EXISTENCE AND ASYMPTOTIC PROPERTIES FOR QUASILINEAR ELLIPTIC EQUATIONS WITH GRADIENT DEPENDENCE

DIEGO AVERNA, DUMITRU MOTREANU, AND ELISABETTA TORNATORE

Abstract. The paper focuses on a Dirichlet problem driven by the \((p,q)\)-Laplacian containing a parameter \(\mu > 0\) in the principal part of the elliptic equation and a (convection) term fully depending on the solution and its gradient. Existence of solutions, uniqueness, a priori estimates, and asymptotic properties as \(\mu \to 0\) and \(\mu \to \infty\) are established under suitable conditions.

1. Introduction

In this paper we focus on the following nonlinear Dirichlet problem driven the \((p,q)\)-Laplacian operator \((P_\mu)\)

\[
\begin{cases}
-\Delta_p u - \mu \Delta_q u = f(x,u,\nabla u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here \(\Omega \subset \mathbb{R}^N\) is a nonempty bounded open set with the boundary \(\partial \Omega\), and \(\mu\) is a positive real parameter. In the statement of problem \((P_\mu)\), with given numbers \(1 < q < p\), \(\Delta_p\) and \(\Delta_q\) stand for the \(p\)-Laplacian and \(q\)-Laplacian, respectively, that is \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) and \(\Delta_q u = \text{div}(|\nabla u|^{q-2} \nabla u)\). The right-hand side of the equation in \((P_\mu)\) is expressed through \(f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\), which is a Carathéodory function, i.e \(f(\cdot,s,\xi)\) is measurable for all \((s,\xi)\) \(\in \mathbb{R} \times \mathbb{R}^N\) and \(f(x,\cdot,\cdot)\) is continuous for a.e. \(x \in \Omega\).

We also examine the limiting case of problem \((P_\mu)\), namely if \(\mu = 0\). In this case \((P_0)\) becomes the problem driven by the \(p\)-Laplacian operator

\[
\begin{cases}
-\Delta_p u = f(x,u,\nabla u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The main point in our study is the fact that the right-hand side of problems \((P_\mu)\) and \((P_0)\) depends on the solution \(u\) and on its gradient \(\nabla u\). The expression \(f(x,u,\nabla u)\) is often called convection term. Due to the presence of the gradient \(\nabla u\) in the term \(f(x,u,\nabla u)\), problems \((P_\mu)\) and \((P_0)\) do not have generally variational structure, so the variational methods are not applicable. In view of this difficulty, problem \((P_\mu)\) in its general form is rarely studied in the literature. It is more investigated problem \((P_0)\) (see [2], [3], [4], [5], [10], [11], and the references therein) and the variational case in problem \((P_\mu)\) where the right-hand side does not depend
on the gradient $\nabla u$, i.e., $f(x, s, \xi) = f(x, s)$ (see [6], [7], [9], and the references therein).

Under only two hypotheses on the function $f(x, s, \xi)$, we show that we have existence of solutions for all problems $(P_\mu)$, with $\mu > 0$, and $(P_0)$. Our approach relies on the theory of pseudomonotone operators for which we refer to the monographs [1], [7], [12]. Adding a further condition, a uniqueness result is also produced. Under the same hypotheses as for the existence part, we establish a priori estimates for the solutions of $(P_\mu)$. Based on them, we look at asymptotic properties of the solution sets of $(P_\mu)$ regarding $\mu$ as parameter. In this respect, a principal objective of the present paper is to show that in the limit as $\mu \to 0$ we obtain a solution of $(P_0)$ that is approached in the space $W^{1,p}_0(\Omega)$ through a sequence of solutions of problems $(P_\mu)$, whereas letting $\mu \to +\infty$ along the solutions of problems $(P_\mu)$ we reach zero in the space $W^{1,q}_0(\Omega)$.

The rest of the paper is organized as follows. Section 2 deals with existence and uniqueness of solution to problem $(P_\mu)$. Section 3 is devoted to the asymptotic properties related to problem $(P_\mu)$ when $\mu \to 0$ and $\mu \to +\infty$.

2. Existence and uniqueness of solution to problem $(P_\mu)$

In the sequel, for every $r \in [1, +\infty]$ we denote by $r'$ its Hölder conjugate, i.e., $r' \equiv \frac{1}{r} + \frac{1}{r'} = 1$. In particular, this applies to the Sobolev critical exponent $p^*$ with its conjugate $(p^*)'$. Recall that $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \leq p$. The strong convergence and the weak convergence are denoted by $\rightarrow$ and $\rightharpoonup$, respectively.

