [22] for an overview on nonlinear fractional equations. These equations arise in mathematical modeling of processes in physics, engineering such as visco-elasticity, among
the papers, see for example [12], [13] and the references therein. We observe that if $\alpha = 1$ the problem (1.1) reduces to the second order boundary value problem

$$\begin{cases}
-u''(x) + u(x) = \lambda f(x, u(x)) & \text{in } [a, b], \\
u(a) = u(b) = 0.
\end{cases}$$

(1.2)

The existence of infinitely many solutions for this problem is widely studied via variational methods and critical point theorems among the papers, see for example [7], [17], (see also [3], [6], [9] for mixed boundary value problem, [5], [8], [16] for the Neumann problem). The variational approach has been also employed to study fractional differential problem when $\frac{1}{2} < \alpha < 1$. For results on the existence of one solution we cite [14], [15], in which the authors studied a fractional boundary value problem by using the Mountain pass theorem, under the usual Ambrosetti-Rabinowitz condition. For results on the multiplicity of solutions see [1], [2], [10], [11], [19], [23]. In [10] and [19] the authors studied the fractional boundary value problem (1.1) with impulsive effects and proved the existence of at least three solutions. In [1] and [2] the authors studied the problem (1.1) obtaining results on the existence of multiple solutions under appropriate conditions of nonlinear term. In [11] the authors studied the same model proposed in [15] under distinct hypotheses on potential function. In [23] the authors studied the following problem

$$\begin{cases}
\frac{1}{\Gamma(\alpha)} \left( 0 D^\alpha_T u(t) \right) = \lambda \nabla W(t, u(t)) & \text{in } [0, T], \\
u(0) = u(T) = 0,
\end{cases}$$

where $W \in C^1([0, T] \times \mathbb{R}^n, \mathbb{R})$ and proved the existence of infinitely many solutions under appropriate subquadratic growth of $W$ via the genus properties in the critical theory. Motivated by these works, in this paper, using a result of Bonanno [4], we present a result on the existence of infinitely many solutions of problem (1.1) under some hypotheses on the behaviour of potential $F$ at infinity (see Theorem 3.1). As a consequence, we obtain the existence of infinitely many solutions for autonomous case (see Corollary 3.1).

2. Preliminaries

In this section, we recall definitions, lemmas and a critical point theorem used in this paper.

Let $(X, \| \cdot \|)$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals and $r \in \mathbb{R}$ be $\Phi - \Psi$ satisfies the *Palais-Smale condition cut off upper at $r$* (in short $(PS)_r^\Phi$-condition) if any sequence $(u_n)_{n \in \mathbb{N}}$ in $X$ such that
\((\alpha_1)\) \(\{I(u_n)\}\) is bounded,
\((\alpha_2)\) \(\lim_{n \to +\infty} ||I'(u_n)||_{X^*} = 0,\)
\((\alpha_3)\) \(\Phi(u_n) < r \quad \forall n \in \mathbb{N},\)
has a convergent subsequence.

When \(r = +\infty\) the previous definition coincides with the classical \((PS)\)-condition, while if \(r < \infty\) such condition is more general than the classical one. We refer to \([4]\) for more details.

We investigate the existence of infinitely many solutions for problem \((1.1)\) by using Theorem 2.1. This theorem is a refinement, due to Bonanno, of the Variational Principle of Ricceri \([18, \text{Theorem 2.5}]\), here we recall it as given in \([4]\).

**Theorem 2.1.** (see \([4, \text{Theorem 7.4}]\)) Let \(X\) be a real Banach space and let \(\Phi, \Psi : X \to \mathbb{R}\) be two continuously Gâteaux differentiable functionals with \(\Phi\) bounded from below, \(\lambda\) be a positive real parameter.

Put, for each \(r \in \mathbb{R}\)
\[
\varphi(r) := \inf_{u \in \Phi^{-1}([-\infty,r[)} \sup_{v \in \Phi^{-1}([-\infty,r[)} \frac{\Psi(v) - \Psi(u)}{r - \Phi(u)}, \quad (2.1)
\]
\[
\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).
\]

One has

(a) If \(\gamma < \infty\) and for each \(\lambda \in \left]0, \frac{1}{\gamma}\right[\), the functional \(\Phi - \lambda \Psi\) satisfies \((PS)^{[r]}\)-condition for all \(r \in \mathbb{R}\), then, for each \(\lambda \in \left]0, \frac{1}{\gamma}\right[\), the following alternative holds: either

\((a_1)\) \(\Phi - \lambda \Psi\) possesses a global minimum, or

\((a_2)\) there is a sequence \(\{u_n\}\) of critical points (local minima) of \(\Phi - \lambda \Psi\) such that \(\lim_{n \to +\infty} \Phi(u_n) = +\infty\).

