Multiple solutions for a Dirichlet problem with $p$-Laplacian and set-valued nonlinearity

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Abstract
The existence of a negative solution, of a positive solution, and of a sign changing
solution to a Dirichlet eigenvalue problem with $p$-Laplacian and multi-valued non-
linearity is investigated via sub- and super-solution methods as well as variational
techniques for non-smooth functions.

Keywords: $p$-Laplacian; generalized gradient; multiple nontrivial solutions; sub- and super-solutions; critical points of non-smooth functions.

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial \Omega$, let $1 < p < +\infty$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in each variable separately. Given a real parameter $\lambda$, consider the problem

$$\begin{cases}
-\Delta_p u = \lambda |u|^{p-2}u - g(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.1)$$

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$. If $p = 2$ then the existence of multiple solutions to (1.1) has been widely investigated; see [2, 18, 1, 19] and the references therein. All these papers treat the case when $(x, t) \mapsto g(x, t)$ does not depend on $x$ and is suitably regular, like, for instance, continuously differentiable [1] or Lipschitz continuous [18, 19]. Roughly speaking, the obtained results read as follows. Let the function $g$ exhibit a superlinear behavior both at zero and at infinity. Under a further technical condition, which may vary from one work to another, Problem (1.1) possesses at least three nontrivial solutions provided $\lambda > \lambda_2$, the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. Combining the method of sub- and super-solutions with variational techniques chiefly based on the second deformation lemma, the very recent papers [4, 17] examine a much more general situation, i.e., $1 < p < +\infty$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ of Carathéodory’s type only.

We next point out that Struwe’s result [19, Theorem 10.5] has been extended to the so-called elliptic hemivariational inequalities in [11].

The same non-smooth framework of [11] is adopted here, but the technical approach exploited is patterned after that of [4]. More precisely, setting, for $g$ merely bounded on bounded sets,

$$G(x, \xi) := \int_0^\xi g(x, t) \, dt, \quad (x, \xi) \in \Omega \times \mathbb{R},$$

we shall be concerned with the problem

$$\begin{cases}
-\Delta_p u \in \lambda |u|^{p-2}u - \partial G(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.2)$$

where $\partial G(x, u(x))$ indicates the generalized gradient of $\xi \mapsto G(x, \xi)$ at the point $u(x)$. Obviously, (1.2) reduces to (1.1) as soon as $g$ satisfies Carathéodory’s conditions. We say that $u \in W_0^{1,p}(\Omega)$ is a solution of (1.2) provided there exists an $\eta \in L^{p/(p-1)}(\Omega)$ such that

$$\eta(x) \in \partial G(x, u(x)) \quad \text{a.e. in } \Omega, \quad (1.3)$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx + \int_\Omega (\eta - \lambda |u|^{p-2}u) \varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega). \quad (1.4)$$
The main result of this paper, Theorem 4.1 below, establishes the existence of at least three nontrivial solutions \( u_-, u_+, u_0 \in C_0^1(\Omega) \) to (1.2) such that \( u_- < 0 < u_+ \), while \( u_0 \) changes of sign, in \( \Omega \). It represents a non-smooth version of Theorem 4.1 in [4] and includes both Theorem 3.9 of [17] and Corollary 3.2 in [11] as special cases. Accordingly, Theorem 4.1 also extends the results of [2, 18, 1, 19] to Problem (1.2). Let us next note that it exhibits significant qualitative properties of the obtained solutions. For other multiplicity results under different assumptions, see [15, 12, 9] and the references given there.

Problems like (1.2) are sometimes called elliptic hemivariational inequalities. They arise in the mathematical formulation of several complicated mechanical and engineering questions, where the relevant energy functionals turn out to be neither convex nor smooth (the so-called super-potentials). The monographs [14, 16, 9, 10] are general items on this subject.

2 Basic assumptions and preliminary results

Let \((X, \| \cdot \|)\) be a real Banach space. Given a set \( V \subseteq X \), write \( \partial V \) for the boundary of \( V \), int\((V)\) for the interior of \( V \), and \( \overline{V} \) for the closure of \( V \). If \( x, z \in X \) and \( \delta > 0 \) then

\[
B_\delta(x) := \{ w \in X : \| w - x \| < \delta \}, \quad [x, z] := \{ (1 - t)x + tz : t \in [0, 1] \}.
\]

The symbol \( X^* \) denotes the dual space of \( X \), while \( \langle \cdot, \cdot \rangle \) indicates the duality pairing between \( X \) and \( X^* \). A function \( \Phi : X \to \mathbb{R} \) is called coercive when

\[
\lim_{\| x \| \to +\infty} \Phi(x) = +\infty.
\]

If to every \( x \in X \) there correspond a neighborhood \( V_x \) of \( x \) and a constant \( L_x \geq 0 \) such that

\[
|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall z, w \in V_x
\]

then we say that \( \Phi \) is locally Lipschitz continuous. In this case, \( \Phi^0(x; z), x, z \in X \), denotes the generalized directional derivative of \( \Phi \) at the point \( x \) along the direction \( z \), i.e.,

\[
\Phi^0(x; z) := \limsup_{w \to x, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.
\]

The generalized gradient of the function \( \Phi \) in \( x \) is the set

\[
\partial\Phi(x) := \{ x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z) \quad \forall z \in X \}.
\]

Proposition 2.1.2 of [6] ensures that \( \partial\Phi(x) \) turns out to be nonempty, convex, in addition to weak* compact, and that

\[
\Phi^0(x; z) = \max \{ \langle x^*, z \rangle : x^* \in \partial\Phi(x) \}, \quad z \in X.
\]
Hence, it makes sense to write

\[ m_\Phi(x) := \min\{\|x^*\|_{X^*} : x^* \in \Phi(x)\} . \]

The classical Palais-Smale condition for \( C^1 \) functions becomes here (cf. [5, Definition 2]):

\( \text{(PS) } \) Every sequence \( \{x_n\} \subseteq X \) such that \( \{\Phi(x_n)\} \) is bounded and \( \lim_{n \to +\infty} m_\Phi(x_n) = 0 \) possesses a convergent subsequence.

We say that \( x \in X \) is a critical point of \( \Phi \) when \( 0 \in \partial \Phi(x) \), namely \( \Phi^0(x;z) \geq 0 \) for all \( z \in X \). Obviously, each local minimizer or maximizer of \( \Phi \) turns out to be a critical point of \( \Phi \). Put

\[ K(\Phi) := \{x \in X : 0 \in \partial \Phi(x)\} . \]

The following non-smooth version of Ambrosetti-Rabinowitz’s Mountain Pass Theorem is essentially due to Chang [5, Theorem 3.4] and will be exploited in Section 4.

**Theorem 2.1.** \( \) Let \( X \) be reflexive and let \( \Phi \) satisfy \( \text{(PS)} \). If there exist \( x_0, x_1 \in X, r > 0 \) such that \( \|x_1 - x_0\| > r \) and \( \max\{\Phi(x_0), \Phi(x_1)\} < \inf_{x \in \partial B_r(x_0)} \Phi(x) \) then \( \Phi \) has a critical point \( \hat{x} \in X \) such that

\[ \inf_{x \in \partial B_r(x_0)} \Phi(x) \leq \Phi(\hat{x}) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) , \]

where \( \Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\} \).

An operator \( A : X \to X^* \) is called coercive when

\[ \lim_{\|x\| \to +\infty} \frac{\langle A(x), x \rangle}{\|x\|} = +\infty . \]

We say that \( A \) is of type \( (S)_+ \) if \( x_n \rightharpoonup x \) in \( X \) and \( \limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0 \) imply \( x_n \to x \).