Consider the Sobolev space $W^{1,p}_0(\Omega)$ endowed with the norm $\|u\| := \|\nabla u\|_{L^p(\Omega)}$ for all $u \in W^{1,p}_0(\Omega)$. In studying problem $(P_\mu)$ we rely on the negative $p$-Laplacian $-\Delta_p : W^{1,p}_0(\Omega) \to (W^{1,p}_0(\Omega))' = W^{-1,p}_0(\Omega)$. It is well known that the operator $-\Delta_p$ is continuous, bounded, pseudomonotone and has the $S_p$-property (see [1], [7]). We denote by $\lambda_{1,p}$ the first eigenvalue of $-\Delta_p$ on $W^{1,p}_0(\Omega)$. It has the variational characterization

$$\lambda_{1,p} = \inf_{u \in W^{1,p}_0(\Omega), u \neq 0} \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p}.$$

Throughout the paper we assume that the nonlinearity $f(x, s, \xi)$ satisfies the hypotheses:

(H1) There exist constants $a_1 \geq 0$, $a_2 \geq 0$, $\alpha \in [0, p^* - 1]$, $\beta \in [0, \frac{p}{p-1}]$ and a function $\sigma \in L^\gamma(\Omega)$, with $\gamma \in [1, p^*]$, such that

$$|f(x, s, \xi)| \leq \sigma(x) + a_1|s|^{\alpha} + a_2|\xi|^\beta \text{ a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H2) there exist constants $d_1 \geq 0$, $d_2 \geq 0$ with $\lambda_{1,p}^{-1}d_1 + d_2 < 1$, and a function $\omega \in L^1(\Omega)$ such that

$$f(x, s, \xi)s \leq \omega(x) + d_1|s|^p + d_2|\xi|^p \text{ a.e. } x \in \Omega, \ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

A (weak) solution of problem $(P_\mu)$ for $\mu \geq 0$ is any $u \in W^{1,p}_0(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx + \mu \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v \, dx = \int_\Omega f(x, u, \nabla u) v \, dx$$

for all $v \in W^{1,p}_0(\Omega)$.
for all $v \in W^{1,p}_0(\Omega)$. According to hypothesis (H1) and Hölder’s inequality, the integrals exist in the definition of weak solution as given in (2.1). Indeed, let us note that
\begin{equation}
(2.2)
    f(x, u, \nabla u) \in L^{r^*} (\Omega), \forall u \in W^{1,p}_0(\Omega),
\end{equation}
with some $r \in [1, \rho^*]$, as can be easily checked by using the growth condition in (H1) and Sobolev embedding theorem.

**Theorem 1.** Assume that conditions (H1) and (H2) hold. Then problem $(P_\mu)$, with $\mu \geq 0$, admits at least one weak solution $u_\mu \in W^{1,p}_0(\Omega)$.

**Proof.** We are going to prove the existence of weak solutions to problem $(P_\mu)$ by means of the theory for pseudomonotone operators. Specifically, corresponding to $(P_\mu)$ we introduce the nonlinear operator $A : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by
\begin{equation}
(2.3)
    A(u) = -\Delta_p u - \mu \Delta_q u - N(u),
\end{equation}
where $N : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ denotes the Nemytskii operator associated to $f$, that is $N(u) = f(x, u, \nabla u)$. It is known from (2.2), that $N(u) \in W^{-1,p'}(\Omega)$ for all $u \in W^{1,p}_0(\Omega)$.

It is clear from the growth condition in (H1) that $A : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is bounded, which means that it maps bounded sets onto bounded sets.