(b) If \(\delta > +\infty\) and for each \(\lambda \in \left]0, \frac{1}{\delta}\right[\), the functional \(\Phi - \lambda \Psi\) satisfies \((PS)^{[r]}\)-condition for some \(r > \inf_X \Phi\) then, for each \(\lambda \in \left]0, \frac{1}{\delta}\right[\), the following alternative holds: either

\((b_1)\) there is a global minimum of \(\Phi\) which is a local minimum of \(\Phi - \lambda \Psi\), or

\((b_2)\) there is a sequence \(\{u_n\}\) of pairwise distinct critical points (local minima) of \(\Phi - \lambda \Psi\), with \(\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi\).

Now we present some lemmas from the theory of fractional calculus that will be used in the paper (see \([20, 21]\)).
DEFINITION 2.1. The left and right Riemann-Liouville fractional integrals of order \( \alpha > 0 \) for a continuous function \( u : [a, b] \to \mathbb{R} \) denoted by \( aD_x^{-\alpha}u(x) \) and \( xD_b^{-\alpha}u(x) \) respectively, are defined by

\[
aD_x^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} u(s) \, ds, \quad x \in [a, b],
\]

\[
xD_b^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} u(s) \, ds, \quad x \in [a, b],
\]

where \( \Gamma \) is the gamma function.

If \( \alpha = 0 \) we put \( aD_x^0u(x) = u(x) \) and \( xD_b^0u(x) = u(x) \).

DEFINITION 2.2. The left and right Riemann-Liouville fractional derivatives of order \( 0 \leq \alpha < 1 \) for a continuous function \( u : [a, b] \to \mathbb{R} \) denoted by \( aD_x^\alpha u(x) \) and \( xD_b^\alpha u(x) \) respectively, are defined by

\[
aD_x^\alpha u(x) = \frac{d}{dx} aD_x^{\alpha-1}u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-s)^{-\alpha} u(s) \, ds, \quad x \in [a, b],
\]

\[
xD_b^\alpha u(x) = \frac{d}{dx} xD_b^{\alpha-1}u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (s-x)^{-\alpha} u(s) \, ds, \quad x \in [a, b].
\]

DEFINITION 2.3. The left and right Caputo fractional derivatives of order \( 0 < \alpha \leq 1 \) for an absolutely continuous function \( u : [a, b] \to \mathbb{R} \) denoted by \( C_x^\alpha u(x) \) and \( xD_b^\alpha u(x) \) respectively, are defined by

\[
C_x^\alpha u(x) = a^\alpha D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-s)^{-\alpha} u'(s) \, ds, \quad x \in [a, b],
\]

\[
x^\alpha D_b^\alpha u(x) = x^\alpha D_b^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (s-x)^{-\alpha} u'(s) \, ds, \quad x \in [a, b].
\]

Note that when \( \alpha = 1 \) then \( a^1 D_x^1 u(x) = u'(x) \) and \( x^1 D_b^1 u(x) = -u'(x) \).

Let \( E_0^\alpha \) (\( 0 < \alpha \leq 1 \)) be the fractional derivative space defined by the closure of \( C_0^\infty([a, b]) \) with respect on the norm

\[
\|u\| = \left( \|u\|_{L^2([a, b])}^2 + \|D_x^\alpha u\|_{L^2([a, b])}^2 \right)^{\frac{1}{2}}.
\]

Now \( E_0^\alpha \) is a reflexive and separable Banach space (for more details see [15]), moreover \( E_0^\alpha \) is a Hilbert space with the inner product

\[
(u, v)_{\alpha} = (u, v)_{L^2([a, b])} + (C_x^{\alpha}D_x^{\alpha}u, C_x^{\alpha}D_x^{\alpha}v)_{L^2([a, b])}, \quad \forall u, v \in E_0^\alpha,
\]

where \( \| \cdot \|_{L^2([a, b])} \) and \( (\cdot, \cdot)_{L^2([a, b])} \) denote the norm and the inner product in \( L^2([a, b]) \) respectively.
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Moreover we use
$$\|u\|_\infty = \max_{x \in [a,b]} |u(x)|, \quad \forall u \in C^0([a,b]).$$