Throughout the paper, \( \Omega \) denotes a bounded domain of the real Euclidean \( N \)-space \( (\mathbb{R}^N, |\cdot|) \), \( N \geq 3 \), with a smooth boundary \( \partial \Omega \), \( p \in ]1, +\infty[ \), and \( p' := p/(p - 1) \). The symbol \( W^{1,p}_0(\Omega) \) indicates the closure of \( C^\infty(\Omega) \) in \( W^{1,p}(\Omega) \). On \( W^{1,p}_0(\Omega) \) we introduce the norm

\[ \|u\| := \left( \int_\Omega |\nabla u(x)|^p dx \right)^{1/p} . \]

Denote by \( p^* \) the critical exponent for the Sobolev embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \). Recall that \( p^* = Np/(N - p) \) if \( p < N \) and \( p^* = +\infty \) if \( p \geq N \). As usual, we write

\[ L^{p^*}(\Omega)_+ := \{u \in L^{p^*}(\Omega) : u(x) \geq 0 \text{ a.e. in } \Omega\} , \]

\[ \text{and } L^p(\Omega)_+ := \{u \in L^p(\Omega) : u(x) \geq 0 \text{ a.e. in } \Omega\} . \]
\[ C^1_0(\Omega)_+ := \{ u \in C^1_0(\Omega) : u(x) \geq 0 \ \forall \ x \in \Omega \}. \]

It is known (see, e.g., [10, Remark 6.2.10]) that
\[ \text{int}(C^1_0(\Omega)_+) = \left\{ u \in C^1_0(\Omega) : u(x) > 0, \ \frac{\partial u}{\partial n}(x) < 0 \ \forall \ x \in \Omega \right\}, \]

with \( u(x) \) being the outward unit normal vector to \( \partial \Omega \) at the point \( x \in \partial \Omega \).

Let \( \lambda_1 \) (respectively, \( \lambda_2 \)) be the first (respectively, the second) eigenvalue of the operator \(-\Delta_p u := -\text{div}(\|\nabla u\|^{p-2} \nabla u)\) in \( W^{1,p}_0(\Omega) \) and let \( A : W^{1,p}_0(\Omega) \rightarrow (W^{1,p}_0(\Omega))^* \) defined by
\[ \langle A(u), v \rangle := \int_{\Omega} \|\nabla u(x)\|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx \ \forall \ u, v \in W^{1,p}_0(\Omega). \quad (2.1) \]

The following properties of \( \lambda_1, \lambda_2, \) and \( A \) can be found in [10, Section 6.2]; vide also [8].

(p1) \( 0 < \lambda_1 < \lambda_2. \)

(p2) There exists an eigenfunction \( \varphi_1 \) corresponding to \( \lambda_1 \) such that \( \varphi_1 \in \text{int}(C^1_0(\Omega)_+) \) as well as \( \|\varphi_1\|_{L^p(\Omega)} = 1. \)

(p3) Let \( S := \{ u \in W^{1,p}_0(\Omega) : \|u\|_{L^p(\Omega)} = 1 \} \) and let \( \Gamma_0 := \{ \gamma \in C^0([-1,1], S) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1 \} \).

Then
\[ \lambda_2 = \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([-1,1])} \|u\|^p. \]

(p4) The operator \( A \) is maximal monotone, coercive, and of type \((S)_+.\)

Finally, to shorten notation, we define, if \( u, v : \Omega \rightarrow \mathbb{R}, \)
\[ \Omega(u \leq v) := \{ x \in \Omega : u(x) \leq v(x) \}, \quad u^+ := \max\{u, 0\} \quad u^- := \min\{u, 0\} . \]

From now on, ‘measurable’ always means Lebesgue measurable. If \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies the conditions
\begin{enumerate}
  \item[(g1)] \( g \) is measurable in each variable separately,
  \item[(g2)] there exist \( a_1 > 0, \ q \in [p, p^*] \) such that
  \[ |g(x, t)| \leq a_1 (1 + |t|^{q-1}) \]
  for almost every \( x \in \Omega \) and every \( t \in \mathbb{R}, \)
\end{enumerate}
then the functions $G(x, \cdot) : \mathbb{R} \to \mathbb{R}$ and $G : L^q(\Omega) \to \mathbb{R}$ given by

$$G(x, \xi) := \int_0^\xi g(x, t)dt \quad \forall \xi \in \mathbb{R},$$

$$G(u) := \int_\Omega G(x, u(x))dx \quad \forall u \in L^q(\Omega), \quad (2.2)$$

respectively, are well defined and locally Lipschitz continuous. So, it makes sense to consider their generalized gradients $\partial G(x, \cdot)$ and $\partial G$. Set, for every $(x, t) \in \Omega \times \mathbb{R},

$$g_1(x, t) := \lim_{\delta \to 0^+} \text{ess inf}_{|\tau-t|<\delta} g(x, \tau), \quad g_2(x, t) := \lim_{\delta \to 0^+} \text{ess sup}_{|\tau-t|<\delta} g(x, \tau).$$

Proposition 1.7 in [14] ensures that

$$\partial G(x, \xi) = [g_1(x, \xi), g_2(x, \xi)], \quad (2.3)$$

while Theorem 4.5.19 of [10] leads to

$$\partial G(u) \subseteq \{w \in L^{q'}(\Omega) : g_1(x, u(x)) \leq w(x) \leq g_2(x, u(x)) \text{ a.e. in } \Omega\}, \quad (2.4)$$

with $q' := q/(q-1)$. The next result is an immediate consequence of [6, Proposition 2.1.5], besides the choice of $q$.

**Lemma 2.1.** Suppose $u_n \to u$ in $W^{1,p}_0(\Omega)$, $w_n \rightharpoonup w$ in $L^{p'}(\Omega)$, and $w_n \in \partial G(u_n)$ for all $n \in \mathbb{N}$. Then $w \in \partial G(u)$.

We will further assume

$$(g_3) \lim_{t \to 0} \frac{g(x, t)}{|t|^{p-2}t} = 0 \text{ uniformly for almost all } x \in \Omega, \text{ and}$$

$$\lim_{|t| \to +\infty} \frac{g(x, t)}{|t|^{p-2}t} = +\infty \text{ uniformly for almost all } x \in \Omega.$$  

**Remark 2.1.** When $p = 2$ while $(x, t) \mapsto g(x, t)$ does not depend on $x$ and is continuous, hypotheses $(g_3)$–$(g_4)$ have been previously introduced in [18, 1]. The very recent paper [11] deals with possibly discontinuous nonlinearities.

