FUNCTIONAL ANALYSIS
HILBERT SPACES

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Remember that when we talk about scientific problems you are completely free to
tell me that I’m wrong, because face to the science we are the same.
(Mauro Picone (1885 - 1977), professor of mathematics analysis Rome,
in an answer to his pupil Ennio De Giorgi (1928 - 1996),
become a big mathematician, too).
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CHAPTER 1

HILBERT SPACES

1. Pre-hilbertian spaces

$K$ will denote or the field of complex numbers $\mathbb{C}$, or the field of real numbers $\mathbb{R}$. The elements of $K$ are called scalars and they will denote by the greek letters. $\alpha$ will denote the CONJUGATE of the scalar $\alpha$. Hence $\alpha = \overline{\alpha} \iff \alpha \in \mathbb{R}$.

**Def. 1.1.** Let $L$ be a vector space on $K$. An INNER PRODUCT ON $L$ is a function from $L \times L \to K$, denoted by $(f, g) \mapsto \langle f, g \rangle$ such that it verifies the following properties:

1. $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$
2. $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$
3. $\langle g, f \rangle = \overline{\langle f, g \rangle}$
4. $\langle f, f \rangle \geq 0$ (note that $\langle f, f \rangle \in \mathbb{R}$ from p3)
5. $\langle f, f \rangle = 0 \iff f = 0$

**Def. 1.2.** If $L$ is a vector space on $K$ supplied with an inner product, then $L$ is called PRE-HILBERTIAN space (su $K$).

From now on $L$ will denote a fixed pre-hilbertian space on $K$.

**Theorem 1.1.** If $\langle \cdot, \cdot \rangle$ is an inner product on a $K$-linear space $L$, then:

1. $\langle f, g_1 + g_2 \rangle = \langle f, g_1 \rangle + \langle f, g_2 \rangle$, $\forall f, g_1, g_2 \in L$
2. $\langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle$, $\forall f, g \in L, \forall \alpha \in K$
3. $\langle 0, g \rangle = 0 = \langle f, 0 \rangle$, $\forall f, g \in L$

Moreover:

$$\langle \sum_{k=1}^{n} \alpha_k f_k, \sum_{h=1}^{m} \beta_h g_h \rangle = \sum_{k=1}^{n} \sum_{h=1}^{m} \alpha_k \overline{\beta_h} \langle f_k, g_h \rangle .$$

**Proof.**

\[ \langle f, g_1 + g_2 \rangle = \langle g_1 + g_2, f \rangle = \langle g_1, f \rangle + \langle g_2, f \rangle = \overline{\langle f, g_1 \rangle + \langle f, g_2 \rangle} = \langle f, g_1 \rangle + \langle f, g_2 \rangle. \]

\[ \langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle = \alpha \langle g, f \rangle = \overline{\alpha \langle g, f \rangle} = \overline{\alpha} \langle g, f \rangle = \alpha \langle f, g \rangle. \]

\[ \langle 0, g \rangle = \langle 0, g \rangle = \langle 0 + 0, g \rangle = \langle 0, g \rangle + \langle 0, g \rangle \implies \langle 0, g \rangle = 0. \] Thus, $\langle f, 0 \rangle = \langle 0, f \rangle = 0$.
Prove by induction.

**Example 1.1.** In the \( n \)-dimensional euclidean space, \( \mathbb{R}^n = \{ a = (\alpha_1, \ldots, \alpha_n) : \alpha_k \in \mathbb{R} \} \) we define:

\[
a + b = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)
\]

\[
\lambda a = (\lambda \alpha_1, \ldots, \lambda \alpha_n)
\]

\[
\langle a, b \rangle = \sum_{k=1}^{n} \alpha_k \beta_k
\]

Prove that the axioms \( p_1 \), \( p_2 \), \( p_3 \), \( p_4 \), \( p_5 \) are obviously verified.

**Example 1.2.** Let \( n \geq 1 \). We consider the \( n \)-dimensional unitary space \( \mathbb{C}^n = \{ a = (\alpha_1, \ldots, \alpha_n) : \alpha_k \in \mathbb{C}, k = 1, \ldots, n \} \) and define:

\[
a + b = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)
\]

\[
\lambda a = (\lambda \alpha_1, \ldots, \lambda \alpha_n)
\]

\[
\langle a, b \rangle = \sum_{k=1}^{n} \alpha_k \beta_k
\]

Verify that all the mentioned conditions in the definition 1.2 are satisfied.

**Example 1.3 (Finite sequences).** Let \( L = \{ a = (\alpha_k)_{k=1}^{\infty} : \alpha_k \in \mathbb{C} \text{ per } k \in \mathbb{N}, \alpha_k = 0 \text{ per } k > n(a) \} \) and we define:

\[
a + b = (\alpha_k + \beta_k)_{k=1}^{\infty}
\]

\[
\lambda a = (\lambda \alpha_k)_{k=1}^{\infty}
\]

\[
\langle a, b \rangle = \sum_{k=1}^{\infty} \alpha_k \beta_k
\]

Note: The series reduces to a finite sum. Verify that \( L \) is a pre-hilbertian space.

**Example 1.4.** Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( I = [a, b] \). We denote by \( C(I) \) the class of all continuous functions \( f : I \to \mathbb{C} \). We define (punctually on \( I \))

\[
(f + g)(x) = f(x) + g(x), \, \forall x \in I
\]

\[
(\lambda f)(x) = \lambda f(x), \, \forall x \in I
\]

Moreover, let:
\[ \langle f, g \rangle = \int_a^b f(x)g(x)dx \]

Prove that \( C(I) \) is a space supplied with an inner product.

The inner product takes in a natural way to the following one:

**Def. 1.3.** For every vector \( f \in L \), let \( \|f\| \) denote the nonnegative real number \( +\sqrt{\langle f, f \rangle} \). Then \( \|f\| \) is called the **NORM OF** \( f \) (**INDUCED BY THE INNER PRODUCT** \( \langle \cdot, \cdot \rangle \) **ON** \( L \)).

A vector with norm 1 is called **UNITY VECTOR**.

**Theorem 1.2 (Cauchy-Schwarz-Bunyakovsky inequality).**

\[ |\langle f, g \rangle| \leq \|f\||g| \]

Moreover:

The equality holds \( \iff f \) and \( g \) are linearly dependent.

Proof. If \( g = 0 \) then the conclusion is trivial. Therefore, assume \( g \neq 0 \) and \( \langle g, g \rangle =: \|g\|^2 > 0 \). Put \( \alpha := \frac{\langle f, g \rangle}{\|g\|^2} \). Hence, one has:

\[
0 \leq \langle f - \alpha g, f - \alpha g \rangle = \|f\|^2 - \alpha \langle f, f \rangle - \alpha \langle g, f \rangle + |\alpha|^2 \|g\|^2 = \|f\|^2 - \frac{|\langle f, g \rangle|^2}{\|g\|^2}
\]

namely \( |\langle f, g \rangle|^2 \leq \|f\|^2 \|g\|^2 \) so, by extracting the square root \( |\langle f, g \rangle| \leq \|f\||g| \).

Moreover if \( f \) and \( g \) are linearly dependent, and taking into account \( f = \alpha g \), one has:

\[
|\langle f, g \rangle| = |\langle \alpha g, g \rangle| = |\alpha||\|g\||^2 = |\alpha||\langle g, g \rangle||g| = \|f\||\|g\|.
\]

On the contrary, if \( |\langle f, g \rangle| = \|f\||\|g\| \), by squaring, we obtain \( |\langle f, g \rangle|^2 = \|f\|^2 \|g\|^2 \).

So, from (1), we obtain \( \alpha \) such that \( \langle f - \alpha g, f - \alpha g \rangle = 0 \) from which \( f - \alpha g = 0 \).

**Theorem 1.3.** If \( \langle L, \langle \cdot, \cdot \rangle \rangle \neq 0 \implies \|f\| = \sup_{\|g\| = 1} |\langle f, g \rangle| \).

Proof. If \( f = 0 \) the assertion is banal because \( \|0\| = 0 = \sup_{\|g\| = 1} |\langle 0, g \rangle| = 0 \)

If \( f \neq 0 \implies \|f\| 
eq 0 \)

\[
\|f\| = \langle f, f/\|f\| \rangle \leq \sup_{\|g\| = 1} |\langle f, g \rangle| \leq \sup_{\|g\| = 1} \|f\||\|g\|| = \|f\|.
\]

**Theorem 1.4.** The norm \( \|\cdot\| \) **INDUCED BY THE INNER PRODUCT** \( \langle \cdot, \cdot \rangle \) **ON** \( L \) has the following properties:

\( N_1 \) \( \|f\| \geq 0, \forall f \in L \)

\( N_2 \) \( \|f\| = 0 \iff f = 0 \)
1. HILBERT SPACES

\[ \|af\| = |\alpha|\|f\|, \forall f \in L, \forall \alpha \in K \]

\[ \|f + g\| \leq \|f\| + \|g\|, \forall f, g \in L \]

Moreover:

The equality in \( N_4 \) is valid \( \iff \) \( g = 0 \) or \( f = \lambda g \) with \( \lambda \geq 0 \) and \( g \neq 0 \).

Proof. Only \( N_4 \) is to prove.

\[
\|f + g\|^2 = \|f\|^2 + 2 \text{Re} \langle f, g \rangle + \|g\|^2 \leq \|f\|^2 + 2 \|f\| |\langle f, g \rangle| + \|g\|^2 \leq \|f\|^2 + 2 \|f\| \|g\| + \|g\|^2 = (\|f\| + \|g\|)^2.
\]

We prove that \( \|f + g\| = \|f\| + \|g\| \) if and only if \( f = \lambda g \) with \( \lambda \geq 0 \) and \( g \neq 0 \).

We suppose \( f = \lambda g \) with \( \lambda \geq 0 \); from this one it follows that:

\[
\|f + g\| = \|\lambda g + g\| = \|\lambda + 1\|g\| = |\lambda + 1|\|g\| = \|\lambda g\| + \|g\| = \|f\| + \|g\|.
\]

Now, we suppose that \( \|f + g\| = \|f\| + \|g\| \); since:

\[
\|f + g\|^2 \leq \|f\|^2 + 2 |\langle f, g \rangle| + \|g\|^2 \leq \|f\|^2 + 2 \|f\| \|g\| + \|g\|^2
\]

from the hypothesis it follows that:

\[ |\langle f, g \rangle| = \|f\| \|g\| \]

and so there exists \( \lambda \) such that \( f = \lambda g \); now, we must prove that \( \lambda \geq 0 \), in fact being

\[
\|f + g\| = \|\lambda g + g\| = \|\lambda + 1\|g\| = |\lambda + 1|\|g\|
\]

and moreover

\[
\|f + g\| = \|f\| + \|g\| = \|\lambda g\| + \|g\| = (|\lambda| + 1)\|g\|
\]

it follows that

\[ |\lambda + 1| = |\lambda| + 1 \]

and this one implies \(^1\)

\[ |\lambda| = \lambda. \]

\[ \square \]

**Theorem 1.5 (Parallelogram law).** The norm \( \|\cdot\| \) induced by the inner product \( \langle \cdot, \cdot \rangle \) on a pre-Hilbertian \( K \)-space \( L \) has the following properties:

\[
\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2, \forall f, g \in L.
\]

\(^1\) Let \( |\lambda + 1| = |\lambda| + 1 \), if we suppose that \( \lambda \) is complex and we square the two terms of the equality knowing that it is true if and only if \( \lambda \) is a nonnegative real complex.
Proof.

\[ \| f + g \|^2 = (f + g, f + g) = \| f \|^2 + (f, g) + (g, f) + \| g \|^2 \]

\[ \| f - g \|^2 = (f - g, f - g) = \| f \|^2 - (f, g) - (g, f) + \| g \|^2 \]

summing member to member we have the thesis. □

**Remark 1.1.** In \( \mathbb{R}^2 \) the Theorem 1.5 is very well visualized: The sum of squares on the diagonals of a parallelogram equals the sum of squares on its sides.

\[ \text{Figure 1. Parallelogram law} \]

**Def. 1.4.** Two vectors \( f, g \in L \) are mutually said ORTHOGONAL (or PERPENDICULAR), \( f \perp g \), if \( (f, g) = 0 \).