We claim that the operator $A$ in (2.3) is pseudomonotone. To this end let $\{u_n\} \subset W^{1,p}_0(\Omega)$ be such that $u_n \rightharpoonup u$ and $\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0$. We provide the proof in the case where $N > p$. The case $N \leq p$ is easier and thus we omit it. It is seen from hypothesis (H1) that $\gamma, \frac{p}{p - \alpha}, \frac{p}{p - \beta} < p'$. Then Rellich’s compact embedding theorem implies that $u_n \to u$ in $L^\gamma(\Omega), L^{\frac{p}{p - \alpha}}(\Omega)$, and $L^{\frac{p}{p - \beta}}(\Omega)$. This, in conjunction with hypothesis (H1) and applying Hölder inequality, leads to
\begin{equation}
(2.4)
    \lim_{n \to \infty} \int_{\Omega} f(x, u_n, \nabla u_n)(u_n - u) \, dx = 0.
\end{equation}

Taking into account (2.3) and (2.4), we infer that
\begin{align*}
\limsup_{n \to +\infty} (-\Delta_p u_n - \mu \Delta_q u_n, u_n - u) &= \limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \\
&\leq 0.
\end{align*}

At this point, the $S_+$-property of the operator $-\Delta_p - \mu \Delta_q$ on the space $W^{1,p}_0(\Omega)$ can be used (see, e.g., [7, Proposition 2.70]) to derive the strong convergence $u_n \to u$ in $W^{1,p}_0(\Omega)$. Now it is straightforward to get that $A(u_n) \rightharpoonup A(u)$ in $W^{-1,p'}(\Omega)$, which ensures in particular that the operator $A$ is pseudomonotone.

Let us prove that $A$ is coercive, which means to have
\begin{equation}
\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.
\end{equation}

On the basis of hypothesis (H2), it turns out that
\begin{align*}
\langle Au, u \rangle &= \|\nabla u\|_{L^p(\Omega)}^p + \mu \|\nabla u\|_{L^q(\Omega)}^q - \int_{\Omega} f(x, u, \nabla u) u \, dx \\
&\geq (1 - d_1 \lambda_{1,p}^{-1} - d_2) \|\nabla u\|_{L^p(\Omega)}^p - \|u\|_{L^1(\Omega)}.
\end{align*}

It follows that $A$ is coercive because $p > 1$ and $\lambda_{1,p}^{-1} d_1 + d_2 < 1$.

Since $A : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is pseudomonotone, bounded and coercive, we can apply the main theorem on pseudomonotone operators (see [1, Theorem 2.99],
Therefore there is at least one element \( u_\mu \in W_0^{1,p}(\Omega) \) such that \( Au_\mu = 0 \), so \( u_\mu \) is a weak solution of problem (\( P_\mu \)), which completes the proof. \( \square \)

The final part of the section deals with the uniqueness of solution to problem (\( P_\mu \)), which can hold only under strong hypotheses (see [8] for the case where \( f \) in (\( P_\mu \)) does not depend on the gradient \( \nabla u \)). We illustrate this topic by presenting a uniqueness result in the case where \( p = 2 \) or \( q = 2 \). Our assumption is as follows:

(U)(a) there exists a constant \( b_1 \geq 0 \) such that
\[
(f(x,s,\xi) - f(x,t,\xi))(s-t) \leq b_1 |s-t|^2 \text{ a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N, \forall s, t \in \mathbb{R};
\]
(U)(b) there exist a function \( \tau \in L^\delta(\Omega) \), with some \( \delta \in [1,p^*] \), and a constant \( b_2 \geq 0 \) such that the function \( f(x,s,\cdot) - \tau(x) \) is linear and
\[
|f(x,s,\xi) - \tau(x)| \leq b_2 |\xi| \text{ a.e. } x \in \Omega, \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}^N.
\]

**Theorem 2.** Assume that conditions (H1), (H2), (U)(a) and (U)(b) hold.

(i) If \( p > 2 > q > 1 \) and \( b_1 \lambda_{1,1}^{1,2} + b_2 \lambda_{1,2}^{1,2} < 1 \), then the solution of problem (\( P_\mu \)) is unique for every \( \mu > 0 \).

(ii) If \( p > q = 2 \), then the solution of problem (\( P_\mu \)) is unique for every \( \mu > b_1 \lambda_{1,1}^{1,2} + b_2 \lambda_{1,2}^{1,2} \).