**Remark 2.2.** Assumption $(g_3)$ forces $g_1(x, 0) \leq 0 \leq g_2(x, 0)$ for almost all $x \in \Omega$. Hence, in view of (2.3), Problem (1.2) always possesses the trivial solution.
A function $u \in W^{1,p}(\Omega)$ is called a sub-solution to (1.2) if $u|_{\partial \Omega} \leq 0$ and there exists an $\eta \in L^{p'}(\Omega)$ such that

$$\eta(x) \in \partial G(x, u(x)) \text{ for almost every } x \in \Omega,$$

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} \eta \varphi \, dx \leq 0 \quad \forall \varphi \in W^{1,p}(\Omega) \cap L^p(\Omega)^+.$$  (2.5)

Likewise, we say that $\overline{u} \in W^{1,p}(\Omega)$ is a super-solution of Problem (1.2) if $\overline{u}|_{\partial \Omega} \geq 0$ and there exists an $\overline{\eta} \in L^{p'}(\Omega)$ fulfilling

$$\overline{\eta}(x) \in \partial G(x, \overline{u}(x)) \text{ for almost every } x \in \Omega,$$

$$\int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx + \int_{\Omega} (\overline{\eta} - \lambda |\overline{u}|^{p-2} \overline{u}) \varphi \, dx \geq 0 \quad \forall \varphi \in W^{1,p}_0(\Omega) \cap L^p(\Omega)^+.$$  (2.7)

Because of $(p_4)$, the operator $A$ given by (2.1) turns out to be maximal monotone and coercive. So, it is also surjective; see for instance [10, Corollary 3.2.21]. Thus, we can find a function $e \in W^{1,p}_0(\Omega)$ such that $-\Delta_p e = 1$. Gathering Theorems 1.5.6 and 1.5.7 of [9] together yields $e \in \text{int}(C_0^1(\overline{\Omega})^+)$. We are in a position now to establish the existence of sub- and super-solutions to Problem (1.2).

**Theorem 2.2.** Let $(g_1)–(g_4)$ be satisfied. Then, for every $\lambda > \lambda_1$ there exists a constant $a_\lambda > 0$ such that $-a_\lambda e$ (respectively, $a_\lambda e$) is a sub-solution (respectively, a super-solution) of (1.2). Moreover, $\varepsilon \varphi_1$ (respectively, $-\varepsilon \varphi_1$) is a sub-solution (respectively, a super-solution) to (1.2) and $\varepsilon \varphi_1 < a_\lambda e$ in $\Omega$ for any sufficiently small $\varepsilon > 0$.

**Proof.** Pick $\lambda > \lambda_1$. Hypothesis $(g_4)$ produces a $t_\lambda > 0$ such that

$$\frac{g(x, t)}{|t|^{p-2}} > \lambda \quad \text{provided } |t| > t_\lambda.$$  (2.9)

Through $(g_2)$ we can find a $c_\lambda > 0$ fulfilling

$$|g(x, t) - \lambda |t|^{p-2}t| \leq c_\lambda \quad \text{for all } |t| \leq t_\lambda.$$  (2.10)

Both the above inequalities hold almost everywhere in $\Omega$. Moreover, combining (2.9) with (2.10) we achieve

$$-\Delta_p (-c_\lambda^{\frac{1}{p-1}} e) + \lambda c_\lambda^{\frac{1}{p-1}} e \geq 0$$

as well as

$$-\Delta_p (c_\lambda^{\frac{1}{p-1}} e) - \lambda c_\lambda^{\frac{1}{p-1}} e \geq 0.$$  (2.12)
Therefore, on account of (2.3), the first conclusion is true once we put \( a_\lambda := \frac{c_1^{1/(p-1)}}{p-1} \).

Next, since \( \lambda > \lambda_1 \), assumption \((g_3)\) yields a \( \delta_\lambda > 0 \) such that
\[
|g(x, t)| < \lambda - \lambda_1 \quad \text{provided} \quad 0 < |t| \leq \delta_\lambda.
\] (2.13)

Fix a positive number \( \varepsilon \leq \frac{\delta_\lambda}{\|\varphi_1\|_{L^\infty(\Omega)}} \). From \((p_2)\) and (2.13), which holds almost everywhere in \( \Omega \), it easily follows
\[-\Delta p(\varepsilon \varphi_1) - \lambda \varepsilon^{p-1} \varphi_1^{p-1} + g_2(x, \varepsilon \varphi_1) < 0.\]

Likewise,
\[-\Delta p(-\varepsilon \varphi_1) + \lambda \varepsilon^{p-1} \varphi_1^{p-1} + g_1(x, -\varepsilon \varphi_1) > 0.\]

Hence, the function \( \varepsilon \varphi_1 \) (respectively, \( -\varepsilon \varphi_1 \)) turns out to be a sub-solution (respectively, a super-solution) of (1.2). Finally, as \( e \in \text{int}(C^1_0(\overline{\Omega})_+) \), for any sufficiently small \( \varepsilon > 0 \) we have
\[ e - \frac{\varepsilon}{a_\lambda} \varphi_1 \in C^1_0(\overline{\Omega})_+, \]
namely \( \varepsilon \varphi_1 \leq a_\lambda e \) in \( \Omega \). This completes the proof.

## 3 Constant-sign solutions

Two non-zero, constant-sign, extremal solutions to Problem (1.2) can be achieved when \( \lambda > \lambda_1 \), the first eigenvalue of \(-\Delta p\) in \( W^{1,p}_0(\Omega) \). The next result represents a preliminary step in this direction.

**Theorem 3.1.** If \((g_1)-(g_4)\) hold true, while \( \lambda > \lambda_1 \), then for every \( \varepsilon > 0 \) small enough (1.2) has a least positive solution \( u_+ \in \text{int}(C^1_0(\overline{\Omega})_+) \cap [\varepsilon \varphi_1, a_\lambda e] \) and a greatest negative solution \( u_- \in -\text{int}(C^1_0(\overline{\Omega})_+) \cap [-a_\lambda e, -\varepsilon \varphi_1] \), with \( a_\lambda > 0 \) given by Theorem 2.2.

**Proof.** Since the reasoning for \( u_- \) and \( u_+ \) are similar, we shall discuss only the case of \( u_+ \).

Let
\[ u := \varepsilon \varphi_1, \quad \overline{u} := a_\lambda e. \]

Theorem 2.2 ensures that \( u \) (respectively, \( \overline{u} \)) turns out to be a sub-solution (respectively, a super-solution) of (1.2) lying in \( \text{int}(C^1_0(\overline{\Omega})_+) \) and that \( u < \overline{u} \) provided \( \varepsilon > 0 \) is sufficiently small. Put
\[ U := \{ u \in W^{1,p}_0(\Omega) : u(x) \leq u(x) \leq \overline{u}(x) \quad \text{for almost every} \quad x \in \Omega \}. \]
Thanks to (g₁), (g₂) the functional $E_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ given by

$$E_\lambda(u) := \frac{1}{p} \|u\|^p - \frac{\lambda}{p} \|u\|_{L^p(\Omega)}^p + G(u) \quad \forall u \in W^{1,p}_0(\Omega),$$

with $G$ as in (2.2), is well defined, locally Lipschitz continuous, weakly sequentially lower semicontinuous, and coercive in $U$. Hence, there exists a $u_\lambda \in U$ fulfilling

$$E_\lambda(u_\lambda) = \inf_{u \in U} E_\lambda(u). \quad (3.1)$$

We claim that $u_\lambda$ solves Problem (1.2). Indeed, pick $v \in W^{1,p}_0(\Omega)$, $\alpha > 0$, and set

$$w(x) := \begin{cases} 
\frac{u(x)}{u_\lambda(x) + \alpha v} & \text{for } x \in \Omega(u_\lambda + \alpha v \leq u), \\
\frac{u_\lambda(x) + \alpha v}{\overline{u}}(x) & \text{when } x \in \Omega(\overline{u} \leq u_\lambda + \alpha v). 
\end{cases}$$