**Proposition 2.1.** ([15] Proposition 3.2) If $\frac{1}{2} < \alpha \leq 1$. For all $u \in E_0^\alpha$ we have
$$\|u\|_\infty \leq \frac{(b-a)^{2\alpha-1}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \|u\|.$$  \hfill (2.2)

**Lemma 2.1.** ([15]) Let $\frac{1}{2} < \alpha \leq 1$ and let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence such that $u_n \rightharpoonup u$ in $E_0^\alpha$. Then \( \{u_n\}_{n \in \mathbb{N}} \) converges strongly to $u$ in $C^0([a,b])$.

We will use the following integration by parts formulae

**Proposition 2.2.** ([20]) Let $0 < \alpha \leq 1$ and $f, g \in L^2([a,b])$. Then
$$\int_a^b x D_b^{\alpha-1} f(x) g(x) dx = \int_a^b f(x) a D_x^{\alpha-1} g(x) dx.$$

**Proposition 2.3.** ([20]) Let $0 < \alpha \leq 1$ and $f, g \in E_0^\alpha$. Then
$$\int_a^b \left( C_a^{\alpha} D_x f(x) \right) \left( a D_x^{\alpha} g(x) \right) dx = \int_a^b x D_0^\alpha \left( C_a^{\alpha} D_x f(x) \right) g(x) dx.$$

Let $AC([a,b])$ be the space of absolutely continuous functions over $[a,b]$ (see [20]).

A function $u \in AC([a,b])$ is said a solution to problem (1.1) if
- $x D_b^\alpha (C_a^{\alpha} D_x u(x))$ exists a.e in $[a,b]$,
- $u$ satisfies (1.1).

A function $u \in E_0^\alpha$ is said a weak solution to problem (1.1) if
$$\int_a^b \left[ (C_a^{\alpha} D_x u(x)) (C_a^{\alpha} D_x v(x)) + u(x)v(x) \right] dx = \lambda \int_a^b f(x, u(x)) v(x) dx \quad (2.3)$$
for every $v \in E_0^\alpha$.

To study problem (1.1), we will use the functionals $\Phi, \Psi : E_0^\alpha \to \mathbb{R}$ defined by
$$\Phi(u) := \frac{\|u\|^2}{2}, \quad \Psi(u) := \int_a^b F(x, u(x)) dx, \quad (2.4)$$
for every $u \in E_0^\alpha$, where $F(x, t) = \int_0^t f(x, \xi) d\xi$ for all $(x, t) \in [a,b] \times \mathbb{R}$.

Clearly, $\Phi$ is coercive, continuous and convex, and hence it is weakly sequentially lower semicontinuous. Moreover $\Phi$ is continuously Gâteaux
differentiable and its Gâteaux derivative admits a continuous inverse. The Gâteaux derivative of $\Phi$ at a point $u \in E_0^\alpha$ is defined by

$$\Phi'(u)(v) = \int_a^b \left( C_a D_x^\alpha u(x) \right) \left( C_a D_x^\alpha v(x) \right) dx + \int_a^b u(x)v(x) dx,$$

for every $v \in E_0^\alpha$.

Now $\Psi$ is continuously Gâteaux differentiable and its Gâteaux derivative at a point $u \in E_0^\alpha$ is defined by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x) dx,$$

for every $v \in E_0^\alpha$.

Moreover, if we assume that $\frac{1}{2} < \alpha \leq 1$ then $\Psi'$ is compact operator. We observe that $\Phi(0) = \Psi(0) = 0$.

A critical point for the functional $I_\lambda := \Phi - \lambda \Psi$ is any $u \in E_0^\alpha$ such that $u(0) = u(b) = 0$.

We can prove the following lemma.

**Lemma 2.2.** If $\frac{1}{2} < \alpha \leq 1$, then $u \in E_0^\alpha$ is a weak solution of (1.1) if and only if it is a solution of (1.1).

**Proof.** By standard arguments, taking into account Proposition 2.3 if $u$ is a solution of problem (1.1) then $u$ is a weak solution.