**Proof.** Since conditions (H1) and (H2) are supposed to be fulfilled, we may apply Theorem 1, which asserts that there exists a solution \( u_\mu \in W_0^{1,p}(\Omega) \) of problem (\( P_\mu \)) for every \( \mu > 0 \). Suppose that \( v_\mu \in W_0^{1,p}(\Omega) \) is a second solution of (\( P_\mu \)). Acting with \( u_\mu - v_\mu \) on the equation in (\( P_\mu \)) gives
\[
\begin{align*}
&(-\Delta_p u_\mu + \Delta_p v_\mu, u_\mu - v_\mu) + \mu(-\Delta_p u_\mu + \Delta_p v_\mu, u_\mu - v_\mu) \\
&= \int_\Omega (f(x,u_\mu,\nabla u_\mu) - f(x,v_\mu,\nabla v_\mu))(u_\mu - v_\mu) \, dx \\
&+ \int_\Omega (f(x,u_\mu,\nabla u_\mu) - f(x,v_\mu,\nabla v_\mu))(u_\mu - v_\mu) \, dx.
\end{align*}
\]

(i) For \( p = 2 \), hypotheses (U)(a) and (U)(b), in conjunction with (2.5), the monotonicity of \(-\Delta_p \) and H"{o}lder inequality imply
\[
\|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \leq b_1 \|u_\mu - v_\mu\|^2_{L^2(\Omega)} + \int_\Omega (f(x,u_\mu,\nabla \frac{1}{2}(u_\mu - v_\mu)^2) - \tau(x)) \, dx
\]
\[
\leq (b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-2}) \|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}.
\]

Using that \( b_1 \lambda_{1,1}^{-1} + b_2 \lambda_{1,2}^{-2} < 1 \), the equality \( u_\mu = v_\mu \) follows.

(ii) For \( p > q = 2 \), arguing as in the case of part (i), we find the estimate
\[
\mu \|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)} \leq (b_1 \lambda_{1,2}^{-1} + b_2 \lambda_{1,2}^{-2}) \|\nabla(u_\mu - v_\mu)\|^2_{L^2(\Omega)}.
\]
The conclusion that \( u_\mu = v_\mu \) ensues provided that \( b_1 \lambda_{1,1}^{-1} + b_2 \lambda_{1,2}^{-2} < \mu \), which completes the proof. \( \square \)

3. **Asymptotic properties as \( \mu \to 0 \) and \( \mu \to +\infty \)**

It is shown in Theorem 1 that problem (\( P_\mu \)) possesses a solution \( u_\mu \in W_0^{1,p}(\Omega) \) for every \( \mu > 0 \). We establish the following a priori estimate.

**Lemma 1.** Assume that conditions (H1) and (H2) hold. Then there exists a constant \( b > 0 \) independent of \( \mu > 0 \) such that
\[
\|\nabla u_\mu\|_{L^p(\Omega)} \leq b, \forall \mu > 0.
\]
Proof. Fix \( \mu > 0 \). Since \( u_\mu \in W_0^{1,p}(\Omega) \) is a solution of \((P_\mu)\), we can insert \( v = u = u_\mu \) in (2.1). Thanks to assumption (H2), for every \( \mu > 0 \) we get the estimate

\[
\|\nabla u_\mu\|_{L^p(\Omega)}^p \leq \frac{1}{\mu} f(x, u_\mu, \nabla u_\mu) u_\mu dx \leq (d_1 \lambda_{1,p}^{-1} + d_2) \|\nabla u_\mu\|_{L^p(\Omega)}^p + \|\omega\|_{L^1(\Omega)}.
\]

We have by hypothesis (H2) that \( \lambda_{1,p}^{-1} d_1 + d_2 < 1 \). Consequently, (3.1) is obtained by choosing

\[
b = \left( \frac{\|\omega\|_{L^1(\Omega)}}{1 - d_1 \lambda_{1,p}^{-1} - d_2} \right)^\frac{1}{p}.
\]

Next, taking advantage that \( \mu \) is a parameter, we consider the limit points of the net \( (u_\mu) \) as \( \mu \to 0 \) and \( \mu \to +\infty \). We start by letting \( \mu \to 0 \) in problem \((P_\mu)\).

Theorem 3. For any sequence \( \mu_n \to 0^+ \), there exists a relabeled subsequence of solutions \( (u_{\mu_n}) \) of the corresponding problems \((P_{\mu_n})\) such that \( u_{\mu_n} \to u \) in \( W_0^{1,p}(\Omega) \), with \( u \in W_0^{1,p}(\Omega) \) weak solution of problem \((P_0)\).

Proof. Set, for simplicity, \( u_n := u_{\mu_n} \). Since \( u_n \) is a weak solution of problem \((P_{\mu_n})\), we can apply Lemma 1 and deduce that the sequence \( (u_n) \) is bounded in \( W_0^{1,p}(\Omega) \). Then along a relabeled subsequence one has that \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) for some \( u \in W_0^{1,p}(\Omega) \).