Obviously, $w \in U$. Consequently, $tw + (1 - t)u_\lambda \in U$ for all $t \in [0,1]$. If $I_{[0,1]}$ denotes the indicator function of $[0,1] \subseteq \mathbb{R}$,

$$f(t) := E_\lambda(tw + (1 - t)u_\lambda) \quad \text{and} \quad \tilde{f}(t) := f(t) + I_{[0,1]}(t) \quad \forall t \in \mathbb{R},$$

then, due to (3.1), the function $\tilde{f}$ attains its minimum at $t = 0$. So, by [16, Proposition 2.1],

$$f^0(0; \tau - 0) + I_{[0,1]}(\tau) - I_{[0,1]}(0) \geq 0 \quad \forall \tau \in \mathbb{R},$$

which means, in particular, $f^0(0;1) \geq 0$. Since $f^0(0;1) = \max \{ z : z \in \partial f(0) \}$, we can find a $z \in \partial f(0) \cap [0, +\infty[$. Using the chain rule [6, Theorem 2.3.10] yields

$$\partial f(0) \subseteq \partial E_\lambda(u_\lambda) \cdot (w - u_\lambda). \quad (3.2)$$

On account of (3.2) and (2.4) there thus exists a $w_\lambda \in L^q(\Omega)$ such that

$$w_\lambda(x) \in \partial G(x, u_\lambda(x)) \quad \text{a.e. in } \Omega, \quad (3.3)$$

$$\langle -\Delta_p u_\lambda, w - u_\lambda \rangle - \lambda \int_\Omega w_\lambda^{-1}(w - u_\lambda) \, dx + \int_\Omega w_\lambda(w - u_\lambda) \, dx = z \geq 0. \quad (3.4)$$

We explicitly note that $w - u_\lambda \in L^q(\Omega)$ because $q \leq p^*$. If $\eta$ (respectively, $\overline{\eta}$) belongs to $L^p(\Omega)$ and satisfies (2.6) (respectively, (2.8)) then, by the choice of $w$, inequality (3.4)
becomes
\[
0 \leq \alpha \int_{\Omega} |\nabla u_\lambda|^p \nabla u_\lambda \cdot \nabla v \, dx - \alpha \int_{\Omega} (\lambda u_\lambda^{p-1} - w_\lambda) v \, dx \\
- \int_{\Omega(\tau \leq u_\lambda + \alpha v)} |\nabla \tau|^p \nabla \tau \cdot \nabla (u_\lambda + \alpha v - \tau) \, dx \\
+ \int_{\Omega(\tau \leq u_\lambda + \alpha v)} (\lambda \tau^{p-1} - \bar{\eta})(u_\lambda + \alpha v - \tau) \, dx \\
+ \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} |\nabla u|^p \nabla u \cdot \nabla (u - u_\lambda - \alpha v) \, dx \\
- \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (\lambda u^{p-1} - \eta)(u - u_\lambda - \alpha v) \, dx \\
+ \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (\lambda \tau^{p-1} - \bar{\eta} - \lambda u_\lambda^{p-1} + w_\lambda)(u - u_\lambda - \alpha v) \, dx \\
+ \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (\lambda u^{p-1} - \eta - \lambda u_\lambda^{p-1} + w_\lambda)(u - u_\lambda - \alpha v) \, dx \\
- \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (|\nabla u_\lambda|^p - |\nabla \underline{u}|^p) \nabla (u_\lambda - \underline{u}) \, dx \\
- \alpha \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (|\nabla u_\lambda|^p - |\nabla \underline{u}|^p) \nabla \underline{v} \, dx \\
+ \int_{\Omega(\tau \leq u_\lambda + \alpha v)} (|\nabla \tau|^p - |\nabla u_\lambda|^p) \nabla (u_\lambda - \tau) \, dx \\
+ \alpha \int_{\Omega(\tau \leq u_\lambda + \alpha v)} (|\nabla \tau|^p - |\nabla u_\lambda|^p) \nabla \underline{v} \, dx \, . \tag{3.5}
\]
Now, putting \( \varphi := (u_\lambda + \alpha v - \bar{u})^+ \) in (2.8) one has
\[
- \int_{\Omega(\tau \leq u_\lambda + \alpha v)} |\nabla \tau|^p \nabla \tau \cdot \nabla (u_\lambda + \alpha v - \tau) \, dx \\
+ \int_{\Omega(\tau \leq u_\lambda + \alpha v)} (\lambda \tau^{p-1} - \bar{\eta})(u_\lambda + \alpha v - \tau) \, dx \leq 0, \tag{3.6}
\]
while (2.6) written for \( \varphi := (\underline{u} - u_\lambda - \alpha v)^+ \) gives
\[
\int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} |\nabla \underline{u}|^p \nabla \underline{u} \cdot \nabla (\underline{u} - u_\lambda - \alpha v) \, dx \\
+ \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (\eta - \lambda u^{p-1})(\underline{u} - u_\lambda - \alpha v) \, dx \leq 0. \tag{3.7}
\]
Since \( \underline{u} \leq u_\lambda \leq \overline{u} \) in \( \Omega \), it results in
\[
\int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (u^{p-2} - u_\lambda^{p-2})(u - u_\lambda - \alpha v) \, dx \leq 0 ,
\] (3.8)
\[
\int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} (\overline{u}^{p-2} - u_\lambda^{p-2})(\overline{u} - u_\lambda - \alpha v) \, dx \leq 0 .
\] (3.9)

Next, assumption \((g_2)\), \((2.3)\), and the continuity of \( u, \overline{u} \) on \( \overline{\Omega} \), ensure that both \( \partial G(x, u(x)) \) and \( \partial G(x, \overline{u}(x)) \) are uniformly bounded with respect to \( x \in \Omega \). So, in view of \((2.5)\), \((2.7)\), and \((3.3)\), there exists a constant \( a_2 > 0 \) fulfilling
\[
\int_{\Omega} (u_\lambda + \alpha v - \underline{u})(-\eta + w_\lambda)(u - u_\lambda - \alpha v) \, dx \leq a_2 \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} v \, dx
\] (3.10)
as well as
\[
\int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} (u_\lambda + \alpha v - \overline{u})(u_\lambda - u_\lambda - \alpha v) \, dx \leq a_2 \int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} v \, dx .
\] (3.11)

Finally, through \((p_4)\) we get
\[
- \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla u|^{p-2}\nabla u) \cdot \nabla (u_\lambda - u) \, dx \leq 0
\] (3.12)
and
\[
\int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} (|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda) \cdot \nabla (u_\lambda - \overline{u}) \, dx \leq 0 .
\] (3.13)

At this point, gathering \((3.5)\)–\((3.13)\) together and dividing by \( \alpha > 0 \) yields
\[
0 \leq \int_{\Omega} |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla v \, dx - \int_{\Omega} (\lambda u_\lambda^{p-1} - w_\lambda) v \, dx
\]
\[
- a_2 \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} v \, dx + a_2 \int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} v \, dx
\]
\[
- \int_{\Omega(u_\lambda + \alpha v \leq \underline{u})} (|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla u_\lambda|^{p-2}\nabla u) \cdot \nabla v \, dx
\]
\[
+ \int_{\Omega(\overline{u} \leq u_\lambda + \alpha v)} (|\nabla \overline{u}|^{p-2}\nabla \overline{u} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda) \cdot \nabla v \, dx .
\] (3.14)
For $\alpha \to 0^+$ inequality (3.14) becomes

$$0 \leq \int_{\Omega} \nabla u_\lambda |p-2\nabla u_\lambda \cdot \nabla v \, dx - \int_{\Omega} (\lambda u_\lambda^{p-1} - w_\lambda) v \, dx,$$

because $u_\lambda$ lies in $U$. As $v \in W^{1,p}_0(\Omega)$ was arbitrary, we actually have

$$-\Delta_p u_\lambda = \lambda u_\lambda^{p-1} - w_\lambda,$$  

(3.15)
i.e., $u_\lambda$ is a positive solution of (1.2). From $u_\lambda, w_\lambda \in L^\infty(\Omega)$ it follows $\Delta_p u \in L^\infty(\Omega)$. Theorem 6.2.7 of [10] forces $u_\lambda \in C^1(\Omega)$. Due to (2.13) and (g2) we can find a constant $\tilde{c}_\lambda > 0$ satisfying

$$|g(x, t)| \leq \tilde{c}_\lambda t^{p-1} \quad \forall t \in [0, \|\bar{u}\|_{L^\infty(\Omega)}]$$  