Conversely, let $u \in E_0^\alpha$ be a weak solution of problem (1.1) i.e. (2.3) holds. Then, taking into account (2.3) and Proposition 2.3 we have

$$\int_a^b \left[ (C_a D_x^\alpha u(x)) (C_a D_x^\alpha v(x)) + u(x)v(x) \right] dx = \lambda \int_a^b f(x, u(x))v(x) dx$$

and

$$\int_a^b (C_a D_x^\alpha u(x)) (C_a D_x^\alpha v(x)) dx = \int_a^b (C_a D_x^\alpha u(x))v(x) dx,$$

for all $v \in C_0^\infty([a, b])$, which imply that

$$xD_b^\alpha \left( C_a D_x^\alpha u(x) \right) + u(x) = \lambda f(x, u(x)),$$

for almost every $x \in [a, b]$. Moreover, since $u \in E_0^\alpha$ we have $u(a) = u(b) = 0$. This completes the proof.

Hence, the critical points for the functional $I_\lambda := \Phi - \lambda \Psi$ are exactly the solutions to problem (1.1).
Now, put
\[ k = \frac{b-a}{3} + \frac{6\alpha^2 - 19\alpha + 16}{2(1-\alpha)^2(3-2\alpha)(2-\alpha)\Gamma^2(1-\alpha)} \left( \frac{b-a}{4} \right)^{1-2\alpha}, \] (2.5)

\[ A := \liminf_{\xi \to +\infty} \int_a^b \max_{|\tau| \leq \xi} F(x, \tau) dx \frac{\xi^2}{\xi^2}, \quad B := \limsup_{\xi \to +\infty} \int_{a+c-n}^{b+c-n} F(x, \xi) dx \frac{\xi^2}{\xi^2}, \] (2.6)

\[ \lambda_1 := \frac{1}{B}, \quad \lambda_2 := \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}A}, \] (2.7)

where we suppose \( \lambda_1 = 0 \) if \( B = \infty \), and \( \lambda_2 = +\infty \) if \( A = 0 \).

3. Main result

Our main result is the following theorem

**Theorem 3.1.** Assume that

(h1) \( F(x, t) \geq 0 \), \( \forall(x, t) \in [a, b] \times \mathbb{R}^+ \),

(h2) \( \liminf_{\xi \to +\infty} \int_{a+c-n}^{b+c-n} F(x, \xi) dx \frac{\xi^2}{\xi^2} < \frac{2(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} \limsup_{\xi \to +\infty} \int_{a+c-n}^{b+c-n} F(x, \xi) dx \frac{\xi^2}{\xi^2} \).

Then, for each \( \lambda \in \lambda_1, \lambda_2 \), where \( \lambda_1, \lambda_2 \) are given by (2.7), the problem (1.1) has a sequence of solutions which is unbounded in \( E^\alpha_0 \).

**Proof.** Our goal is to apply Theorem 2.1. Consider the Sobolev space \( E^\alpha_0 \) and the operators defined in (2.4), taking into account that all the regularity assumptions on \( \Phi \) and \( \Psi \) are satisfied.

Pick \( \lambda \in \lambda_1, \lambda_2 \). By using (2.6), let \( \{c_n\}_{n \in \mathbb{N}} \) be a real sequence such that \( \lim_{n \to +\infty} c_n = +\infty \) and

\[ \lim_{n \to +\infty} \int_a^b \max_{|\xi| \leq c_n} F(x, \xi) dx \frac{\xi^2}{c_n^2} = A. \]

Put \( r_n = \frac{(2\alpha-1)\Gamma^2(\alpha)}{2(b-a)^{2\alpha-1}} c_n^2 \) for all \( n \in \mathbb{N} \), taking into account (2.2), one has \( ||v||_\infty \leq c_n \) for all \( v \in X \) such that \( ||v||^2 \leq 2r_n \).
Hence, for all \( n \in \mathbb{N} \), one has

\[
\varphi(r_n) = \inf_{u \in \Phi^{-1}(]-\infty,r_n[)} \frac{\Psi(v) - \Psi(u)}{r_n - \Phi(u)} \leq \sup_{||v|| \leq 2r_n} \int_a^b F(x,v(x))dx \leq \frac{\int_a^b \max_{|\xi| \leq c_n} F(x,\xi)dx}{r_n},
\]

therefore, since from (h2) one has \( A < \infty \), we obtain

\[
\gamma := \liminf_{n \to \infty} \varphi(r_n) \leq \frac{2(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} A < \infty.
\]

Now, we claim that the functional \( I_\lambda = \Phi - \lambda \Psi \) is unbounded from below.