Following the same reasoning based on hypothesis (H1) as in the proof of Theorem 1, we can show the validity of relation (2.4). Through the equation in \((P_{\mu_n})\), the fact that \( \mu_n \to 0^+ \) and using (2.4) we are led to

\[
\lim_{n \to +\infty} \langle -\Delta_p u_n, u_n - u \rangle = 0.
\]

Recalling that the operator \( -\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \) satisfies the \( S_+ \)-property, we conclude that \( u_n \to u \) in \( W_0^{1,p}(\Omega) \). As arrived at the strong convergence \( u_n \to u \) in \( W_0^{1,p}(\Omega) \), we can pass to the limit in the equation in problem \((P_{\mu_n})\) as \( n \to \infty \). Specifically, \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) implies that \( \nabla u_n \to \nabla u \) in \( L^p(\Omega)^N \), so the growth condition in assumption (H1) and Krasnoselskii’s theorem ensure

\[
N(u_n) = f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \to N(u) = f(\cdot, u(\cdot), \nabla u(\cdot))
\]

in \( L^q(\Omega) \) as \( n \to \infty \), for some \( \rho \in [1, p^*] \). Bearing in mind that \( -\Delta_p u_n \to -\Delta_p u \) in \( W^{-1,p'}(\Omega) \), \( \mu_n \to 0^+ \), and (3.2), letting \( n \to \infty \) in the equation of \((P_{\mu_n})\) allows us to see that \( u \) is a weak solution of problem \((P_0)\), which completes the proof. \( \square \)

We turn to the asymptotic property as \( \mu \to +\infty \).

Theorem 4. For any sequence \( \mu_n \to +\infty \), the sequence of solutions \( (u_{\mu_n}) \) of the corresponding problems \((P_{\mu_n})\) satisfies \( u_{\mu_n} \to 0 \) in \( W_0^{1,p}(\Omega) \).

Proof. Proceeding as in the proof of Theorem 3, we set \( u_n := u_{\mu_n} \) and apply Lemma 1 to derive that the sequence \( (u_n) \) is bounded in \( W_0^{1,p}(\Omega) \), so up to a relabeled subsequence we have \( u_n \to u \) in \( W_0^{1,p}(\Omega) \) for some \( u \in W_0^{1,p}(\Omega) \).

We note that \( u_n \) satisfies

\[
\begin{cases}
-\nabla u_n - \Delta_q u_n = \frac{1}{\mu_n} f(x, u_n, \nabla u_n) & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

If we act with \( u_n - u \) in (3.3), we find that

\[
\lim_{n \to +\infty} \langle -\Delta_q u_n, u_n - u \rangle = 0.
\]
This follows from (3.3) because $\mu_n \to +\infty$, the sequence $(\Delta_p u_n)$ is bounded in $W^{-1,p'}(\Omega)$, and the sequence $(f(\cdot, u_n(\cdot), \nabla u_n(\cdot)))$ is bounded in $L^{r'}(\Omega)$, for some $r \in [1, p^*]$ (arguing as for (2.4) in the proof of Theorem 1). Then the $S_+$-property of the operator $-\Delta_q : W^{1,q}_0(\Omega) \to W^{-1,q'}(\Omega)$ guarantees that $u_n \to u$ in $W^{1,q}_0(\Omega)$. Letting $n \to \infty$ in (3.3) entails $\Delta_q u = 0$, so $u = 0$. Taking into account that the preceding argument applies for every convergent subsequence of $(u_n)$, we conclude that for the whole sequence we have that $u_n \to 0$ in $W^{1,q}_0(\Omega)$. The proof is thus complete. \hfill \Box

**Acknowledgement.** The authors are grateful to L. Fresse and V.V. Motreanu for their useful comments.

**REFERENCES**


**Dipartimento di Matematica e Informatica, Università degli studi di Palermo, 90123 Palermo, Italy**

*E-mail address: diego.averna@unipa.it*

**Département de Mathématiques, Université de Perpignan, 66860 Perpignan, France**

*E-mail address: motreanu@univ-perp.fr*

**Dipartimento di Matematica e Informatica, Università degli studi di Palermo, 90123 Palermo, Italy**

*E-mail address: elisa.tornatore@unipa.it*