(3.16)

Hence, by (3.15), (3.3), and (3.16),

$$\Delta_p u_\lambda \leq (\lambda + \tilde{c}_\lambda) u_\lambda^{p-1}.$$  

(3.17)

The Vázquez Maximum Principle [9, Theorem 1.5.7] thus provides $u_\lambda \in \text{int} (C^1_0(\Omega)_+)$. Denote by $\mathcal{U}$ the set of all solutions $u \in \text{int} (C^1_0(\Omega)_+)$ to Problem (1.2) such that $u \leq u \leq \bar{u}$ in $\Omega$. Since $u_\lambda \in \mathcal{U}$, one has $\mathcal{U} \neq \emptyset$. Arguing exactly as in the proofs of [3, Lemma 4.23] and [3, Corollary 4.24], and using [9, Theorem 1.5.7] once more, we then see that $\mathcal{U}$ possesses a least element, say $u_+$, with respect to the pointwise usual order.

Two extremal solutions of (1.2) having opposite constant sign can now be obtained via Theorem 3.1.

**Theorem 3.2.** Let $(g_1)-(g_4)$ be fulfilled. Then for every $\lambda > \lambda_1$ there exist a least positive solution $u_+ \in \text{int} (C^1_0(\Omega)_+) \cap [0, a_\lambda e]$ and a greatest negative solution $u_- \in -\text{int} (C^1_0(\Omega)_+) \cap [-a_\lambda e, 0]$ to Problem (1.2), where $a_\lambda > 0$ is given by Theorem 2.2.

**Proof.** Fix $\lambda > \lambda_1$. Since the reasoning for $u_-$ and $u_+$ are similar, we shall discuss only the case of $u_+$. Keep the same notation introduced in the proof of Theorem 3.1. By that result, for every positive integer $n$ sufficiently large there is a least solution $u_n \in \text{int} (C^1_0(\Omega)_+) \cap [\frac{1}{n} \varphi_1, \bar{u}]$ to (1.2). The minimality property of $u_n$ gives

$$u_n \downarrow u_+ \quad \text{pointwise in } \Omega$$  

(3.18)

for some $u_+ : \Omega \to \mathbb{R}$ satisfying $0 \leq u_+ \leq \bar{u}$. We claim that

the function $u_+$ turns out to be a solution of Problem (1.2).

(3.19)
In fact, from (1.4), with \( u := u_n \), it follows
\[
\langle -\Delta_p u_n, \varphi \rangle = \int_{\Omega} (\lambda |u_n|^{p-2} u_n - \eta_n) \varphi \, dx \quad \forall \varphi \in W_0^{1,p}(\Omega), \tag{3.20}
\]
where \( \eta_n \in L^p(\Omega) \) and \( \eta_n(x) \in \partial G(x, u_n(x)) \) for almost all \( x \in \Omega \). If \( \varphi := u_n \) then
\[
\|u_n\|^p = \int_{\Omega} (\lambda |u_n|^p - \eta_n) u_n \, dx, \quad n \in \mathbb{N}. \tag{3.21}
\]
Due to \((g_2)\), besides the inequality \( 0 \leq u_n \leq \bar{u} \), the sets \( \partial G(x, u_n(x)), x \in \Omega, n \in \mathbb{N} \), are uniformly bounded. Hence, there exists an \( a_3 > 0 \) such that
\[
|\eta_n(x)| \leq a_3 \quad \text{a.e. in } \Omega, \quad \forall n \in \mathbb{N}. \tag{3.22}
\]
Thus, by (3.21), the sequence \( \{u_n\} \subseteq W_0^{1,p}(\Omega) \) is bounded too. Taking a subsequence when necessary, we may suppose
\[
u_n \rightharpoonup u_+ \text{ in } W_0^{1,p}(\Omega), \quad u_n \to u_+ \text{ in } L^p(\Omega). \tag{3.23}
\]
On account of (3.20) with \( \varphi := u_n - u_+ \) one has
\[
\langle -\Delta_p u_n, u_n - u_+ \rangle = \lambda \int_{\Omega} (|u_n|^p - |u_n|^{p-2} u_n u_+) \, dx - \int_{\Omega} \eta_n(u_n - u_+) \, dx \quad \forall n \in \mathbb{N}.
\]
Now, through (3.23), (3.18), and the Lebesgue Dominated Convergence Theorem, this forces \( \lim_{n \to \infty} \langle A(u_n), u_n - u_+ \rangle = 0 \). Thanks to \((p_4)\) we then obtain
\[
u_n \to u_+ \text{ in } W_0^{1,p}(\Omega). \tag{3.24}
\]
Using (3.22) yields a function \( \eta_+ \in L^p(\Omega) \) such that \( \eta_n \rightharpoonup \eta_+ \text{ in } L^p(\Omega) \). By (3.24), Lemma 2.1 can be applied to get \( \eta_+(x) \in \partial G(x, u_+(x)) \) for almost every \( x \in \Omega \). From (3.20) it finally follows
\[
\langle -\Delta_p u_+, \varphi \rangle = \int_{\Omega} (\lambda |u_+|^{p-2} u_+ - \eta_+) \varphi \, dx \quad \forall \varphi \in W_0^{1,p}(\Omega),
\]
namely
\[
-\Delta_p u_+ = \lambda |u_+|^{p-2} u_+ - \eta_+,
\]
and (3.19) is proved.
Next, since \( u_+ \in L^\infty(\Omega) \), assumption \((g_2)\) produces \( \Delta_p u \in L^\infty(\Omega) \). Owing to (3.16) we achieve, as before,
\[
\Delta_p u_+ \leq (\lambda + \tilde{c}_\lambda) u_+^{p-1}.
\]
The Vázquez Maximum Principle [9, Theorem 1.5.7] ensures that either $u_+ \equiv 0$ or $u_+ \in \text{int}(C_0^1(\Omega)_+)$. If the assertion
\begin{equation}
 u_+ \in \text{int}(C_0^1(\Omega)_+)
\end{equation}
were false then $u_+ \equiv 0$. Accordingly, in view of (3.18),
\begin{equation}
 u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega .
\end{equation}
Setting
\[ \tilde{u}_n = \frac{u_n}{|u_n|}, \quad n \in \mathbb{N}, \]
we may suppose (along a relabelled subsequence, when necessary)
\begin{equation}
 \tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } W_0^{1,p}(\Omega), \quad \tilde{u}_n \to \tilde{u} \quad \text{in } L^p(\Omega),
\end{equation}
as well as
\begin{equation}
 |\tilde{u}_n(x)| \leq w(x) \quad \forall \ n \in \mathbb{N}, \quad \tilde{u}_n(x) \to \tilde{u}(x) \quad \text{for almost all } x \in \Omega ,
\end{equation}
with $w \in L^p(\Omega)_+$. Through (3.20) it results in
\begin{equation}
 \langle -\Delta_p \tilde{u}_n, \varphi \rangle = \lambda \int_{\Omega} \tilde{u}_n^{p-1}\varphi \ dx - \int_{\Omega} \frac{\eta_n}{u_n^{p-1}}\tilde{u}_n^{p-1}\varphi \ dx \quad \forall \ \varphi \in W_0^{1,p}(\Omega). \tag{3.29}
\end{equation}
If $\varphi := \tilde{u}_n - \tilde{u}$ then
\begin{equation}
 \langle -\Delta_p \tilde{u}_n, \tilde{u}_n - \tilde{u} \rangle = \lambda \int_{\Omega} \tilde{u}_n^{p-1}(\tilde{u}_n - \tilde{u}) \ dx - \int_{\Omega} \frac{\eta_n}{u_n^{p-1}}\tilde{u}_n^{p-1}(\tilde{u}_n - \tilde{u}) \ dx. \tag{3.30}
\end{equation}
By (3.16), (3.28) there exists a constant $\tilde{c}_\lambda > 0$ fulfilling
\[ \frac{|\eta_n(x)|}{u_n(x)^{p-1}}|\tilde{u}_n(x)^{p-1}| \leq \tilde{c}_\lambda w(x)^p |\tilde{u}_n(x) - \tilde{u}(x)| \leq 2\tilde{c}_\lambda w(x)^p \]
almost everywhere in $\Omega$. Due to (3.28), besides the Lebesgue Dominated Convergence Theorem, we get
\[ \lim_{n \to \infty} \int_{\Omega} \frac{\eta_n}{u_n^{p-1}}\tilde{u}_n^{p-1}(\tilde{u}_n - \tilde{u}) \ dx = 0. \]
Hence, from (3.30) and (3.28) again it follows $\lim_{n \to \infty} \langle A(\tilde{u}_n), \tilde{u}_n - \tilde{u} \rangle = 0$, which, on account of $(p_4)$, forces
\begin{equation}
 \tilde{u}_n \to \tilde{u} \quad \text{in } W_0^{1,p}(\Omega). \tag{3.31}
\end{equation}
So, in particular, \( \|\tilde{u}\| = 1 \). Gathering (3.29), (3.31), (3.26), and (g3) together provides