A family \( F = \{f_\sigma\}_{\sigma \in \Sigma} \subset L \) is said ORTHOGONAL if \( f_{\sigma_1} \perp f_{\sigma_2} \forall \sigma_1, \sigma_2 \in \Sigma \) with \( \sigma_1 \neq \sigma_2 \).

A family \( F = \{f_\sigma\}_{\sigma \in \Sigma} \subset L \) is called ORTHONORMAL if \( F \) is orthogonal and \( \| f_\sigma \| = 1, \forall \sigma \in \Sigma \).

**Def. 1.5.** A family \( F = \{f_1, \ldots, f_n\} \subset L \) is said LINEARLY INDEPENDENT if the relation

\[ \alpha_1 f_1 + \ldots + \alpha_n f_n = 0, \forall \alpha_i \in K \]

implies \( \alpha_i = 0, i = 1, \ldots, n \). If this one doesn’t happen the family is said LINEARLY DEPENDENT.

Analogously an infinite family \( F = \{f_\sigma\}_{\sigma \in \Sigma} \subset L \) will be said LINEARLY INDEPENDENT if however taken its distinct elements \( n, \forall n \in \mathbb{N} \) they result linearly independent.
Example 1.5. We consider the space $C(I)$ like in the example 1.4. For $k \in \mathbb{Z}$ we define:

$$e_k(x) = \frac{1}{\sqrt{b-a}} e^{2\pi i k \frac{x-a}{b-a}}, \text{ for every } x \in [a,b].$$

Then $e_k \in C(I)$ and moreover $\{e_k : k \in \mathbb{Z}\}$ is an orthonormal family. In fact:

$$\langle e_k, e_n \rangle = \int_a^b e_k(x) e_n(x) dx = \frac{1}{b-a} \int_a^b e^{2\pi i (k-n)(x-a)/(b-a)} dx = \delta_{n,k} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

(remember $e^{2\pi i} = \cos 2\pi + i \sin 2\pi$).

Theorem 1.6. An orthogonal family $F \subset (L, \langle \cdot, \cdot \rangle)$ of nonzero vectors is linearly independent.

Proof. Let $F = \{f_n\}$ be the orthogonal family of nonzero vectors and we suppose that it’s linearly dependent.

$$\sum_{i=1}^n \alpha_i f_i = 0 \implies \alpha_i \neq 0, \text{ for at least an } i$$

we suppose $\alpha_n \neq 0$.

We make the inner product for $f_n$, after we have taken to the second member the term containing $f_n$

$$\left\langle \sum_{i=1}^{n-1} \alpha_i f_i, f_n \right\rangle = \langle -\alpha_n f_n, f_n \rangle$$

We have so ($f_i \perp f_n, i = 1, \ldots, n-1$, nonzero $f_n$)

$$0 = \alpha_n.$$

□

Theorem 1.7 (Pythagorean theorem). If $f, g \in (L, \langle \cdot, \cdot \rangle), f \perp g \implies \|f \pm g\|^2 = \|f\|^2 + \|g\|^2$.

Proof. $\|f \pm g\|^2 = \langle f \pm g, f \pm g \rangle = \|f\|^2 \pm \langle f, g \rangle \pm \langle g, f \rangle + \|g\|^2 = \|f\|^2 + \|g\|^2$. □

Corollary 1.1. If $f, g \in (L, \langle \cdot, \cdot \rangle), f \perp g, \|f\| = \|g\| = 1 \implies \|f - g\| = \sqrt{2}$

$^2$In mathematics for **Kronecker delta** (with its name it remembers the german mathematician Leopold Kronecker (1823-1891)) we intend a function of two variables (in particular a function of two variables on integers), which holds 1 if their values coincide, while it holds 0 on the contrary case.
Corollary 1.2. If $\{f_k\}_{k=1}^n \subset (L, \langle \ldots \rangle)$ is an orthogonal family of vectors $\implies \|\sum_{k=1}^n f_k\|^2 = \sum_{k=1}^n \|f_k\|^2$.

Theorem 1.8 (Bessel inequality). Let $\{e_k\}_{k=1}^n$ be an orthonormal family of vectors. Then for every $f \in L$ it results:

$$\|f\|^2 \geq \sum_{k=1}^n |\langle f, e_k \rangle|^2$$

Proof. Let $g = f - \sum_{k=1}^n \langle f, e_k \rangle e_k$. Then for every $h = 1, \ldots, n$ it results:

$$\langle g, e_h \rangle = \langle f, e_h \rangle - \sum_{k=1}^n \langle f, e_k \rangle \langle e_k, e_h \rangle = \langle f, e_h \rangle - \langle f, e_h \rangle = 0$$

Then $g \perp e_h$ for $h = 1, \ldots, n$ and the vectors $g, \langle f, e_1 \rangle e_1, \ldots, \langle f, e_n \rangle e_n$ form an orthogonal family.

Hence from the Corollary 1.2 we have:

$$\|f\|^2 = \|g + \sum_{k=1}^n \langle f, e_k \rangle e_k\|^2 = \|g\|^2 + \sum_{k=1}^n |\langle f, e_k \rangle|^2 \geq \sum_{k=1}^n |\langle f, e_k \rangle|^2.$$ 

□

Exercise 1.1. If $\{e_k\}_{k=1}^n$ is an orthonormal family of vectors, then $\|f\|^2 = \sum_{k=1}^n |\langle f, e_k \rangle|^2 \iff f = \sum_{k=1}^n \langle f, e_k \rangle e_k$.

Theorem 1.9. If $\{e_k\}_{k=1}^n$ is an orthonormal family of vectors, then for every $f \in L$, $\|f - \sum_{k=1}^n \alpha_k e_k\|^2$ is minimized when $\alpha_k = \langle f, e_k \rangle$ for $k = 1, \ldots, n$.

Proof. $\|f - \sum_{k=1}^n \alpha_k e_k\|^2 = \langle f - \sum_{k=1}^n \alpha_k e_k, f - \sum_{k=1}^n \alpha_k e_k \rangle = \|f\|^2 - \sum_{k=1}^n \overline{\alpha_k} \langle f, e_k \rangle - \sum_{k=1}^n \alpha_k \langle e_k, f \rangle + \sum_{k=1}^n \alpha_k \overline{\alpha_k} e_k$.

The last term which is on the right is $\sum_{k=1}^n |\alpha_k|^2$ because the $e_k$ are orthonormal.

Moreover $-\overline{\alpha_k} \langle f, e_k \rangle - \alpha_k \langle e_k, f \rangle + |\alpha_k|^2 = -\overline{\alpha_k} \langle f, e_k \rangle - \alpha_k \overline{\langle f, e_k \rangle} + |\alpha_k|^2 = 3 - 2 \text{Re}(\overline{\alpha_k} \langle f, e_k \rangle) + |\alpha_k|^2 = -|\langle f, e_k \rangle|^2 + |\alpha_k - \langle f, e_k \rangle|^2$.

So:

$$0 \leq \|f - \sum_{k=1}^n \alpha_k e_k\|^2 = \|f\|^2 - \sum_{k=1}^n |\langle f, e_k \rangle|^2 + \sum_{k=1}^n |\alpha_k - \langle f, e_k \rangle|^2.$$ 

□

Remark 1.2. The pre-hilbertian spaces are a natural extension of the spaces $\mathbb{R}^n$ since many geometric properties of $\mathbb{R}^n$ continue to be valid.

\footnote{Remember that: $|\alpha_k - \langle f, e_k \rangle|^2 = |\alpha_k|^2 + |\langle f, e_k \rangle|^2 - 2 \text{Re}(\overline{\alpha_k} \langle f, e_k \rangle)$.}
2. Normed linear spaces

$K$ will denote $\mathbb{C}$ or $\mathbb{R}$ yet.

**Def. 2.1.** A vector space $L$ on $K$ is said **NORMED LINEAR SPACE** (on $K$) if there exists a function from $L$ to $\mathbb{R}$, said **NORM**, satisfying the following conditions:

- $N_1$) $\|f\| \geq 0$
- $N_2$) $\|f\| = 0 \iff f = 0$
- $N_3$) $\|\alpha f\| = |\alpha|\|f\|$
- $N_4$) $\|f + g\| \leq \|f\| + \|g\|

From the Theorem 1.4 hence every pre-hilbertian space is a normed space. But there are normed spaces which aren’t pre-hilbertian, as the following example shows.

**Example 2.1.** Let $L = C([0, 2\pi])$. We define:

$$\|f\| = \sup\{|f(x)| : x \in [0, 2\pi]\}$$

Verify that it is a norm (also said the **UNIFORM NORM**).

However this norm isn’t induced by any inner product on $L$. In fact if it wasn’t it should be valid the parallelogram law.

Choosing, for example:

$$f(x) = \max\{|\sin x, 0\} \text{ and } g(x) = \max\{-\sin x, 0\}$$

we have:

$$\|f + g\| = \|f - g\| = \|f\| = \|g\| = 1$$

and so:

$$2 = \|f + g\|^2 + \|f - g\|^2 \neq 2\|f\|^2 + 2\|g\|^2 = 4$$

**Theorem 2.1.** If $(L, \|\cdot\|)$ is a normed $K$-space and the norm satisfies the parallelogram law, then $L$ becomes a pre-hilbertian $K$-space defining the inner product $\langle \cdot, \cdot \rangle$:

$$\langle f, g \rangle = \frac{1}{4}[\|f + g\|^2 - \|f - g\|^2], \text{ if } K = \mathbb{R}$$
\[ \langle f, g \rangle = \frac{1}{4}[\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2], \text{ if } K = \mathbb{C}. \]

Moreover: the inner product \( \langle \cdot, \cdot \rangle \) induces the norm \( \|\cdot\| \).

N.S.C. 2.1. so that a norm is induced by an inner product is that the norm satisfies the parallelogram law.

From now on we will denote with \( L \) a normed linear space on \( K \).

**Theorem 2.2.** \( |\|f\| - \|g\| | \leq \|f - g\| \).

**Proof.**

\[ \|f\| = \|(f - g) + g\| \leq \|f - g\| + \|g\| \]

\[ \|g\| = \|(g - f) + f\| \leq \|f - g\| + \|f\| \]

whence the assertion. \( \square \)

A normed linear space is a topological vector space with the topology induced by the metric \( d(f, g) = \|f - g\| \). Prove that \( d \) is a distance on \( L \).

**Def. 2.2.** The space \( L \) is said SEPARABLE if it contains a countable subset \( A \) which is EVERYWHERE DENSE IN \( L \) (that is \( \overline{A} = L \)).

**Example 2.2.** \( \mathbb{C}^n \) is separable for every \( n \geq 1 \). In fact the set of the vectors with rational complex coordinates that is \( \alpha_k = \beta_k + i\gamma_k \), with \( \beta_k, \gamma_k \in \mathbb{Q} \) is countable and everywhere dense in \( \mathbb{C}^n \).

**Def. 2.3.** A sequence \( (f_n)_{n=1}^{\infty} \subset L \) CONVERGES TO VECTOR \( f \), said its LIMIT \( \lim_{n \to \infty} f_n = f \), if:

\[ \forall \varepsilon > 0 \exists \overline{n} = \overline{n}(\varepsilon) : \forall n \geq \overline{n} \implies \|f - f_n\| < \varepsilon. \]

A series \( \sum_{k=1}^{\infty} g_k \) CONVERGES TO A VECTOR \( g \), said its SUM, \( \sum_{k=1}^{\infty} g_k = g \), if \( \lim_{n \to \infty} \sum_{k=1}^{n} g_k = g \).

A sequence or series that doesn’t converge is said DIVERGENT.

**Remark 2.1.** The following formulations for \( \lim_{n \to \infty} f_n = f \) are equivalent:

a) \( \forall \varepsilon > 0 \exists \) open \( \varepsilon \)-sphere of \( f \) containing all vectors \( f_n \) for sufficiently big \( n \).

b) \( \lim_{n \to \infty} \|f_n - f\| = 0 \).

The following formulations for \( \sum_{k=1}^{\infty} g_k = g \) are equivalent:

c) The sequence of the partial sums \( \left( \sum_{k=1}^{n} g_k \right)_n \) converges to \( g \).

d) \( \lim_{n \to \infty} \|g - \sum_{k=1}^{n} g_k\| = 0 \).

**Exercise 2.1.** Prove that a convergent sequence determines univocally its limit.

\footnote{The norm is said induced if \( \|f\|^2 = \langle f, f \rangle, \forall f \in L \) (Def. 1.3).}
Theorem 2.3. Let \( f \in L \) and \( U \subset L \). Then \( f \in \overline{U} \iff \) there exists a sequence \( (f_n)_{n=1}^\infty \subset U \) convergent to \( f \).

