ON THE RIESZ REPRESENTATION THEOREM FOR BOUNDED LINEAR FUNCTIONALS

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ABSTRACT

In this paper we give a new proof of the Riesz representation theorem, which characterises the dual space of a Hilbert space.

Let $H$ be a Hilbert space and let $y \in H$. Then the mapping $\phi_y$ defined by

$$\phi_y(x) = \langle x, y \rangle \quad (x \in H)$$

is a continuous linear functional on $H$ and $\|\phi_y\| = \|y\|$.

**Theorem**. (Riesz representation theorem). For any continuous linear functional $\phi$ on $H$ there exists a unique $y \in H$ such that $\phi = \phi_y$.

In this note we present a proof of the theorem which is different from the proofs given in the most commonly used textbooks on elementary Hilbert space theory (see for example the following theorems: [1, 3.25], [2, IV.4.5], [4, I.17.3], [11, 6.8], [6, 2.3.1], [10, 4.11] and [9, 12.5]).

The theorem was originally proved independently by Riesz [7] and Fréchet [3] in the case where $H = L^2[0, 1]$. The proof of the general case was given later by Riesz [8]. All the subsequent proofs in the literature quoted above were essentially repetitions of Riesz's proof. The argument in such proofs is very simple, and we will now repeat it for the convenience of the reader and to indicate why we felt the need to give a different proof.

Let $\phi$ be a continuous linear functional on $H$ and let $M = \ker \phi$ be the nullspace of $\phi$. If $\phi$ admits a representing vector $y$, then $y$ must necessarily lie in $M^\perp$. If $M = H$, then it is necessary and sufficient to take $y = 0$. If $M$ is not all of $H$, then let $z$ be any non-zero vector in $M^\perp$ and let

$$y = \frac{\phi(z)}{\|z\|^2} \quad (1)$$

Then $y \in M^\perp$ and, for any $x$ in $H$, $x - \frac{\phi(x)}{\phi(z)} z$ is in $M$, so

$$0 = \left( x - \frac{\phi(x)}{\phi(z)} z, y \right)$$

$$= \langle x, y \rangle - \frac{\phi(x)}{\phi(z)} \left( z, \frac{\phi(z)}{\|z\|^2} z \right)$$

$$= \phi_y(x) - \phi(x),$$

that is, $\phi = \phi_y$. To show the uniqueness of $y$, one simply notes that
\[
\|\phi_x - \phi_y\| = \|\phi_{x-y}\| = \|y - y\|.
\]

The proof above is coordinate-free, and so is the statement of the theorem. Halmos [5, problem 3] gives a co-ordinated proof, claiming that the coordinate-free proofs ‘are so elegant that they conceal what is really going on’.

What we personally find unsatisfactory about the proof above is the way the representing vector $y$ is constructed. Although formula (1) undoubtedly defines a representing vector $y$ for $\phi$ for any choice of a non-zero vector $z$ in $M^\perp$, one would wish to find a more intrinsic way of characterising $y$, independently of the arbitrary choice of some other vector $z$. We will now propose an alternative coordinate-free proof of the theorem, where such a characterisation is accomplished.

Let $\phi$ be a continuous linear functional on $H$. Then
\[
K_\phi = \{x \in H : \phi(x) = \|\phi\|^2\}
\]
is a non-empty closed convex set in $H$. Therefore $K_\phi$ contains a unique element of smallest norm. To prove the Riesz representation theorem it is then sufficient to prove the following result, which gives simultaneously the existence and the uniqueness of a representing vector for a given continuous linear functional.

**Proposition.** Let $H$ be a Hilbert space and let $\phi$ be a continuous linear functional on $H$. Then $\phi = \phi_y$ if and only if $y$ is the element of smallest norm in $K_\phi$.

**Proof.** If $\phi \equiv 0$, then the statement is trivial, so we may and will assume that $\phi$ is not the zero functional.

If $\phi = \phi_y$ for some $y$ in $H$, then $y \neq 0$ and we can write
\[
K_\phi = \{x \in H : \langle x, y \rangle = \|y\|^2\}.
\]

Hence $y \in K_\phi$. Moreover, for any $x$ in $K_\phi$, one has
\[
\|y\|^2 = \langle x, y \rangle \leq \|x\| \|y\|,
\]
whence $\|y\| \leq \|x\|$. Therefore $y$ is the element of smallest norm in $K_\phi$.

Conversely, let $y$ be the element of smallest norm in $K_\phi$. Note that $y \neq 0$, since $\phi(y) = \|\phi\|^2 \neq 0$. We claim that $\ker \phi \subseteq \ker \phi_y$. Indeed, if $\ker \phi = \{0\}$, then there is nothing to prove. If $\ker \phi \neq \{0\}$, then take $w \neq 0$ in $\ker \phi$. Then
\[
\phi\left(y - \frac{\langle y, w \rangle}{\|w\|^2} w\right) = \phi(y) = \|\phi\|^2,
\]
that is, $y - \frac{\langle y, w \rangle}{\|w\|^2} w \in K_\phi$. Therefore
\[
\|y\|^2 \leq \left\|y - \frac{\langle y, w \rangle}{\|w\|^2} w\right\|^2
\]

\[
= \|y\|^2 - \frac{\|\langle y, w \rangle\|^2}{\|w\|^2},
\]

that is, $\phi = \phi_y$. This completes the proof.
so \( \langle y, w \rangle = 0 \), that is, \( w \in \text{Ker}\phi_y \) and our claim is proved.

Now for any \( x \in H \)

\[
x - \frac{\phi(x)}{\|\phi\|^2} y \in \text{Ker}\phi \subseteq \text{Ker}\phi_y,
\]

so

\[
0 = \left( x - \frac{\phi(x)}{\|\phi\|^2} y, y \right)
\]

\[
= \phi_y(x) - \frac{\|y\|^2}{\|\phi\|^2} \phi(x)
\]

that is,

\[
\phi(x) = \frac{\|\phi\|^2}{\|y\|^2} \phi_y(x)
\]

\[
= \phi_y(x) \text{ for all } x \in H,
\]

where

\[
y' = \frac{\|\phi\|^2}{\|y\|^2} y.
\]

By the first part of the proof, since \( \phi = \phi_y \), then \( y' \) is the element of smallest norm in \( K_\phi \) and thus \( y' = y \). Hence \( \phi = \phi_y \).

The proof is now complete. ■

REFERENCES