\[
\langle -\Delta_p \tilde{u}, \varphi \rangle = \lambda \int_{\Omega} \tilde{u}^{p-1} \varphi \, dx \quad \forall \varphi \in W_0^{1,p}(\Omega),
\]

i.e., \( \tilde{u} \) is an eigenfunction of \( -\Delta_p \) in \( W_0^{1,p}(\Omega) \) corresponding to the eigenvalue \( \lambda > \lambda_1 \). By [10, Proposition 6.2.15], the function \( \tilde{u} \) must change sign in \( \Omega \), whereas (3.28) and (3.26) imply \( \tilde{u}(x) \geq 0 \) for almost all \( x \in \Omega \). Therefore, (3.25) holds.

Let us finally verify that \( u_+ \) is the least positive solution of (1.2) within \([0,\bar{u}]\).

If \( u \in W_0^{1,p}(\Omega) \cap [0,\bar{u}] \), \( u > 0 \) in \( \Omega \), and \( u \) satisfies (1.2) then, by [10, Theorem 6.2.7], \( u \in C_0^1(\Omega) \). The same argument employed before regarding \( u_\lambda \) and \( u_+ \) yields now \( u \in \text{int}(C_0^1(\Omega)_+) \). Consequently, \( u \in [\frac{1}{n}\varphi_1,\bar{u}] \) for any sufficiently large \( n \). Since \( u_n \) is the least solution of (1.2) in \([\frac{1}{n}\varphi_1,\bar{u}]\) we have \( u_n \leq u \). As \( n \) was arbitrary, (3.18) leads to \( u_+ \leq u \), and the conclusion follows.

\section{Sign-changing solutions}

A third non-zero, sign-changing solution to (1.2) can be obtained when \( \lambda > \lambda_2 \), the second eigenvalue of \( -\Delta_p \) in \( W_0^{1,p}(\Omega) \), as the next result shows.

\begin{thm}
Under assumptions (g1)–(g4), for every \( \lambda > \lambda_2 \) Problem (1.2) possesses a positive solution \( u_+ \in \text{int}(C_0^1(\Omega)_+) \), a negative one \( u_- \in -\text{int}(C_0^1(\Omega)_+) \), and a nontrivial sign-changing solution \( u_0 \in C_0^1(\Omega) \).
\end{thm}

\textbf{Proof.} Fix \( \lambda > \lambda_2 \). If \( u_+ \) and \( u_- \) are given by Theorem 3.2 then there exist \( \eta_+, \eta_- \in L^p(\Omega) \) such that

\[
-\Delta_p u_+(x) = \lambda u_+(x)^{p-1} - \eta_+(x), \quad -\Delta_p u_-(x) = \lambda|u_-(x)|^{p-2}u_-(x) - \eta_-(x),
\]

as well as

\[
\eta_+(x) \in \partial G(x,u_+(x)) , \quad \eta_-(x) \in \partial G(x,u_-(x))
\]

for almost every \( x \in \Omega \). Define, whenever \( (x,t) \in \Omega \times \mathbb{R} \),

\[
\tau_+(x,t) := \begin{cases} 0 & \text{if } t < 0 , \\ t & \text{for } 0 \leq t \leq u_+(x) , \\ u_+(x) & \text{when } t > u_+(x) , \end{cases} \quad \tau_-(x,t) := \begin{cases} u_-(x) & \text{if } t < u_-(x) , \\ t & \text{for } u_-(x) \leq t \leq 0 , \\ 0 & \text{when } t > 0 , \end{cases}
\]

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\[ \tau_0(x, t) := \begin{cases} u_-(x) & \text{if } t < u_-(x), \\ t & \text{for } u_-(x) \leq t \leq u_+(x), \\ u_+(x) & \text{when } t > u_+(x), \end{cases} \]

and

\[ g_+(x, t) := \begin{cases} 0 & \text{if } t < 0, \\ g(x, t) & \text{for } 0 \leq t \leq u_+(x), \\ \eta_+(x) & \text{when } t > u_+(x), \end{cases} \]
\[ g_-(x, t) := \begin{cases} \eta_-(x) & \text{if } t < u_-(x), \\ g(x, t) & \text{for } u_-(x) \leq t \leq 0, \\ 0 & \text{when } t > 0, \end{cases} \]
\[ g_0(x, t) := \begin{cases} \eta_-(x) & \text{if } t < u_-(x), \\ g(x, t) & \text{for } u_-(x) \leq t \leq u_+(x), \\ \eta_+(x) & \text{when } t > u_+(x). \end{cases} \]

Moreover, set, provided \( u \in W_0^{1,p}(\Omega) \),

\[ E_+(u) := \frac{1}{p} \|u\|^p - \int_{\Omega} \left( \int_0^{u(x)} (\lambda \tau_+(x, t))^{p-1} - g_+(x, t) \right) dt \right) dx, \]
\[ E_-(u) := \frac{1}{p} \|u\|^p - \int_{\Omega} \left( \int_0^{u(x)} (\lambda \tau_-(x, t))^{p-2} \tau_-(x, t) - g_-(x, t) \right) dt \right) dx, \]
\[ E_0(u) := \frac{1}{p} \|u\|^p - \int_{\Omega} \left( \int_0^{u(x)} (\lambda \tau_0(x, t))^{p-2} \tau_0(x, t) - g_0(x, t) \right) dt \right) dx. \]