Proof. If \( f \in \overline{U} \), then \( \forall \ n \geq 1 \) the open \( \frac{1}{n} \)-sphere of \( f \) contains at least a vector \( f_n \in U \). From \( \|f - f_n\| < \frac{1}{n} \) it follows \( \lim_{n \to \infty} f_n = f \).

The converse: if there exists \( (f_n)_{n=1}^\infty \subset U \) such that \( f = \lim_{n \to \infty} f_n \) and if \( f \notin U \) and for every \( \varepsilon > 0 \) it follows, from \( \|f - f_n\| < \varepsilon \) for every \( n > \overline{n}(\varepsilon) \), that \( f_n \neq f \) is contained in the \( \varepsilon \)-sphere of \( f \), that is, \( f \) is an accumulation point for \( U \) and hence \( f \in \overline{U} \). \( \square \)

Theorem 2.4. If \( \lim_{n \to \infty} f_n = f \), \( \lim_{n \to \infty} \alpha_n = \alpha \), \( \lim_{n \to \infty} g_n = g \), \( \lim_{n \to \infty} \beta_n = \beta \) then the following properties hold:

a) \( \lim_{n \to \infty} (\alpha_n f_n + \beta_n g_n) = \alpha f + \beta g \)

b) \( \lim_{n \to \infty} \|f_n\| = \|f\| \)

c) If \( L \) is pre-hilbertian and the norm is induced by the inner product \( \langle \cdot, \cdot \rangle \), then

\[
\lim_{n \to \infty} \langle f_n, g_n \rangle = \langle f, g \rangle.
\]

Proof. a): \( \|\alpha f + \beta g - (\alpha_n f_n + \beta_n g_n)\| \leq \|\alpha f - \alpha_n f_n\| + \|\beta g - \beta_n g_n\| \leq |\alpha - \alpha_n||f| + |\beta - \beta_n||g| + |\beta_n||g - g_n| \)

Since the sequences \((\alpha_n)_{n=1}^\infty\) and \((\beta_n)_{n=1}^\infty\) are bounded it follows a).

b): \( \|f\| = \|f_n\| \) from the Theorem 2.2, hence we have b).

c): \( |\langle f, g \rangle - \langle f_n, g_n \rangle| \leq \|\langle f - f_n, g_n \rangle + \|f_n - f\||g_n\| \)

Since \( \lim_{n \to \infty} \|g_n\| = \|g\| \) it follows that the sequence \((\|g_n\|)_{n=1}^\infty\) is bounded. Therefore we have c). \( \square \)

Def. 2.4. A sequence \((f_n)_{n=1}^\infty \subset L\) is said CAUCHY (or FUNDAMENTAL) sequence if

\[
\forall \ \varepsilon > 0 \ \exists \ \overline{n} = \overline{n}(\varepsilon) : \forall \ m, n \geq \overline{n} \implies \|f_m - f_n\| < \varepsilon.
\]

Notation: \( \lim_{m, n \to \infty} \|f_m - f_n\| = 0 \)

Exercise 2.2. Every Cauchy sequence is bounded.

Theorem 2.5. Every convergent sequence in \( L \) is a Cauchy sequence.

Proof. For exercise. \( \square \)

Def. 2.5. A normed linear space is said COMPLETE if every Cauchy sequence in \( L \) converges to any vector of \( L \).

A complete and normed linear space on \( K \) is said BANACH SPACE ON \( K \).

Remark 2.2. From the previous definition it follows that in a Banach space a sequence \((f_n)_{n=1}^\infty \) converges if and only if it is a Cauchy sequence; thus a series \( \sum_{n=1}^\infty g_k \) converges in a Banach space if and only if the sequence of the partial sums is a Cauchy sequence, that is if and only if \( \lim_{m, n \to \infty} \|\sum_{k=m}^n g_k\| = 0 \).
Example 2.3. Let \( L = \mathbb{C}^n \). Prove that \( L \) is complete in the norm induced by the inner product given in the example 1.2.

Example 2.4. Let \( L \) be the pre-hilbertian space of the example 1.3. The space \( L \) of all finite sequences of complex numbers isn’t complete in the norm induced by the inner product.

In fact let \( a_n = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots\} \) for \( n \geq 1 \).

Then for every \( n > m \) we have

\[
\|a_m - a_n\|^2 = \sum_{k=m+1}^{n} \frac{1}{k^2}.
\]

Since the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges it follows that \( \lim_{m,n \to \infty} \|a_m - a_n\| = 0 \).

But on the other hand, the sequence \( (a_n)_{n=1}^{\infty} \) can’t converge to an element

\[
\alpha = \{\alpha_1, \ldots, \alpha_j, 0, \ldots\} \in L
\]

In fact for every \( n > j \) we should have \( \|\alpha - a_n\|^2 \geq \frac{1}{(j+1)^2} > 0 \).

Example 2.5. Let \( C(I) \) be the pre-hilbertian space of the example 1.4. \( C(I) \) isn’t complete. For simplicity let \( I[-1,1] \).

For every \( n \geq 1 \) we define:

\[
f_n(x) = \begin{cases} 
0 & \text{for } -1 \leq x \leq 0 \\
px & \text{for } 0 < x < \frac{1}{n} \\
1 & \text{for } \frac{1}{n} \leq x \leq 1.
\end{cases}
\]
Then we have:

\[ f_m(x) - f_n(x) = 0 \quad \text{for} \quad -1 \leq x \leq 0 \]

\[ |f_m(x) - f_n(x)| \leq 1 \quad \text{for} \quad 0 \leq x \leq \max\{\frac{1}{m}, \frac{1}{n}\} \]

\[ f_m(x) - f_n(x) = 0 \quad \text{for} \quad \max\{\frac{1}{m}, \frac{1}{n}\} \leq x \leq 1 \]

In conclusion:

\[ \|f_m - f_n\|^2 = \int_{-1}^{1} |f_m(x) - f_n(x)|^2 \, dx \leq \max\{\frac{1}{m}, \frac{1}{n}\} \]

hence \( \lim_{m,n \to \infty} \|f_m - f_n\| = 0 \).

If \( f \in C[-1,1] \) was such that:

\[ \lim_{n \to \infty} \|f - f_n\|^2 = \lim_{n \to \infty} \int_{-1}^{1} |f(x) - f_n(x)|^2 \, dx = 0 \]

then necessarily:

\[ f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 < x \leq 1. \end{cases} \]

In fact \( \int_{a}^{b} |f_n(x) - f(x)|^2 \, dx \leq \int_{-1}^{1} |f_n(x) - f(x)|^2 \, dx \) for every interval \([a, b] \subset [-1,1]\)

hence \( \int_{a}^{b} |f_n(x) - f(x)|^2 \, dx \to 0 \). In particular

\[ \int_{-1}^{0} |f_n(x) - f(x)|^2 \, dx = \int_{-1}^{0} |f(x)|^2 \, dx \to 0 \]

in other words:

\[ \int_{-1}^{0} |f(x)|^2 \, dx = 0. \]

Since \( f \) is continuous it follows that \( f(x) = 0 \quad \forall \ x \in [-1,0] \).

Let \( 0 < \varepsilon < 1 \). We have:

\[ \int_{\varepsilon}^{1} |f_n(x) - f(x)|^2 \, dx \to 0. \]
2. NORMED LINEAR SPACES

But for $n > \frac{1}{\varepsilon}$ it results $f_n(x) = 1$ for $x \in [\varepsilon, 1]$, so

$$\int_{\varepsilon}^{1} |f_n(x) - f(x)|^2 dx = \int_{\varepsilon}^{1} |1 - f(x)|^2 dx \text{ for } n > \frac{1}{\varepsilon}.$$

For $n \to \infty$ it follows that: $\int_{\varepsilon}^{1} |1 - f(x)|^2 dx = 0$ so $f(x) = 1$ on $[\varepsilon, 1]$. For the arbitrariness of $\varepsilon > 0$ it follows that $f(x) = 1$ for $x \in [0, 1]$. And this is impossible for the continuity of $f$.

**Example 2.6.** Let $L = C(I)$ of the example 2.1. We observe that a sequence $(f_n)_{n=1}^{\infty}$ converges to $f$ in $C(I)$ $\iff$ it converges uniformly to $f$ in $I$. In fact the relation $\|f - f_n\| < \varepsilon$ is equivalent to $|f(x) - f_n(x)| < \varepsilon$, $\forall x \in I$.

$C(I)$ is complete. In fact, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $C(I)$. For every $x \in I$ we have $|f_n(x) - f_m(x)| \leq \|f_m - f_n\|$. Then for every $x \in I$ the sequence of numbers $(f_n(x))_{n=1}^{\infty}$ converges to a number that we will denote by $f(x)$.

The function $f : x \to f(x)$ defined on $I$ is continuous and $(f_n)_{n=1}^{\infty}$ converges to $f$ in the space $C(I)$. In fact for every $\varepsilon > 0$ there exists an index $\bar{n} = \bar{n}(\varepsilon)$ such that $\|f_n - f_m\| < \varepsilon$ for $n, m > \bar{n}$ and so $|f_n(x) - f_m(x)| < \varepsilon$ for every $x \in I$ and $n, m > \bar{n}$.

It follows that:

(2) \[ |f(x) - f_n(x)| < \varepsilon \text{ for every } x \in I \text{ and } n > \bar{n} \]

that is the sequence $(f_n)_{n=1}^{\infty}$ of continuous functions converges uniformly to $f$, so $f$ is continuous. From (2) we have that $\|f - f_n\| < \varepsilon$ for $n > \bar{n}$ and the proof is complete.

**Theorem 2.6.** Let $L$ be a normed linear space on $K$. Then there exists a linear space $L_1$ on $K$ with the norm $\|\|_1$, said the completion of $L$, with the following properties:

a) $L \subseteq L_1$

b) $\|f\|_1 = \|f\|$ for every $f \in L$

c) $L$ is everywhere dense in $L_1$

d) $L_1$ is complete.

If the norm $\|\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$, then the norm $\|\|_1$ is induced by an inner product $\langle \cdot, \cdot \rangle_1$ in $L$ and $\langle f, g \rangle = \langle f, g \rangle_1$ for every $f$ and $g \in L$.

Remark to the Theorem 2.6.

The space $L_1$ with the properties a) b) c) d) is univocally determined by $L$. This justifies the name the completion of $L$. The base idea of the proof is the same of that one used in the construction of the field of the real numbers beginning from the field of the rational numbers.

Talking by intuition, to every Cauchy sequence in $L$ that doesn’t converge to a point of $L$ we associate a new point (its limit point) that we add to $L$ and the norm of this one is univocally determined by the condition of continuity (Theorem 2.4 b)); the same holds for the linear operations (Theorem 2.4 a)).
DEF. 2.6. A pre-hilbertian space on $\mathbb{C}$ (or $\mathbb{R}$) which is complete with respect to the norm induced by the inner product is said complex (or real) HILBERT SPACE.

3. The Hilbert space $l_2$

Now we will give a Hilbert space which is the completion of pre-hilbertian space of all finite sequences of complex numbers (cfr. Example 1.3).

We consider the set $l_2$ of all sequences to summable square of complex numbers:

$$l_2 = \{a = (\alpha_k)_{k=1}^{\infty} : \alpha_k \in \mathbb{C} \text{ for every } k, \Sigma_{k=1}^{\infty} |\alpha_k|^2 < \infty\}$$

**Theorem 3.1.** For every $a = (\alpha_k)_{k=1}^{\infty} \in l_2$ and $b = (\beta_k)_{k=1}^{\infty} \in l_2$ we define:

$$a + b = (\alpha_k + \beta_k)_{k=1}^{\infty}$$

$$\lambda a = (\lambda \alpha_k)_{k=1}^{\infty}$$

Then $l_2$ is a complex linear space.

Proof. Using the Cauchy inequality in $\mathbb{C}^2$ we have:

$$|\langle (\alpha_k, \beta_k), (1, 1) \rangle|^2 = |\alpha_k + \beta_k|^2 \leq (|\alpha_k|^2 + |\beta_k|^2)(1 + 1) = 2(|\alpha_k|^2 + |\beta_k|^2)$$

so:

$$\Sigma_{k=1}^{\infty} |\alpha_k + \beta_k|^2 \leq 2(\Sigma_{k=1}^{\infty} |\alpha_k|^2 + \Sigma_{k=1}^{\infty} |\beta_k|^2) < \infty.$$ 

So $a + b \in l_2$.