By using (2.6), let \( \{d_n\}_{n \in \mathbb{N}} \) be a real sequence such that \( \lim_{n \to \infty} d_n = +\infty \) and

\[
\limsup_{n \to (+\infty)} \frac{\int_a^b F(x,d_n)dx}{d_n^2} = B.
\]

For all \( n \in \mathbb{N} \), we consider the function \( \bar{u}_n \in E_0^\alpha \) defined by

\[
\bar{u}_n(x) := \begin{cases}
\frac{4d_n}{b-a}(x-a) & \text{if } a \leq x \leq a + \frac{b-a}{4}, \\
d_n & \text{if } a + \frac{b-a}{4} < x \leq b - \frac{b-a}{4}, \\
\frac{4d_n}{b-a}(b-x) & \text{if } b - \frac{b-a}{4} < x \leq b.
\end{cases}
\]

Clearly, one has

\[
\bar{u}_n'(x) := \begin{cases}
\frac{4d_n}{b-a} & \text{if } a < x < a + \frac{b-a}{4}, \\
0 & \text{if } a + \frac{b-a}{4} < x < b - \frac{b-a}{4}, \\
-\frac{4d_n}{b-a} & \text{if } b - \frac{b-a}{4} < x < b,
\end{cases}
\]

and

\[
\left\{ \begin{array}{ll}
\frac{4d_n}{(b-a)^{1-\alpha}} & \text{if } a \leq x \leq a + \frac{b-a}{4}, \\
\frac{4d_n}{(b-a)^{1-\alpha}} & \text{if } a + \frac{b-a}{4} < x \leq b - \frac{b-a}{4}, \\
\frac{4d_n}{(1-\alpha)(b-a)} & \text{if } b - \frac{b-a}{4} < x \leq b,
\end{array} \right.
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_a^x (x - \tau)^{-(1-\alpha)} \bar{u}_n'(\tau)d\tau.
\]
so that
\[ \Phi(\bar{u}_n) = kd_n^2 , \quad \text{(3.3)} \]

where \( k \) is given by \((2.5)\).

Taking into account \((h_1)\), we have
\[ \int_a^b F(x, \bar{u}_n(x)) \, dx \geq \int_{a+\frac{b-a}{4}}^{b-\frac{a+b}{4}} F(x, d_n) \, dx . \quad \text{(3.4)} \]

Then, for all \( n \in \mathbb{N} \)
\[ \Phi(\bar{u}_n) - \lambda \Psi(\bar{u}_n) \leq kd_n^2 - \lambda \int_{a+\frac{b-a}{4}}^{b-\frac{a+b}{4}} F(x, d_n) \, dx . \]

Now, if \( B < \infty \), we fix \( \epsilon \in ]\frac{k}{\lambda B}, 1[ \), from \((3.1)\) there exists \( \nu_\epsilon \in \mathbb{N} \) such that
\[ \int_{a+\frac{b-a}{4}}^{b-\frac{a+b}{4}} F(x, d_n) \, dx > \epsilon B d_n^2 , \quad \forall n > \nu_\epsilon , \]

therefore
\[ \Phi(\bar{u}_n) - \lambda \Psi(\bar{u}_n) \leq (k - \lambda \epsilon B)d_n^2 , \quad \forall n > \nu_\epsilon , \]

by the choice of \( \epsilon \), one has
\[ \lim_{n \to \infty} \left[ \Phi(\bar{u}_n) - \lambda \Psi(\bar{u}_n) \right] = -\infty . \]

On the other hand, if \( B = +\infty \), we fix
\[ M > \frac{k}{\lambda} , \]

from \((3.1)\) there exists \( \nu_M \in \mathbb{N} \) such that
\[ \int_{a+\frac{b-a}{4}}^{b-\frac{a+b}{4}} F(x, d_n) \, dx > M d_n^2 , \quad \forall n > \nu_M , \]

therefore
\[ \Phi(\bar{u}_n) - \lambda \Psi(\bar{u}_n) \leq (k - \lambda M)d_n^2 , \quad \forall n > \nu_M \]

by the choice of \( M \), one has
\[ \lim_{n \to \infty} \left[ \Phi(\bar{u}_n) - \lambda \Psi(\bar{u}_n) \right] = -\infty . \]

Hence, our claim is proved.