Due to \((g_2)\), besides the regularity properties of \( u_+ \) and \( u_- \), the functionals \( E_+, E_-, E_0 : W_0^{1,p}(\Omega) \to \mathbb{R} \) are locally Lipschitz continuous. We claim that

each critical point of \( E_+ \) belongs to \([0, u_+]\). \hfill (4.1)

In fact, if \( u \in W_0^{1,p}(\Omega) \) fulfills \( 0 \in \partial E_+(u) \) then \(-\Delta_p u = \tau_+(x, u)^{p-1} - w\) for some \( w \in L^p(\Omega) \) such that \( w(x) \in \partial G_+(x, u(x)) \) almost everywhere in \( \Omega \), with

\[ G_+(x, \xi) := \int_0^{\xi} g_+(x, t) dt, \quad (x, \xi) \in \Omega \times \mathbb{R}. \]

Choosing the test function \( \varphi := (u - u_+)^+ \) gives

\[ \langle A(u) - A(u_+), (u - u_+)^+ \rangle = \int_{\Omega} [\lambda \tau_+(x, u)^{p-1} - w - \lambda u_+^{p-1} + \eta_+](u - u_+)^+ dx = 0. \]
On account of \((p_1)\) this implies \(u \leq u_+\). Similarly, from
\[
\langle A(u), -u^- \rangle = - \int_{\Omega} [\lambda \tau_+(x, u)^{p-1} - w] u^- \, dx = 0,
\]
it follows \(u \geq 0\). Hence, assertion \((4.1)\) holds.

An easy verification ensures that the functional \(E_+\) is bounded below, weakly sequentially lower semicontinuous, and coercive. So, there exists a \(v_+ \in W^{1,p}_0(\Omega)\) satisfying
\[
E_+(v_+) = \inf_{u \in W^{1,p}_0(\Omega)} E_+(u),
\]
which forces both \(v_+ \in [0, u_+]\), on account of \((4.1)\), and \(-\Delta_p v_+ \in \lambda v_+^{p-1} - \partial G_+(x, v_+)\). Since \(\partial G_+(x, v_+(x)) \subset \partial G(x, v_+(x))\), \(x \in \Omega\), the function \(v_+\) turns out to be a solution of \((1.2)\). Moreover, \(v_+ \neq 0\). Indeed, through \((3.25)\) we get
\[
t \varphi_1 \leq u_+, \quad t \|\varphi_1\|_{L^\infty(\Omega)} \leq \delta_\lambda,
\]
with \(\delta_\lambda\) as in \((2.13)\), provided \(t > 0\) is sufficiently small. Consequently, by \((4.2)\), \((p_2)\), and \((2.13)\),
\[
E_+(v_+) \leq E_+(t \varphi_1) = \frac{\lambda_1}{p} t^p - \int_{\Omega} \left( \int_0^{t \varphi_1(x)} (\lambda s^{p-1} - g(x, s)) \, ds \right) \, dx < 0,
\]
namely \(v_+ \neq 0\). At this point, the same argument exploited in the proof of Theorem 3.2 to achieve \((3.25)\) shows here that
\[
v_+ \in \operatorname{int} \left( C^1_0(\overline{\Omega}) \right).
\]
Gathering \((4.4)\), the inequality \(v_+ \leq u_+\), and Theorem 3.2 together produces \(v_+ = u_+\). Thus, due to \((4.2)\), besides \((4.4)\), the function \(u_+\) is a local minimizer of \(E_0\) in \(C^1_0(\Omega)\). Proposition 4.6.10 of [9] guarantees that \(u_+\) enjoys the same property in the space \(W^{1,p}_0(\Omega)\). Likewise, replacing the functional \(E_+\) with \(E_-\) one realizes that \(u_-\) is a local minimizer of \(E_0\) in \(W^{1,p}_0(\Omega)\).

Since \(E_0\) is bounded below, weakly sequentially lower semicontinuous, and coercive, there exists a \(v_0 \in W^{1,p}_0(\Omega)\) fulfilling \(E_0(v_0) = \inf_{u \in W^{1,p}_0(\Omega)} E_0(u)\). Moreover, as before, it results in \(v_0 \neq 0\) and
\[
each \text{critical point of } E_0 \text{ belongs to } [u_-, u_+].
\]
Therefore, \(v_0 \in [u_-, u_+]\) and \(v_0\) is a nontrivial solution of \((1.2)\). Without loss of generality we may suppose \(v_0 = u_+\) or \(v_0 = u_-\), because otherwise the extremality of \(u_+\) and \(u_-\)
established in Theorem 3.2 would force a changing of sign for \( v_0 \), which completes the proof. So, let \( v_0 = u_+ \) (a similar reasoning applies when \( v_0 = u_- \)). We may assume also that \( u_- \) is a strict local minimizer of \( E_0 \). In fact, if this were false then infinitely many sign-changing solutions to (1.2) might be found via (4.5) besides the extremality of \( u_+ \), \( u_- \), and the conclusion follows. Pick a \( \rho \in ]0, \| u_+ - u_- \| [ \) such that

\[
E_0(u_+) \leq E_0(u_-) < \inf_{u \in \partial B_{\rho}(u_-)} E_0(u), \tag{4.6}
\]

The functional \( E_0 \) satisfies condition (PS), because it is bounded below, locally Lipschitz continuous, and coercive; see e.g. [13, Corollary 2.4]. Bearing in mind (4.6), Theorem 2.1 can be applied. Hence, there is an \( u_0 \in W^{1,p}_0(\Omega) \) complying with \( 0 \in \partial E_0(u_0) \) and

\[
\inf_{u \in \partial B_{\rho}(u_-)} E_0(u) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} E_0(\gamma(t)), \tag{4.7}
\]

where

\[
\Gamma := \{ \gamma \in C^0([-1,1], W^{1,p}_0(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+ \}.
\]

By (4.5) one has \( \partial G_0(x, u_0(x)) \subseteq G(x, u_0(x)) \), \( x \in \Omega \), i.e., \( u_0 \) solves (1.2). Moreover, thanks to (4.6)–(4.7), \( u_0 \neq u_- \) and \( u_0 \neq u_+ \). The conclusion is thus achieved once we show that \( u_0 \in C^1(\Omega) \setminus \{0\} \). Let us start with \( u_0 \neq 0 \). This immediately comes out from the inequality

\[
E_0(u_0) < 0, \tag{4.8}
\]

which, in view of (4.7), holds as soon as we construct a \( \hat{\gamma} \in \Gamma \) such that

\[
E_0(\hat{\gamma}(t)) < 0 \quad \forall t \in [-1,1]. \tag{4.9}
\]

Set \( S := \{ u \in W^{1,p}_0(\Omega) : \| u \|_{L^p(\Omega)} = 1 \} \) and fix \( \mu \in ]0, \lambda - \lambda_2[. \) Assumption (g3) yields a \( \delta_\mu > 0 \) such that

\[
\frac{|g(x,t)|}{|t|^{p-1}} \leq \mu \quad \text{provided } 0 < |t| \leq \delta_\mu. \tag{4.10}
\]