Analogously from $\Sigma_{k=1}^{\infty} |\lambda \alpha_k|^2 = |\lambda|^2 \Sigma_{k=1}^{\infty} |\alpha_k|^2 < \infty$ we conclude that $\lambda a \in l_2$.

All required conditions in the definition of linear space on $\mathbb{C}$ are obviously verified.

□

**Theorem 3.2.** For every $a = (\alpha_k)_{k=1}^{\infty} \in l_2$ and $b = (\beta_k)_{k=1}^{\infty} \in l_2$ the series $\sum_{k=1}^{\infty} |\alpha_k \bar{\beta}_k|$ converges. With the inner product defined by:

$$\langle a, b \rangle = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$$

$l_2$ is a pre-hilbertian space on $\mathbb{C}$.

Proof. From $0 \leq (|\alpha_k| - |\beta_k|)^2$ we conclude:

$$2|\alpha_k \bar{\beta}_k| \leq |\alpha_k|^2 + |\beta_k|^2$$

$$\sum_{k=1}^{\infty} |\alpha_k \bar{\beta}_k| \leq \frac{1}{2}(\sum_{k=1}^{\infty} |\alpha_k|^2 + \sum_{k=1}^{\infty} |\beta_k|^2) < \infty.$$
Hence the series \( \sum_{k=1}^{\infty} |\alpha_k| \sum_{j=1}^{\infty} |\beta_j| \) converges. Let prove easily that \( l^2 \) is pre-hilbertian. \( \square \)

**Theorem 3.3.** \( l^2 \) is complete with respect to the norm induced by the inner product.

Proof. Let \( a_n = (\alpha_{k,n})_{k=1}^{\infty} \) and let \( (a_n)_{n=1}^{\infty} \) be a Cauchy sequence in \( l^2 \). From

\[
|\alpha_{k,m} - \alpha_{k,n}|^2 \leq \sum_{k=1}^{\infty} |\alpha_{k,m} - \alpha_{k,n}|^2 = \|a_m - a_n\|^2
\]

we have that for every \( k \geq 1 \) the sequence \( (\alpha_{k,n})_{n=1}^{\infty} \) is a Cauchy sequence in \( C \) and so it converges to a number \( \alpha_k \in C \).

Let \( a = (\alpha_k)_{k=1}^{\infty} \). For every \( \varepsilon > 0 \) we have:

\[
\|a_m - a_n\|^2 = \sum_{k=1}^{\infty} |\alpha_{k,m} - \alpha_{k,n}|^2 < \varepsilon^2 \quad \text{for every } m, n > n(\varepsilon)
\]

and so for every fixed index \( h \geq 1 \) we have:

\[
\sum_{k=1}^{h} |\alpha_{k,m} - \alpha_{k,n}|^2 < \varepsilon^2 \quad \text{for every } m, n > n(\varepsilon).
\]

For \( m \to +\infty \) we obtain:

\[
\sum_{k=1}^{h} |\alpha_{k} - \alpha_{k,n}|^2 < \varepsilon^2 \quad \text{for every } n > n(\varepsilon) \quad \text{and } h \geq 1
\]

and for \( h \to \infty \) we have:

\[
\sum_{k=1}^{\infty} |\alpha_{k} - \alpha_{k,n}|^2 < \varepsilon^2 \quad \text{for every } n > n(\varepsilon)
\]

the sequence \( a - a_n = (\alpha_k - \alpha_{k,n})_{k=1}^{\infty} \) so it is in \( l^2 \) and since \( l^2 \) is a linear space also the sequence \( a = (a - a_n) + a_n \) is in \( l^2 \). Hence from (3) we conclude that

\[
\|a - a_n\| \leq \varepsilon \quad \text{for every } n > n(\varepsilon), \quad \lim_{n \to \infty} a_n = a.
\]

Thus every Cauchy sequence in \( l^2 \) converges to an element of \( l^2 \). \( \square \)

**Theorem 3.4.** \( l^2 \) contains a countable orthonormal family.

Proof. Let \( e_n = (\delta_{n,k})_{k=1}^{\infty} \) for \( n \geq 1 \). Then \( (e_m, e_n) = \delta_{m,n} \) for every \( 1 \leq m \leq n \). \( \square \)

**Theorem 3.5.** \( l^2 \) is separable.
Proof. Let \( l' \) be the set of all finite sequences of complex rational numbers, that is:

\[
    l' = \{ a' = (\alpha'_k)_{k=1}^{\infty} : \alpha'_k \in \mathbb{C}, \Re \alpha'_k, \Im \alpha'_k \in \mathbb{Q}, \text{ for } k \geq 1, \alpha'_k = 0 \text{ for } k > n(a') \}
\]

Obviously \( l' \subset l_2 \) is countable. In fact the set of all complex rational numbers has the same cardinality of the set of all pairs of rational numbers and so it’s countable. For every \( n \geq 1 \), the subset of \( l' \) constituted by those elements \( a' = (\alpha'_k)_{k=1}^{\infty} \in l' \) such that \( \alpha'_k = 0 \) for every \( k \geq n \) is countable. The set \( l' \) is the countable union of all these countable subsets for \( n \geq 1 \) and so it’s countable.

The set \( l' \) is everywhere dense in \( l_2 \). In fact let \( a = (\alpha_k)_{k=1}^{\infty} \in l_2 \) and let \( \varepsilon > 0 \). We choose \( n \geq 1 \) such that \( \sum_{k=n+1}^{\infty} |\alpha_k|^2 < \frac{1}{2} \varepsilon^2 \) and \( a' = (\alpha'_k)_{k=1}^{\infty} \in l' \) such that \( \alpha'_k = 0 \) for \( k \geq n+1 \) and \( |\alpha_k - \alpha'_k|^2 < \varepsilon^2 \frac{2}{2n} \) for \( 1 \leq k \leq n \) (this is possible because the complex rational numbers are everywhere dense in \( \mathbb{C} \)).

Then we have:

\[
    \|a - a'\|^2 = \sum_{k=1}^{n} |\alpha_k - \alpha'_k|^2 + \sum_{k=n+1}^{\infty} |\alpha_k - \alpha'_k|^2 < n \frac{\varepsilon^2}{2n} + \varepsilon^2 \frac{2}{2}.
\]

\[ \square \]

**Remark 3.1.** Moreover, the proof of the Theorem 3.5 proves that the prehilbertian space of all finite sequences of complex numbers (cfr. Example 1.3) is everywhere dense in \( l_2 \). By virtue of the Theorem 2.6 it follows that \( l_2 \) is its completion.

**Remark 3.2.** Modifying opportunely \( l_2 \) it is possible to give an example of nonseparable Hilbert space. Instead of the sequences \((\alpha_k)_{k=1}^{\infty}\) of complex numbers we consider the family \( \{\alpha_x\}_{x \in \mathbb{R}} \) of complex numbers. A such family \( a = \{\alpha_x\}_{x \in \mathbb{R}} \) can be visualized as a function of \( \mathbb{R} \) if we define \( a(x) = \alpha_x \).

Let \( L \) be the set of all functions \( a \) of \( \mathbb{R} \) such that:

(a) The function \( a \) is zero in \( \mathbb{R} \) except for a set of points (indexes) that is countable and that can depend from \( a \).

(b) The sum of the squares on the absolute values of the values of \( a \) in these points is finite. That is:

\[
    \sum_{x \in \mathbb{R}} |a(x)|^2 = \sum_{x \in \mathbb{R}} |\alpha_x|^2 < \infty.
\]

We punctually define:

\[
    (a + b)(x) = a(x) + b(x) = \{\alpha_x + \beta_x\}_{x \in \mathbb{R}}
\]

\[
    (\lambda a)(x) = \lambda a(x) = \lambda \alpha_x.
\]
4. THE HILBERT SPACE $L_2$

Then reasoning as in the proofs of the Theorems 3.1, 3.2, 3.3 prove that $L$ is a complex linear space, that

$$\langle a, b \rangle = \sum_{x \in \mathbb{R}} a(x)\overline{b(x)} = \sum_{x \in \mathbb{R}} \alpha_x \overline{\beta_x}$$

properly defines an inner product in $L$ and that $L$ is a Hilbert space with respect to this inner product.

We denote by $e_y$, $y \in \mathbb{R}$ the function defined on $\mathbb{R}$ by $e_y(x) = \delta_{y,x}$. The family $\{e_y : y \in \mathbb{R}\} \subset L$ that has the cardinality of the continuous is orthonormal. By virtue of the Corollary 1.1 it results $\|e_y - e_z\| = \sqrt{2}$ for $y \neq z$. Let $B_y$ be the open sphere with centre $e_y$ and radius $\frac{1}{2} \sqrt{2}$. For $z \neq y$ the spheres $B_z$ and $B_y$ are disjoint. In fact if $a \in B_z \cap B_y$ we had $\|e_z - e_y\| \leq \|e_z - a\| + \|a - e_y\| < \frac{1}{2} \sqrt{2} + \frac{1}{2} \sqrt{2} = \sqrt{2}$ and this is false.

Now, let $L'$ be any subset of everywhere dense $L$ in $L$. Then every sphere $B_y$, $y \in \mathbb{R}$, must contain at least an element of $L'$. Since the sphere $B_y$ are pairwise disjoint, $L'$ must contain at least so many elements as the points $y$ of $\mathbb{R}$. So every subset of everywhere dense $L$ in $L$ has at least the cardinality of the continuous and so $L$ is certainly nonseparable.

**Exercise 3.1.** Let $U = \{a = (\alpha_k)_{k=1}^{\infty} : \alpha_k \in \mathbb{C}, |\alpha_k| < \frac{1}{k}, k \geq 1\}$.

Prove that:

a) $U \subset l_2$

b) Every sequence $(a_n)_{n=1}^{\infty} \subset U$ contains a convergent subsequence

c) For $n \geq 1$ let $e_n = (\delta_{n,k})_{k=1}^{\infty}$; then every subsequence of the sequence $(e_n)_{n=1}^{\infty}$ doesn’t converge.

4. The Hilbert space $L_2$

Now, we will give a Hilbert space that is the completion of the pre-hilbertian space of the continuous functions $C[a,b]$ (cfr. example 1.4).

We denote by $M[a,b]$ the set of all functions to measurable complex values on $[a,b]$ and by $L_1[a,b]$ the subset of all Lebesgue integrable functions on $[a,b]$. Let $L_2[a,b]$ be the subset of all functions to complex values and to summable square on $[a,b]$, that is:

$$L_2[a,b] = \{f \in M[a,b] : \int_{a}^{b} |f(x)|^2 dx < \infty\}.$$  

Note: Two functions which coincide a.e. will be identified.

**Theorem 4.1.** The set $L_2[a,b]$ is a complex linear space under the addition and the multiplication for scalars.
1. HILBERT SPACES

Proof. If \( f \in L^2 \), \( g \in L^2 \) and \( \lambda \in \mathbb{C} \) the functions \( f + g, \lambda f \in M[a,b] \). Moreover:

\[
|f(x) + g(x)|^2 \leq 2(|f(x)|^2 + |g(x)|^2)
\]

\[
\int_a^b |f(x) + g(x)|^2 dx \leq 2\left\{\int_a^b |f(x)|^2 dx + \int_a^b |g(x)|^2 dx\right\} < \infty
\]

\[
\int_a^b \lambda f(x)^2 dx = |\lambda|^2 \int_a^b |f(x)|^2 dx < \infty
\]

In conclusion \( f + g, \lambda f \in L^2[a,b] \). Moreover \( L^2[a,b] \) verifies the mentioned conditions in the definition of complex linear space. \( \square \)

**Theorem 4.2.** If \( f, g \in L^2[a,b] \), the function \( fg \) is integrable on \([a,b]\) with the inner product defined by:

\[
\langle f, g \rangle = \int_a^b f(x)\overline{g}(x)dx,
\]

\( L^2[a,b] \) is a complex pre-hilbertian space.

Proof. The function \( fg \) is Lebesgue measurable on \([a,b]\). Moreover:

\[
2|f(x)\overline{g}(x)| \leq |f(x)|^2 + |g(x)|^2
\]

\[
\int_a^b |f(x)\overline{g}(x)| dx \leq \frac{1}{2} \left\{\int_a^b |f(x)|^2 dx + \int_a^b |g(x)|^2 dx\right\} < \infty.
\]

So \( fg \in L^1[a,b] \).