Since all assumptions of Theorem \([2.1]\) are verified, the functional \( I_\lambda = \Phi - \lambda \Psi \) admits a sequence \( \{ u_n \}_{n \in \mathbb{N}} \) of critical points such that \( \lim_{n \to \infty} ||u_n|| = +\infty \) and the conclusion is achieved. \( \square \)
Remark 3.1. In Theorem 3.1 we can replace $r \to +\infty$ by $r \to 0^+$, applying in the proof part (c) of Theorem 2.1 instead of (b). In this case a sequence of pairwise distinct solutions to the problem (1.1) which converges uniformly to zero is obtained.

Now, we point out a special case of Theorem 3.1.

Corollary 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive continuous function and let $F$ be a primitive of $f$ with $F(0) = 0$. Assume that

$$\lim inf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0, \quad \lim sup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$  

Then, the problem

$$\begin{cases}
D_{\alpha}^{\beta} \left( \frac{C_{\alpha} D_{\alpha}^{\beta} u(x)}{\alpha} \right) + u(x) = f(u(x)) & \text{in } ]0,1[, \\
u(0) = u(1) = 0,
\end{cases}$$

for all $\alpha \in ]\frac{1}{2},1[,$ possesses a sequence of pairwise distinct solutions which is unbounded in $E_{\alpha}^{\beta}$.

Proof. Since $f$ is positive one has that $\max_{|\tau| \leq \xi} F(\tau) = F(\xi)$ for every $\xi \in \mathbb{R}_+$.

Therefore

$$\lim inf_{\xi \to +\infty} \frac{\int_0^1 \max_{|\tau| \leq \xi} F(\tau) dx}{\xi^2} = 0,$$

on the other hand, we have

$$\frac{2}{(2\alpha - 1)\Gamma(\alpha)} \lim sup_{\xi \to +\infty} \int_{\frac{3}{4}}^{\frac{3}{2}} \frac{F(\xi) dx}{\xi^2} = +\infty.$$  

Then we have $\lambda_1 = 0$ and $\lambda_2 = +\infty$ and all assumptions of Theorem 3.1 are satisfied and the proof is complete. $\square$

Now, we present one example that illustrates our result.

Example 3.1. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases}
x^2 e^{2(\sin \ln x + 1)} & \text{if } x > 0, \\
0 & \text{otherwise},
\end{cases}$$
we denote by \( f \) the derivative of \( F \)

\[
f(x) = \begin{cases} 
2xe^{2(\sin \ln x + 1)}(1 + \cos \ln x) & \text{if } x > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Since \( f \) is nonnegative one has that \( \max_{|\tau| \leq \xi} F(\tau) = F(\xi) \) for every \( \xi \in \mathbb{R}_+ \). By a simple computation, we obtain

\[
\lim \inf_{\xi \to +\infty} \frac{\max F(\tau)}{\xi^2} = \lim \inf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 1,
\]

\[
\lim \sup_{\xi \to +\infty} \frac{F(\xi)}{2\xi^2} = \frac{e^4}{2}.
\]

Hence, from Theorem 3.1 for each \( \lambda \in ]\frac{1}{2}, \frac{(2\alpha-1)\Gamma^2(\alpha)}{2}[, \) and for each \( \alpha \in ]\frac{1}{2}, 1[ \) the problem

\[
\begin{cases}
\varepsilon \, D_0^\alpha \left( C \, D_0^\alpha u(x) \right) + u(x) = \lambda f(u(x)) & \text{in } ]0,1[, \\
u(0) = u(1) = 0,
\end{cases}
\]

has a sequence of solutions which is unbounded in \( E^\alpha_{0} \).

**Remark 3.2.** We observe that the results of [23] (for \( n = 1 \)), can not be applied to example 3.1 because the authors require a subquadratic growth on the nonlinear term as \( |u| \to \infty \) while we have \( \lim_{x \to +\infty} \frac{F(x)}{x^2} = +\infty \) (\( 1 < \theta < 2 \)).

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**References**


1 Dept. of Mathematics and Computer Science, University of Palermo
Via Archirafi, 90123 - Palermo, ITALY

Dept. of Mathematics and Computer Science, University of Palermo
Via Archirafi, 90123 - Palermo, ITALY

Dept. of Mathematics and Computer Science, University of Palermo
Via Archirafi, 90123 - Palermo, ITALY

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e-mail: diego.averna@unipa.it
e-mail: angela.sciammetta@unipa.it
e-mail: elisa.tornatore@unipa.it