If \( \rho_0 \in ]0, \lambda - \lambda_2 - \mu[ \) then, due to (p3), there exists a \( \gamma \in \Gamma_0 \) satisfying

\[
\max_{t \in [-1,1]} \| \gamma(t) \|^p < \lambda_2 + \frac{\rho_0}{2}. 
\]

Now, define \( S_C := S \cap C^1_0(\Omega) \) and consider on \( S_C \) the topology induced by that of \( C^1_0(\Omega) \). Clearly, \( S_C \) is a dense subset of \( S \). So, given \( r > 0 \), with

\[
r \leq (\lambda_2 + \rho_0)^{1/p} - \left(\lambda_2 + \frac{\rho_0}{2}\right)^{1/p},
\]

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we can find a $\gamma_0 \in C^0([-1, 1], S_C)$ such that $\gamma_0(-1) = -\varphi_1$, $\gamma_0(1) = \varphi_1$, and

$$\max_{t \in [-1, 1]} \|\gamma(t) - \gamma_0(t)\| < r.$$  

This evidently forces

$$\max_{t \in [-1, 1]} \|\gamma(t)\|^p < \lambda_2 + \rho_0.$$  

(4.11)

Let $\varepsilon_1 > 0$ fulfil

$$\varepsilon_1 \max_{x \in \Omega} |u(x)| \leq \delta_{\mu} \quad \forall u \in \gamma_0([-1, 1]).$$  

(4.12)

Since $u_+, -u_- \in \text{int}(C_0^1(\Omega)_+)$, to every $u \in \gamma_0([-1, 1])$ and every bounded neighborhood $V_u$ of $u$ in $C_0^1(\Omega)$ there corresponds a $\nu > 0$ such that

$$u_+ - \frac{1}{m}v \in \text{int}(C_0^1(\Omega)_+), \quad -u_- + \frac{1}{n}v \in \text{int}(C_0^1(\Omega)_+) \quad \text{whenever } m, n \geq \nu, \; v \in V_u.$$  

Through the compactness of $\gamma_0([-1, 1])$ in $C_0^1(\Omega)$ we thus obtain an $\varepsilon_0 > 0$ satisfying

$$u_-(x) \leq \varepsilon u(x) \leq u_+(x) \quad \text{for all } x \in \Omega, \; u \in \gamma_0([-1, 1]), \; \varepsilon \in ]0, \varepsilon_0[.$$  

(4.13)

The function $t \mapsto \varepsilon \gamma(t), \; t \in [0, 1]$, is a continuous path in $S_C$ joining $-\varepsilon \varphi_1$ and $\varepsilon \varphi_1$. Moreover, if $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ then, (4.11), (4.13), (4.12), and (4.10) yield

$$E_0(\varepsilon \gamma(t)) = \frac{\varepsilon^p}{p} \|\gamma(t)\|^p - \frac{\varepsilon^p}{p} \lambda + \int_{\Omega} \left( \int_0^{\varepsilon \gamma(t)(x)} g(x, \tau_0(x, s)) \, ds \right) \, dx$$

$$\leq \frac{\varepsilon^p}{p} (\lambda_2 + \rho_0 - \lambda) + \int_{\Omega} \left( \int_0^{\varepsilon \gamma(t)(x)} g(x, s) \, ds \right) \, dx$$

$$\leq \frac{\varepsilon^p}{p} (\lambda_2 + \rho_0 - \lambda + \mu) < 0 \quad \forall t \in [-1, 1].$$  

(4.14)

Next, write

$$a_4 := E_+(u_+), \quad U_+ := \{u \in W_0^{1,p}(\Omega) : E_+(u) < 0\}.$$  

One clearly has $a_4 < 0$, because $u_+ = v_+$ and $E_+(v_+) < 0$ by (4.3). Hence, $U_+$ turns out to be nonempty. Moreover, $a_4 = \inf_{u \in W_0^{1,p}(\Omega)} E_+(u)$ on account of (4.2). Gathering (4.1) and Theorem 3.2 together ensures that $E_+$ has no critical point $u$ with $a_4 < E_+(u) < 0$ and that $K(E_+ \cap E_+^{-1}(a_4)) = \{u_+\}$. Finally, since $E_+$ is bounded below, locally Lipschitz continuous, and coercive, it satisfies condition (PS). So, thanks to Theorem 2.10 of [7], there exists a continuous function $h : [0, 1] \times U_+ \to U_+$ fulfilling

$$h(0, \cdot) = \text{id}|_{U_+}, \quad h(1, U_+) = \{u_+\}, \quad E_+(h(t, u)) \leq E_+(u) \quad \forall (t, u) \in [0, 1] \times U_+.$$
Let $\gamma_+: [0, 1] \to U_+$ defined by $\gamma_+(t) := h(t, \varepsilon \phi_1)^+$ for every $t \in [0, 1]$. Then $\gamma_+(0) = \varepsilon \phi_1$, $\gamma_+(1) = u_+$, as well as

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \leq E_+(h(t, \varepsilon \phi_1)) \leq E_+(\varepsilon \phi_1) < 0, \quad t \in [0, 1]. \quad (4.15)$$

In a similar way, but with $E_-$ in place of $E_+$, we can construct a continuous function $\gamma_-: [0, 1] \to W^{1,p}_0(\Omega)$ such that $\gamma_-(0) = -\varepsilon \phi_1$, $\gamma_-(1) = u_-$, and

$$E_0(\gamma_-(t)) < 0 \quad \forall \, t \in [0, 1]. \quad (4.16)$$

Concatenating $\gamma_-$, $\gamma_0$, and $\gamma_+$ produces a path $\hat{\gamma} \in \Gamma$ which, in view of (4.14)–(4.16), satisfies (4.9). This shows (4.8) and, consequently, $u_0 \neq 0$. To complete the proof we simply note that via the same argument, based on Theorem 6.2.7 of [10], exploited before it results in $u_0 \in C^1_0(\Omega)$. □

**Remark 4.1.** The preceding proof is patterned after that of [4, Theorem 4.1]; see also [17, Theorem 3.9]. If the function $t \mapsto g(x, t)$ is continuous on $\mathbb{R}$ then $\partial G(x, \xi) = \{g(x, \xi)\}$, and Problem (1.2) reduces to (1.1). However, even in this setting Theorem 4.1 is more general than Theorem 3.9 of [17], because we do not assume that $g(x, t) \geq 0$ for all $t \in \mathbb{R}$.

**Remark 4.2.** Theorem 4.1 improves Corollary 3.2 of [11]. In fact, let $p = 2$, let $u \in W^{1,2}_0(\Omega)$ be a solution of (1.2) with $g(x, t) \equiv g(t)$, $(x, t) \in \Omega \times \mathbb{R}$, and let $\eta \in L^p(\Omega)$ fulfil (1.3)–(1.4).

By the definition of $\partial G(u(x))$ we have, for any $\varphi \in W^{1,2}_0(\Omega)$,

$$\eta(x) \varphi(x) \leq G^0(u(x); \varphi(x)) \quad \text{a.e. in } \Omega.$$

Hence, due to (1.4),

$$- \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \lambda \int_{\Omega} u \varphi \, dx = \int_{\Omega} \eta \varphi \, dx \leq \int_{\Omega} G^0(u(x); \varphi(x)) \, dx \quad \forall \, \varphi \in W^{1,2}_0(\Omega),$$

i.e., $u$ turns out to be a solution of the hemivariational inequality studied in [11]. Since the hypotheses of [11, Corollary 3.2] imply (g1)–(g4), the assertion follows.

**References**