All properties of the inner product are easily verified. \( \square \)

**Corollary 4.1.** \( L^2[a,b] \subset L^1[a,b] \).

Proof. Let \( f \in L^2[a,b] \). From the Hölder inequality it follows:

\[
\int_a^b |f(x)1| dx \leq \left(\int_a^b |f(x)|^2\right)^{\frac{1}{2}} (b-a)^{\frac{1}{2}} < \infty.
\]

So \( L^2[a,b] \subset L^1[a,b] \). \( \square \)

**Theorem 4.3.** \( L^2[a,b] \) is complete with respect to the norm induced by the inner product.

**Theorem 4.4.** \( L^2[a,b] \) contains a countable orthonormal family.

---

5. \( 0 \leq (|f(x)| - |g(x)|)^2 = |f(x)|^2 + |g(x)|^2 - 2|f(x)||g(x)|, \) \( |f(x) + g(x)|^2 \leq (|f(x)| + |g(x)|)^2 = |f(x)|^2 + |g(x)|^2 + 2|f(x)||g(x)| \leq 2(|f(x)|^2 + |g(x)|^2). \)
Proof. The family \( \{e_k\}_{k=-\infty}^{+\infty} \subset C[a, b] \) defined by:

\[
e_k(x) = \frac{1}{\sqrt{b-a}} e^{2\pi i k \frac{x-a}{b-a}}, x \in [a, b]
\]

has the required properties. \( \square \)

**Theorem 4.5.** \( L_2[a, b] \) is separable.

Proof. (sketch) The countable and everywhere dense set in \( L_2[a, b] \) is the set \( L' \) of all linear combinations of the functions \( e_k \) with rational coefficients. \( \square \)

**Theorem 4.6.** \( L_2[a, b] \) is the completion of the pre-hilbertian space of the continuous functions \( C[a, b] \) (cfr. example 1.4).

**Remark 4.1.** If we alone consider real functions and real scalars we obtain the real Hilbert space \( L^r_2[a, b] \).

Using the opportune shrewdnesses we can prove the theorems already proved for \( L_2[a, b] \) also for \( L^r_2[a, b] \).

**Remark 4.2.** The spaces \( L_2[a, +\infty[, L_2] - \infty, b], L_2] - \infty, +\infty[ \) are also the Hilbert spaces. Using the opportune shrewdnesses we can prove the theorems already proved for \( L_2[a, b] \) also for \( L_2[a, +\infty[, L_2] - \infty, b], L_2] - \infty, +\infty[ \). Excepted the Corollary 4.1 which is valid only for finite \( a, b \).

Analogously for \( L^r_2[a, +\infty[, L^r_2] - \infty, b], L^r_2] - \infty, +\infty[ \).
GEOMETRY OF THE HILBERT SPACES

1. Subspaces

In the whole chapter we will denote by \( H \) a fixed Hilbert space.

**Def. 1.1.** A nonempty subset \( M \) of \( H \) is said LINEAR MANIFOLD if:

\[
\begin{align*}
  f + g & \in M, \text{ for every } f, g \in M \\
  \lambda f & \in M, \text{ for every } f \in M \text{ and } \lambda \in K
\end{align*}
\]

A closed linear manifold is said SUBSPACE.

A SUBSPACE is said PROPER if it doesn’t coincide to \( H \).

It’s known that in \( \mathbb{R}^n \) and \( \mathbb{C}^n \) every linear manifold is closed. In this case the notions of linear manifold and of subspace coincide (this holds for finite dimensional spaces, too, cfr. Theorem 1.4 Banach spaces). In general this isn’t true in all Hilbert spaces.

**Example 1.1.** Let \( H = l_2 \) and let \( M \) be the subset of all finite sequences of \( l_2 \). Obviously, \( M \) is a linear manifold, but since \( M \) is everywhere dense in \( l_2 \) it results \( \overline{M} = l_2 \neq M \) (let see Remark 3.1), so \( M \) isn’t closed.

**Example 1.2.** Let \( H = l_2 \) and let \( M \) be the subset of every \( a = (\alpha_k)_{k=1}^{\infty} \in l_2 \) with \( \alpha_1 = 0 \). Obviously, \( M \) is a linear manifold.

It is easy to prove that \( M \) is a subspace.

In fact if \( b = (\beta_k)_{k=1}^{\infty} \in l_2 \) is an accumulation point for \( M \) then for every \( \varepsilon > 0 \) there exists an element \( a \in M \) such that: \( \|b - a\| < \varepsilon \). Since \( |\beta_1| = |\beta_1 - \alpha_1| \leq \|b - a\| \) it follows that \( \beta_1 = 0 \) and so \( b \in M \).

**Example 1.3.** Let \( H = L_2[a,b] \) and let \( M = C[a,b] \subset L_2[a,b] \) as in the example 1.4. \( M \) is again a linear manifold, but it isn’t a subspace: in fact \( \overline{M} = H \neq M \) (cfr. Theorem 4.6)

**Example 1.4.** Let \( H = L_2[a,b] \) \( (-\infty \leq a < b \leq +\infty) \) and let \( Y \) be a subset of the Lebesgue measurable interval \([a,b]\).

We define \( M = \{ f \in L_2[a,b] : f(x) = 0 \text{ for almost every } x \in Y \} \) (remember that two functions which are a.e. equal in \([a,b]\) are identified). Obviously \( M \) is a linear manifold. We prove that \( M \) is a subspace.
2. GEOMETRY OF THE HILBERT SPACES

If $Y$ is a set of Lebesgue zero measure we have $M = L_2[a,b]$ and so there is nothing to prove. We suppose then that $Y$ has Lebesgue positive measure. If $g \in M$ then for every $\varepsilon > 0$ there exists $f \in M$ such that:

$$\| g - f \|^2 = \int_a^b |g(x) - f(x)|^2 dx < \varepsilon.$$  

This one implies:

$$\int_Y |g(x)|^2 dx = \int_Y |g(x) - f(x)|^2 dx \leq \int_a^b |g(x) - f(x)|^2 dx < \varepsilon.$$  

So $\int_Y |g(x)|^2 dx = 0$ for the arbitrariness of $\varepsilon$, whence $g(x) = 0$ a.e. in $Y$ that is $g \in M$ and so $M$ is closed.

**Remark 1.1.** All the whole space $H$ and the subset 0 constituted from the only element 0 are obviously the subspaces. They are said BANAL.

All the other subspaces are said NONBANAL SUBSPACES.

**Remark 1.2.** All that one which we will say in this paragraph it can be applied to anyone Banach space $L$.

The notion of subspace is important for various reasons. The one is in the fact that a subspace $M$ of a Hilbert space is complete. In fact if $(f_n)_{n=1}^\infty \subset M$ is a Cauchy sequence then it converges to any element $f \in H$ that, since $M$ is closed, it must be an element of $M$.

Moreover, as we will prove, the closure of a linear manifold is a subspace. Since $M$ can be contained in different subspaces, we can say that $\overline{M}$ is the smallest subspace containing $M$.

More generally for every given subset $A$ of $H$ there exists an unique subspace, the smallest subspace containing $A$.

**Theorem 1.1.** Let $M$ be a linear manifold of $H$. Then $\overline{M}$ is a subspace.

Proof. We can prove that if $f, g \in \overline{M}$ and $\lambda \in K$ then:

$$f + g \in \overline{M} \quad \lambda f \in \overline{M}.$$  

Let $\varepsilon > 0$. We choose $f_1, g_1 \in M$ such that:

$$\| f - f_1 \| < \frac{1}{2} \varepsilon \quad \text{and} \quad \| g - g_1 \| < \frac{1}{2} \varepsilon.$$  

Then $f_1 + g_1 \in M$ and $\|(f + g) - (f_1 + g_1)\| \leq \|f - f_1\| + \|g - g_1\| < \varepsilon$. Thus if $f + g \notin M$ then $f + g$ is an accumulation point for $M$ and so $f + g \in \overline{M}$.

Analogously let prove that $\lambda f \in \overline{M}$. \qed
**Theorem 1.2.** Let \( \{M_\sigma\}_{\sigma \in \Sigma} \) a nonempty family of linear manifolds. Then the set \( M = \cap_{\sigma \in \Sigma} M_\sigma \) is a linear manifold.

If \( \{M_\sigma\}_{\sigma \in \Sigma} \) is a family of subspaces, then \( M \) is a subspace.

Proof. From the definition of intersection of linear manifolds it follows that if \( f, g \in M = \cap_{\sigma \in \Sigma} M_\sigma \), \( \lambda \in K \implies f + g \in M \), \( \lambda f \in M \).

If \( \{M_\sigma\}_{\sigma \in \Sigma} \) is a family of subspaces, then \( M \) is closed. \( \square \)

**Theorem 1.3.** Let \( A \) be a subset of \( H \). Then there exists a unique subspace \( M \) with following properties:

a) \( A \subset M \)

b) If \( N \supset A \) is a subspace then \( N \supseteq M \).

Proof. We consider the family \( \{M_\sigma\}_{\sigma \in \Sigma} \) of all subspaces containing \( A \). Such family is nonempty because \( H \) belongs to it. Let \( M = \cap_{\sigma \in \Sigma} M_\sigma \). Then from the Theorem 1.2 \( M \) is a subspace that contains \( A \) and \( M \subset M_\sigma \) for every \( \sigma \in \Sigma \). \( \square \)

**Def.** 1.2. If \( A \) is a subset of \( H \), then the subspace \( M \) associated to \( A \) through the Theorem 1.3 is said SUBSPACE GENERATED BY \( A \) (or EXTENSION OF \( A \)) and we will use the notation:

\[
M = \bigvee A
\]

**Theorem 1.4.** If \( A \) is a subset of \( H \), then:

\[
\bigvee A = \left\{ \sum_{k=1}^{n} \alpha_k f_k : f_k \in A, \alpha_k \in K, \text{ for } 1 \leq k \leq n, n \geq 1 \right\}
\]

Proof. The set of all finite linear combinations of elements of \( A \) is obviously a linear manifold contained in \( \bigvee A \). The closure of this set is a subspace contained in \( \bigvee A \) from the Theorem 1.1. But from the property b) of Theorem 1.3 it must coincide to \( \bigvee A \). \( \square \)

If \( M_1, M_2, \ldots \) are subspaces, we denote by \( M_1 \bigvee M_2 \) and \( \bigvee_{k=1}^{\infty} M_k \) the SUB-SPACES GENERATED BY \( M_1 \cup M_2 \) and \( \cup_{k=1}^{\infty} M_k \).

**Def.** 1.3. If \( M_1 \) and \( M_2 \) are linear manifolds then the set:

\[
M_1 + M_2 = \{ f_1 + f_2 : f_1 \in M_1, f_2 \in M_2 \}
\]

is said SUM (VECTOR) OF \( M_1 \) AND \( M_2 \).

If \( (M_k)_{k=1}^{\infty} \) is a sequence of linear manifolds, then the set:

\[
\sum_{k=1}^{\infty} M_k = \{ f : f \in H, f = \sum_{k=1}^{\infty} f_k, f_k \in M_k \forall k \}
\]

is said SUM (VECTOR) OF THE LINEAR MANIFOLDS \( M_k \).
In the other words, \( \sum_{k=1}^{\infty} M_k \) is the set of the sums of all convergent series \( \sum_{k=1}^{\infty} f_k \) with \( f_k \in M_k \) for every \( k \). Obviously this set is a linear manifold, as also the set \( M_1 + M_2 \). The sum vector of two linear manifold can be considered as a particular case of a sum vector of a sequence of linear manifolds, letting \( M_k = 0 \) for \( k > 2 \). Analogously we can write:

\[
\sum_{k=1}^{n} M_k = M_1 + M_2 + \ldots + M_n
\]

letting \( M_k = 0 \) for \( k > n \).

There exists a simple relation between sum vector and generated subspace:

**Theorem 1.5.** If \( (M_k)_{k=1}^{\infty} \) is a sequence of linear manifolds, then:

\[
M_1 \bigvee M_2 = M_1 + M_2
\]

\[
\bigvee_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} M_k
\]

**Exercise 1.1.** Let \( H = l_2 \), we define:

\[
M_1 = \{ a = (\alpha_k)_{k=1}^{\infty} \in l_2 : \alpha_{2k} = 0, k = 1, 2, \ldots \}
\]

\[
M_2 = \{ b = (\beta_k)_{k=1}^{\infty} \in l_2 : \beta_{2k-1} = \delta_k \cos \frac{1}{k}, \beta_{2k} = \delta_k \sin \frac{1}{k}, k = 1, 2, \ldots \}
\]

and let \( c = (\gamma_k)_{k=1}^{\infty} \) where \( \gamma_{2k-1} = 0, \gamma_{2k} = \sin \frac{1}{k}, \) for \( k = 1, 2, \ldots \).

Prove the following affirmations:

a) \( M_1 \) and \( M_2 \) are two subspaces.

b) \( M_1 \bigvee M_2 = l_2 \).

c) \( c \in l_2 \).

d) \( c \not\in M_1 + M_2 \).

2. Orthogonal subspaces

A way to obtain a subspace, proved in §1, is to start from an arbitrary subset \( A \) of \( H \) and to consider the subspace generated by \( A, \bigvee A \).

An other way, as we will say, is to consider the set of all orthogonal vectors to every vector of \( A \).
2. ORTHOGONAL SUBSPACES

DEF. 2.1. A VECTOR \( g \) is said ORTHOGONAL TO A SUBSET \( A \subset H \), \( g \perp A \), if \( g \perp f \) for every \( f \in A \).

Two SUBSETS \( A \) and \( B \) OF \( H \) are MUTUALLY said ORTHOGONAL, \( A \perp B \), if \( f \perp g \) for every \( f \in A \) e \( g \in B \).

The set: \( A^\perp = \{ g \in H : g \perp A \} \) is said ORTHOGONAL COMPLEMENT OF \( A \) (IN \( H \)).

If \( N \) and \( M \) are subspaces and \( M \subset N \), the set \( (N - M = 1)\{ g \in N : g \perp M \} = N \cap M^\perp \)

is said the ORTHOGONAL COMPLEMENT OF \( M \) in \( N \).

REMARK 2.1. If \( M \) is a subspace, we can write \( M^\perp = H - M \).

REMARK 2.2. If \( A \) and \( B \) are two subsets of \( H \), then \( A \subset B \Rightarrow A^\perp \supset B^\perp \).

THEOREM 2.1. If \( A \) is a subset of \( H \), then \( A^\perp \) is a subspace and \( A \cap A^\perp \) is the subspace \( 0 = \{ 0 \} \) or it is empty (if \( 0 \notin A \)).

Proof. Let \( g_1, g_2 \in A^\perp \). Since \( \langle f, g_1 \rangle = \langle f, g_2 \rangle = 0 \) for every \( f \in A \) it follows that:

\[
\langle f, \alpha_1 g_1 + \alpha_2 g_2 \rangle = \overline{\alpha_1} \langle f, g_1 \rangle + \overline{\alpha_2} \langle f, g_2 \rangle = 0, \forall f \in A
\]

and so \( \alpha_1 g_1 + \alpha_2 g_2 \in A^\perp \).

Hence \( A^\perp \) is a linear manifold.

Let \( g \in A^\perp \) and let \( g = \lim_{n \to \infty} g_n \), with \( g_n \in A^\perp \) for \( n \geq 1 \). Then for every \( f \in A \), it results:

\[
\langle f, g \rangle = \lim_{n \to \infty} \langle f, g_n \rangle = 0, \text{and so } g \in A^\perp.
\]

Thus \( A^\perp \) is closed and it is a subspace. If \( A \cap A^\perp \neq \emptyset \) and if \( f \in A \cap A^\perp \Rightarrow f \perp f \) that is \( \langle f, f \rangle = 0 \) and so \( f = 0 \). \( \square \)

THEOREM 2.2. If \( M \) and \( N \) are two orthogonal subspaces, then:

\[
M \bigvee N = M + N.
\]

Proof. In virtue by the Theorem 1.5 it is sufficient to prove that \( M + N \) is closed.

We suppose that \( f \in M + N \) \( f = \lim_{n \to \infty} f_n \), where \( f_n = g_n + h_n \) with \( g_n \in M \) and \( h_n \in N \) for \( n \geq 1 \).

Since \( g_n \perp h_n \) from the Phytagorean theorem it follows:

\[
\|f_m - f_n\|^2 = \|g_m - g_n\|^2 + \|h_m - h_n\|^2
\]

From the convergence of \( f_n \) to \( f \) it follows that:

\[\text{Let see Remark 2.3.}\]

\[\text{Let see Theorem 2.8.}\]
\[ \lim_{m,n \to \infty} ||f_m - f_n|| = 0 \quad \text{and so} \quad \lim_{m,n \to \infty} ||g_m - g_n|| = 0 \quad \text{and} \quad \lim_{m,n \to \infty} ||h_m - h_n|| = 0. \]

The sequences \((g_n)_{n=1}^{\infty}\) and \((h_n)_{n=1}^{\infty}\) are the Cauchy sequences and they respectively converge in \(H\) to \(g\) and \(h\). Since \(M\) and \(N\) are closed it must be that \(g \in M\) and \(h \in N\). Therefore

\[ f = \lim_{n \to \infty} f_n = \lim_{n \to \infty} (g_n + h_n) = g + h \in M + N. \]

\[ \square \]

**Theorem 2.3.** Let \(\{g_k\}_{k=1}^{\infty}\) be an orthogonal family of vectors.

a) The series \(\sum_{k=1}^{\infty} g_k\) converges if and only if \(\sum_{k=1}^{\infty} ||g_k||^2 < \infty\).

b) If the series \(\sum_{k=1}^{\infty} g_k = f\) converges then \(\sum_{k=1}^{\infty} ||g_k||^2 = ||f||^2\).

**Proof.** We suppose that \(\sum_{k=1}^{\infty} ||g_k||^2 < \infty\). Then:

\[ \lim_{m,n \to \infty} ||\sum_{k=m}^{n} g_k||^2 = \lim_{m,n \to \infty} \sum_{k=m}^{n} ||g_k||^2 = 0 \]

and from the completeness of \(H\) the series \(\sum_{k=1}^{\infty} g_k\) converges.

The converse if \(\sum_{k=1}^{\infty} g_k = f\), it results:

\[ ||f||^2 = \langle f, f \rangle = \left\langle \lim_{n \to \infty} \sum_{k=1}^{n} g_k, \lim_{m \to \infty} \sum_{h=1}^{m} g_n \right\rangle = \lim_{n \to \infty} \sum_{k=1}^{n} \langle g_k, g_k \rangle = \sum_{k=1}^{\infty} ||g_k||^2. \]

\[ \square \]

**Theorem 2.4.** If \((M_k)_{k=1}^{\infty}\) is a sequence of mutually orthogonal subspaces, then:

\[ \bigvee_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} M_k. \]

**Proof.** It's sufficient to prove that \(\bigvee_{k=1}^{\infty} M_k\) is closed. We suppose so that \(f \in \bigvee_{k=1}^{\infty} M_k\), \(f = \lim_{n \to \infty} f_n\) and \(f_n = \sum_{k=1}^{\infty} g_{n,k}\) where \(g_{n,k} \in M_k\) for every \(n \geq 1\) and \(k \geq 1\).

Reasoning analogously to what we made in the proof of the Theorem 2.2, we can write:

\[ \langle f_m - f_n, f_m - f_n \rangle = \sum_{k=1}^{\infty} ||g_{m,k} - g_{n,k}||^2 < \varepsilon^2 \quad \text{per} \quad m \geq n(\varepsilon) \quad \text{and} \quad n \geq n(\varepsilon). \]

So:
Therefore there exist the limits:

\[ g_k = \lim_{n \to \infty} g_{n,k} \in M_k \text{ for } k \geq 1. \]

From (4) we conclude that:

\[ \sum_{k=1}^{h} \|g_{m,k} - g_{n,k}\|^2 < \varepsilon^2 \text{ for } m \geq n(\varepsilon), n \geq n(\varepsilon), \text{ for } h \geq 1. \]

(5)

\[ \sum_{k=1}^{\infty} \|g_k - g_{n,k}\|^2 \leq \varepsilon^2 \text{ for } n \geq n(\varepsilon). \]

Hence from the Theorem 2.3 a) the series \( \sum_{k=1}^{\infty} (g_k - g_{n,k}) \) converges and therefore also the following series converges:

\[ \sum_{k=1}^{\infty} (g_k - g_{n,k}) + \sum_{k=1}^{\infty} g_{n,k} = \sum_{k=1}^{\infty} g_k \]

Moreover, (5) proves that the sum of the last series is the limit of the sequence \( (f_n)_{n=1}^{\infty} \):

\[ f = \lim_{n \to \infty} f_n = \sum_{n=1}^{\infty} g_k \in M_k. \]

**Theorem 2.5.** Let \((M_k)_{k=1}^{\infty}\) a sequence of mutually orthogonal subspaces. Then for every \( f \in \bigvee_{k=1}^{\infty} M_k \) there exists one unique vector \( f_k \in M_k \) for every \( k \geq 1 \) such that \( f = \sum_{k=1}^{\infty} f_k \).

Proof. We suppose that \( f = \sum_{k=1}^{\infty} f'_k = \sum_{k=1}^{\infty} f''_k \) with \( f'_k, f''_k \in M_k \) for every \( k \geq 1 \). Then:\n\[ \sum_{k=1}^{\infty} (f'_k - f''_k) = 0 \] hence from the Theorem 2.3 b) it follows that \( \sum_{k=1}^{\infty} \|f'_k - f''_k\|^2 = 0 \) therefore \( f'_k = f''_k \) for every \( k \geq 1 \). \( \square \)

**Corollary 2.1.** If \( M_1 \) and \( M_2 \) are two orthogonal subspaces, for every vector \( f = M_1 \bigvee M_2 = M_1 + M_2 \) the decomposition of \( f \) in the \( f = f_1 + f_2 \) with \( f_1 \in M_1, f_2 \in M_2 \) is unique.

**Theorem 2.6.** Let \( M \) a subspace and let \( f \in H \). If \( \delta = \inf \{\|f - g\| : g \in M\} \) then there exists a unique vector \( P_M f \in M \), SAID PROJECTION OF \( f \) ON \( M \), such that:

\[ \|f - P_M f\| = \delta. \]
Proof. Let \((g_n)_{n=1}^{\infty} \subset M\) be a sequence of vectors of \(M\) such that \(\lim_{n \to \infty} \|f - g_n\| = \delta\). Applying the parallelogram law to the vectors \((f - g_m)\) and \((f - g_n)\) it results:

\[
\|2f - (g_m + g_n)\|^2 + \|g_m - g_n\|^2 = 2\|f - g_m\|^2 + 2\|f - g_n\|^2
\]

and that is:

\[
\|g_m - g_n\|^2 = 2\|f - g_m\|^2 + 2\|f - g_n\|^2 - 4\|f - \frac{1}{2}(g_m + g_n)\|^2.
\]

Since \(\frac{1}{2}(g_m + g_n) \in M\) it necessarily follows that \(\|f - \frac{1}{2}(g_m + g_n)\| \geq \delta\) and so:

\[
\|g_m - g_n\|^2 \leq 2\|f - g_m\|^2 + 2\|f - g_n\|^2 - 4\delta^2.
\]

For \(m, n \to \infty\) the second member of the inequality tends to 0. Therefore in \(M\) there exists \(\lim_{n \to \infty} g_n = PMf\) and it results \(\|f - PMf\| = \lim_{n \to \infty} \|f - g_n\| = \delta\).

In the end we suppose that \(f_1, f_2 \in M\) are such that:

\[
\|f - f_1\| = \|f - f_2\| = \delta.
\]

Applying the parallelogram law to \((f - f_1)\) and \((f - f_2)\) it results:

\[
\|2f - (f_1 + f_2)\|^2 + \|f_1 - f_2\|^2 = 2\|f - f_1\|^2 + 2\|f - f_2\|^2
\]

and that is

\[
\|f_1 - f_2\|^2 = 4\delta^2 - 4\|f - \frac{1}{2}(f_1 + f_2)\|^2 \leq 0
\]

in that \(\frac{1}{2}(f_1 + f_2) \in M\) and so \(\|f - \frac{1}{2}(f_1 + f_2)\| \geq \delta\). □

**Theorem 2.7.** Let \(M\) be a subspace and let \(f \in H\). We indicate by \(PMf \in M\) the projection of \(f\) on \(M\). Then \((f - PMf) \perp M\).

![Figure 1. \(f - PMf\) is orthogonal to \(M\)](image)
Proof. Let \( f_0 = f - P_M f \) then from the Theorem 2.6 we have \( \|f_0\| = \delta \). For every \( g \in M \) and for every \( \alpha \in K \) we have \( P_M f + \alpha g \in M \) and so \( \delta^2 = \|f_0\|^2 \leq \|f - P_M f - \alpha g\|^2 = \|f_0 - \alpha g\|^2 = \|f_0\|^2 - \alpha \langle f_0, f_0 \rangle - \alpha \langle f_0, g \rangle + |\alpha|^2 \|g\|^2 \),

\[
0 \leq -\alpha \langle f_0, f_0 \rangle - \alpha \langle f_0, g \rangle + |\alpha|^2 \|g\|^2
\]

We suppose that there exists \( g \in M \) such that \( \langle f_0, g \rangle \neq 0 \) (that implies \( g \neq 0 \)). Choosing \( \alpha = \frac{\langle f_0, g \rangle}{\|g\|^2} \) we will obtain

\[
0 \leq -2 \left| \frac{\langle f_0, g \rangle}{\|g\|^2} \right|^2 + \left| \frac{\langle f_0, g \rangle}{\|g\|^2} \right|^2 = -\left| \frac{\langle f_0, g \rangle}{\|g\|^2} \right|^2
\]

which is a contradiction. \( \square \)

**Corollary 2.2.** Let \( M \) be a linear manifold contained in a subspace \( N \). Then \( \overline{M} \neq N \iff \) there exists a vector \( f \in N \) different from 0 and orthogonal to \( M \).

Proof. If \( \overline{M} \neq N \), then we take anyone vector \( f \), different from zero of \( N \) but not of \( M \). Hence the vector \( f_0 = f - P_M f \) has all required properties from the Theorem 2.7.

The converse, if \( \overline{M} = N \) we indicate by \( f \) a vector of orthogonal \( N \) to \( M \). We will prove that \( f \) must be necessarily 0. In fact \( f = \lim_{n \to \infty} f_n, \ f_n \in M \) for \( n \geq 1 \). Therefore

\[
\langle f, f \rangle = \lim_{n \to \infty} \langle f, f_n \rangle = 0.
\]

\( \square \)

**Corollary 2.3.** Let \( M \) be a linear manifold. Then:

\[
\overline{M} = H \iff M^\perp = 0.
\]

**Theorem 2.8 (Projection theorem).** If \( M \) is a subspace, then \( H = M + M^\perp \).

Proof. For every \( f \in H \) we can write \( f = P_M f + (f - P_M f) \) where \( P_M f \in M \) from the Theorem 2.6 and \( (f - P_M f) \in M^\perp \) from the Theorem 2.7. \( \square \)

From the corollary of the Theorem 2.5 we can write the projection theorem as it follows:

**Theorem 2.9.** If \( M \) is a subspace, then every vector \( f \in H \) can be decomposed in the sum \( f = f_1 + f_2 \) where \( f_1(= P_M f) \in M \) and \( f_2(= P_{M^\perp} f) \in M^\perp \).

**Remark 2.3.** If in the Theorem 2.8 to place of \( H \) we consider a subspace \( N \supset M \), then the set \( M^\perp = \{ f \in H : f \perp M \} \) must be substituted with \( N \cap M^\perp = \{ f \in N : f \perp M \} \). The following corollary justifies the notation \( N \cap M^\perp = N - M \).
COROLLARY 2.4. If $N$ and $M$ are two subspaces with $M \subset N$, then:

$$N = M + (N \cap M^\perp) = M + (N - M).$$

EXAMPLE 2.1. Let $H = L^2[a, b]$ ($-\infty \leq a < b \leq +\infty$) and let $Y$ be a Lebesgue measurable subset of $]a, b[$. For every $f \in H$ we define the functions $f_1, f_2 \in M[a, b]$

$$f_1(x) = \begin{cases} 0 & \text{if } x \in Y \\ f(x) & \text{if } x \in ]a, b[ \setminus Y \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \in ]a, b[ \setminus Y. \end{cases}$$

From $\int_a^b |f_k(x)|^2 \, dx \leq \int_a^b |f(x)|^2 \, dx < \infty$ for $k = 1, 2$ we conclude that $f_k \in L^2[a, b]$, $k = 1, 2$. Moreover it results:

$$f_1 \in M_1 = \{ g \in L^2[a, b] : g(x) = 0 \text{ a.e. in } Y \}$$

$$f_2 \in M_2 = \{ g \in L^2[a, b] : g(x) = 0 \text{ a.e. in } ]a, b[ \setminus Y \}$$

$$f = f_1 + f_2$$

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x)\overline{f_2(x)} \, dx = 0$$

Since the decomposition of anyone vector $f$ in the sum of a vector of $M_k^\perp$ and of a vector of $M_k$ is unique from the Theorem 2.9, we conclude that:

$$f_k = P_{M_k} f \text{ for } k = 1, 2 \text{ and } M_2 = M_1^\perp$$

Later on we will write $A^{\perp\perp}$ in the place of $(A^\perp)^\perp$.

THEOREM 2.10. Let $A$ be a subset of $H$. Then $\bigvee A = A^{\perp\perp}$ and consequently $A^\perp = A^{\perp\perp\perp}$. In particular: $A$ is a subspace $\iff A = A^{\perp\perp}$.

Proof. Let $f \in \bigvee A$ the limit of convergent sequence $(f_n)_n$ where $f_n$ is a finite linear combination of vectors of $A$ (cfr. Theorem 1.4). Then for every $g \in A^\perp$ we have $\langle f, g \rangle = \lim_{n \to \infty} \langle f_n, g \rangle = 0$ and so $f \perp g$. This one implies that $\bigvee A \perp A^\perp$ and $\bigvee A \subset A^{\perp\perp}$. If $\bigvee A \neq A^{\perp\perp}$ then from the Corollary 2.2 there should exist a vector different from zero $f \in A^{\perp\perp} \cap A^\perp$. This one, however, is impossible from the Theorem 2.1.

The second part of the theorem is obtained substituting $A$ with $A^\perp$. \□

EXERCISE 2.1. Let $H = l^2$. For every $n \geq 1$ let $e_n = (\delta_{n,k})_{k=1}^\infty \in l^2$ and let $A = (e_{2n-1} + e_{2n})_{n=1}^\infty$.

a) Identify $\bigvee A$ and $A^\perp$ in $l^2$. 
b) \( a = (\alpha_k)_{k=1}^\infty \in l_2 \) then \( P_{\mathcal{F}} a = (\beta_k)_{k=1}^\infty \), where \( \beta_{2n-1} = \beta_{2n} = \frac{1}{2}(\alpha_{2n-1} + \alpha_n) \)
for \( n \geq 1 \); \( P_{\mathcal{F}^\perp} a = (\gamma)_{k=1}^\infty \) with \( \gamma_{2n-1} = -\gamma_{2n} = \frac{1}{2}(\alpha_{2n-1} - \alpha_{2n}) \) for \( n \geq 1 \).

3. Bases

**Def. 3.1.** Let \( M \) be a subspace of a Hilbert space \( H \). An orthonormal family \( \{e_\sigma\}_{\sigma \in \Sigma} \subset M \) is said MAXIMAL in \( M \) if the only orthogonal vector to every \( e_\sigma \), \( \sigma \in \Sigma \), in \( M \), is the vector 0.

A maximal orthonormal family in \( M \) is said a BASE of \( M \).

We remember that an orthonormal family is linearly independent (cfr. Theorem 1.6).

**Example 3.1.** Let \( H = \mathbb{C}^n \). Then every \( n \)-ple of orthonormal vectors is a base.

**Example 3.2.** Let \( H = l_2 \). For every integer \( k \) let \( e_k = (0, \ldots, 0, 1, 0, \ldots) \)
where the \( k \)-th component is 1 and the others are 0. Then \( \{e_k\}_{k=1}^\infty \) is a base, said standard base of \( l_2 \). In fact if \( \langle f, e_k \rangle = 0 \) for every \( k \geq 1 \) then all components of \( f \)
must be 0.

**Example 3.3.** Let \( H = L_2[0,1] \). For every integer \( k \) let \( e_k = e^{2\pi ikx} \). Then \( \{e_k\}_{k=-\infty}^{+\infty} \) is an orthonormal family (cfr. Example 1.5). We suppose that for any \( f \in H \) let \( \langle f, e_k \rangle = 0 \) for every integer \( k \). We remember that the set \( L' \) of all finite linear combinations \( \sum_{k=-m}^{m} \alpha_k' e_k \) with complex rational coefficients \( \alpha_k' \) is everywhere dense (cfr. Theorem 4.5). Obviously \( \langle f, g \rangle = 0 \) for every vector \( g \in L' \). If \( f \neq 0 \) we should choose a vector \( g \in L' \) such that:

\[ \|f - g\| < \|f\|. \]

This one obviously implies:

\[ \|f\|^2 = \langle f, f \rangle = \langle f, f \rangle - \langle f, g \rangle = \langle f, f - g \rangle \leq \|f\| \|f - g\| < \|f\|^2 \]

and this is a contradiction.

So \( \{e_k\}_{k=-\infty}^{+\infty} \) is a base.

**Example 3.4.** Let \( L_2[0,1] \) and let \( e_0 = 1, f_k(x) = \sqrt{2} \cos 2\pi kx, g_k(x) = \sqrt{2} \sin 2\pi kx \)
for \( k = 1, 2, \ldots \). Prove that the family \( \mathcal{F} = \{e_0\} \cup \{f_k\}_{k=1}^\infty \cup \{g_k\}_{k=1}^\infty \) is orthonormal.

This one follows by the fact that \( f_k = \frac{e_k + e_{-k}}{\sqrt{2}} \) and \( g_k = \frac{e_k - e_{-k}}{\sqrt{2}} \) for \( k \geq 1 \).

If \( f \) is an orthogonal vector to \( e_0 \) and to \( f_k \) and \( g_k \) \( \forall k \geq 1 \) then \( f \perp e_k \) for every integer \( k \). Since the family \( \{e_k\}_{k=-\infty}^{+\infty} \) is a base, we conclude that \( f = 0 \). Consequently \( \mathcal{F} \) is a base.

Does every Hilbert space admit a base? We will enunciate that for every separable Hilbert space the answer is yes.
Theorem 3.1 (Gram-Schmidt orthogonalization). Let $\mathcal{F} = \{f_k\}_{k=1}^\chi$ be a countable family of linearly independent vectors. Then there exists an orthogonal family $\mathcal{G} = \{g_k\}_{k=1}^\chi$ (with the same cardinality of $\mathcal{F}$) such that $g_k \neq 0$ and $g_k$ is a linear combination of $f_1, \ldots, f_k$ for every $k$.

Proof. We will build the family $\mathcal{G}$ by induction. Let $g_1 = f_1$ (for hypothesis $f_1 \neq 0$). We suppose that $g_1, \ldots, g_{k-1}$ are mutually orthogonal nonzero vectors verifying the properties of the theorem. We define:

\[ g_k = f_k - \sum_{h=1}^{k-1} \frac{\langle f_k, g_h \rangle}{\|g_h\|^2} g_h. \]

It results, for $1 \leq n \leq k - 1$:

\[ \langle g_k, g_n \rangle = \langle f_k, g_n \rangle - \sum_{h=1}^{k-1} \frac{\langle f_k, g_h \rangle}{\|g_h\|^2} \langle g_h, g_n \rangle = \langle f_k, g_n \rangle - \frac{\langle f_k, g_n \rangle}{\|g_n\|^2} \langle g_n, g_n \rangle = 0. \]

Moreover $g_k$ is a linear combination of $f_1, \ldots, f_k$. Since $f_1, \ldots, f_k$ are linearly independent then (6) implies $g_k \neq 0$. \qed

Corollary 3.1. Let $\mathcal{F} = \{f_k\}_{k=1}^\chi$ and $\mathcal{G} = \{g_k\}_{k=1}^\chi$ as in the Theorem 3.1. Then the following affirmations hold:

a) $f_k$ is a linear combination of $g_1, \ldots, g_k$ for $1 \leq k \leq \chi$

b) $\bigvee \{f_k\}_{k=1}^\chi = \bigvee \{g_k\}_{k=1}^\chi$

c) The family $\{e_k = \frac{g_k}{\|g_k\|}\}_{k=1}^\chi$ is an orthonormal family verifying the Theorem 3.1

d) If $\{h_k\}_{k=1}^\chi$ is an other orthogonal family of nonzero vectors verifying the Theorem 3.1 then $h_k = \alpha_k g_k$ and $\alpha_k \neq 0$ for $1 \leq k \leq \chi$.

Theorem 3.2. A Hilbert space $H$ is separable $\iff$ it has a countable (finite or infinite) base.

Theorem 3.3. Let $\{e_k\}_{k=1}^\chi$ be an orthonormal family of $H$. The following affirmations are equivalent:

a) $\{e_k\}_{k=1}^\chi$ is a base

b) $f \perp e_k$ for every $k \geq 1 \implies f = 0$

c) $H = \bigvee \{e_k\}_{k=1}^\chi$

d) $f = \sum_{k=1}^{\chi} \langle f, e_k \rangle e_k$ for every $f \in H$ (Fourier series)

e) $\langle f, g \rangle = \sum_{k=1}^{\chi} \langle f, e_k \rangle \langle g, e_k \rangle$ for every $f, g \in H$ (Parseval identity)

f) $\|f\|^2 = \sum_{k=1}^{\chi} |\langle f, e_k \rangle|^2$ for every $f \in H$ (Parseval identity).

Remark 3.1. In d) the vector $f$ is represented in a Fourier series with respect to the base $\{e_k\}_{k=1}^\chi$. In this representation $\langle f, e_k \rangle$ is said the Fourier coefficient corresponding to $e_k$. Though the affirmation f) is said Parseval identity, this name is
sometimes assigned to e), too. In fact, only in appearance, this affirmation is more general than f) both the affirmations are equivalent from Theorem 3.3.

**Remark 3.2.** All affirmations of the Theorem 3.3 remain true even if the index $k$ moves from 1 to $\chi$.

**Example 3.5.** Let $H = L_2[0, 1]$ and let consider the base of the example 3.4. For every real function $f \in L_2[0, 1]$ we define:

$$
\alpha_0 = \int_0^1 f(x)dx, \alpha_k = \int_0^1 f(x) \cos 2k\pi x dx, \beta_k = \int_0^1 f(x) \sin 2k\pi x dx,
$$

for $k = 1, 2, \ldots$.

In the notations of the example 3.4 we have:

$$
\alpha_0 = \langle f, e_0 \rangle, \alpha_k = \frac{1}{\sqrt{2}} \langle f, f_k \rangle, \beta_k = \frac{1}{\sqrt{2}} \langle f, g_k \rangle, k = 1, 2, \ldots.
$$

From the Parseval identity we obtain:

$$
\int_0^1 f^2(x)dx = \alpha_0^2 + 2\sum_{k=1}^{\infty} (\alpha_k^2 + \beta_k^2)
$$

The following two corollaries of the Theorem 3.3 are two of the different versions of the so called Riesz-Fisher theorem.

**Corollary 3.2.** If $H$ is a separable complex (real) Hilbert space and if $\{e_k\}_{k=1}^\chi$ is a base, then:

$$
H = \left\{ \sum_{k=1}^\chi \alpha_k e_k : \alpha_k \in \mathbb{C} (\text{or } \mathbb{R}) \text{ for } \chi \geq 1, \sum_{k=1}^\chi |\alpha_k|^2 < \chi \right\}.
$$

Proof. If $\sum_{k=1}^\chi |\alpha_k|^2 < \chi$ then from the Theorem 2.3 a) the series (or the finite sum) $\sum_{k=1}^\chi \alpha_k e_k$ converges. The converse every vector of $H$ admits a Fourier representation having the properties d) and f) from the Theorem 3.3.

**Corollary 3.3.** Let $\{e_k\}_{k=1}^\chi$ be a base of a separable complex (or real) Hilbert space and let $\{\alpha_k\}_{k=1}^\chi$ be a sequence in $\mathbb{C}$ (or in $\mathbb{R}$) such that $\sum_{k=1}^\chi |\alpha_k|^2 < +\chi$. Then there exists one unique vector $f \in H$ such that: $\langle f, e_k \rangle = \alpha_k$, for every $k \geq 1$.

Proof. The vector $f = \sum_{k=1}^\chi \alpha_k e_k$ verifies the requests and from the Theorem 3.3 d) is unique.

**Theorem 3.4.** Two any bases of separable Hilbert space $H$ have the same cardinal number.

**Def. 3.2.** The cardinal number of a base of a Hilbert space $H$ is said the DIMENSION OF $H$. 
\[ \mathbb{R}^n \text{ and } \mathbb{C}^n \text{ have dimension } n. \]
\[ l_2 \text{ and } L_2[a, b] \text{ have dimension } \chi_0. \]

4. Isomorphisms

We will prove that every infinite-dimensional and separable Hilbert space is a copy of \( l_2 \).

**Theorem 4.1.** If two separable Hilbert spaces, \( H \) and \( H' \), have the same (finite or infinite) dimension, then there exists a bijective application:
\[
U : H \to H', \quad f \mapsto Uf
\]
such that for every \( f, g \in H \) and \( \forall \lambda \in \mathbb{C} \), it has:
\[ a) \quad U(f + g) = Uf + Ug \]
\[ b) \quad U(\lambda f) = \lambda Uf \]
\[ c) \quad \langle Uf, Ug \rangle = \langle f, g \rangle \]
(Note: \( \implies \|Uf\| = ||f|| \))

**Proof.** \( \{e_k\}_{k=1}^\chi \) and \( \{e'_k\}_{k=1}^\chi \) bases for respectively \( H \) and \( H' \) (\( \chi \leq \infty \)). For every \( f \in H \) we define:
\[ f' = Uf = \sum_{k=1}^\chi \langle f, e_k \rangle e'_k \]

This definition is significative since from the Parseval identity
\[ \sum_{k=1}^\chi |\langle f, e_k \rangle|^2 = \|f\|^2 < \infty \]
and by virtue of the Theorem 2.3 the series \( \sum_{k=1}^\chi \langle f, e_k \rangle e'_k \) converges in \( H' \). Moreover: \( Ue_k = e'_k \), \( \forall k \geq 1 \).

Prove that \( U \) verifies a), b) and c) and that is a bijection. \( \square \)

Intuitively, the Theorem 4.1 says that we can identify, through \( U \), the elements of \( H \) and \( H' \) such that anyone of these spaces appears (algebraically and topologically) a perfect copy of the other one.

**Def. 4.1.** An **APPLICATION** \( A : D \to H' \), where \( D \) is a linear manifold of \( H \), and \( H \) and \( H' \) are two Hilbert \( K \)-spaces is said **LINEAR** if:
\[ a) \quad A(f + g) = Af + Ag, \quad \forall f, g \in D \]
\[ b) \quad A(\lambda f) = \lambda Af, \quad \forall f \in D, \quad \forall \lambda \in K. \]

**Def. 4.2.** An **APPLICATION** \( A : D \to H' \), where \( D \) is a linear manifold of \( H \), and \( H \) and \( H' \) are two Hilbert \( K \)-spaces is said **ISOMETRY** if:
\[ c) \quad \langle Af, Ag \rangle = \langle f, g \rangle, \quad \forall f, g \in D. \]

**Def. 4.3.** An isometric linear and surjective application \( A : H \to H' \), where \( H \) and \( H' \) are Hilbert \( K \)-space, is said an **ISOMORPHISM OF \( H \) ON \( H' \)**.

An isomorphism of \( H \) on itself is said an **AUTOMORPHISM**.
Remark 4.1. A linear isometry is automatically injective \(^3\). So an isomorphism is bijective.

If there exists an isomorphism of \(H\) on \(H'\), then there exists an isomorphism of \(H'\) on \(H\).

Def. 4.4. Two Hilbert spaces \(H\) and \(H'\) are said isomorphic if there exists an isomorphism of a space on the other one.

Corollary 4.1. Two separable Hilbert spaces are isomorphic \(\iff\) they have the same dimension.

Proof. The sufficient part follows from the Theorem 4.1.

The converse, let \(U : H \to H'\) be an isomorphism and let \(\{e_k\}_{k=1}^{\chi}\) be a base for \(H\). Then \(\{Ue_k\}_{k=1}^{\chi}\) is an orthonormal family of \(H'\) and hence the dimension of \(H'\) is at least \(\chi\). Repeating the reasoning in the other way, we conclude that \(H\) and \(H'\) must have the same dimension. \(\square\)

Corollary 4.2. Every infinitely dimensional and separable Hilbert space is isomorphic to \(l_2\).

\(^3\)for \(g, g \in H, f \neq g \Rightarrow \langle A(f - g), A(f - g) \rangle = \langle f - g, f - g \rangle \neq 0 \Rightarrow A(f - g) \neq 0 \Rightarrow 0 \neq A(f - g) = Af - Ag.\)
CHAPTER 3

BOUNDED AND LINEAR OPERATORS

1. Bounded and linear applications (operators)

We will indicate by $H$ and $H'$ two Hilbert spaces and by $D \subset H$, $D \neq \{0\}$, a linear manifold.

**Def. 1.1.** An APPLICATION (OPERATOR) $A : D \to H'$ is said LINEAR if:

a) $A(f + g) = Af + Ag$, $\forall$ $f, g \in D$

b) $A(\lambda f) = \lambda Af$, $\forall$ $f \in D$, $\forall$ $\lambda \in K$.

**Def. 1.2.** An APPLICATION (OPERATOR) $A : D \to H'$ is said BOUNDED if there exists a real number $k \geq 0$ such that:

$$\|Af\| \leq k\|f\|, \; \forall \; f \in D$$

**Def. 1.3.** Let define NORM of a bounded application $A : D \to H'$ the nonnegative real number $\|A\|:

$$\|A\| = \sup_{f \in D, f \neq 0} \frac{\|Af\|}{\|f\|}$$

**Remark 1.1.** The existence of the norm of a bounded application $A : D \to H'$ is assured by the fact that every superiorly bounded nonempty subset of real numbers admits an upper bound in $\mathbb{R}$.

**Remark 1.2.** Strictly we could define the norm of an application $A : D \to H'$, even if $A$ wasn’t bounded. In this case we should have:

$$+\infty = \sup_{f \in D, f \neq 0} \frac{\|Af\|}{\|f\|} = 1 \sup_{f \in D, \|f\|=1} \|Af\|$$

but we don’t interesting this case.

**Theorem 1.1.** If $A : D \to H'$ is a linear application, we have:

$$\|Af\| \leq \|A\||f|, \; \forall \; f \in D$$

\(^1\)In these passages we make use of the linearity
and
\[
\|A\| = \inf \{ k \in \mathbb{R}_0^+ : \|Af\| \leq k\|f\|, \forall f \in D \}.
\]

Remark 1.3. The norm of a bounded application \(A : D \to H'\) is zero \(\iff\) \(Af = 0' \in H', \forall f \in D\).

Theorem 1.2. If \(A : D \to H'\) is a bounded and linear application, we have:
\[
\|A\| = \sup_{f \in D, f \neq 0} \frac{\|Af\|}{\|f\|} = \sup_{f \in D, f \neq 0} \frac{\|Af\|}{\|f\|} \leq \sup_{f \in D, \|f\| = 1} \|A\| \leq \sup_{f \in D, \|f\| \leq 1} \|Af\| \leq 1 \sup_{f \in D, 0 < \|f\| \leq 1} \frac{\|Af\|}{\|f\|} \leq \|A\|
\]

Example 1.1. Let \(U : H \to H'\) be an isomorphism. Since \(\|Uf\| = \|f\|, \forall f \in H\) it follows that: the linear application \(U\) is bounded and \(\|U\| = 1\).

Example 1.2. Let \(M\) be a subspace of \(H\) and let \(P_M : H \to M\) be the projection on \(M\). \(P_M\) is a bounded and linear application and:
\[
\|P_M\| = \begin{cases} 0 & \text{if } M = \{0\} \\ 1 & \text{if } M \neq \{0\}. \end{cases}
\]

In fact (cfr. Theorem 2.9):
\[(7) \quad f = P_Mf + P_M^\perp f, \ P_Mf \in M, \ P_M^\perp f \in M^\perp, \ \text{for every } f \in H.\]

From the equations:
\[
f + g = (P_Mf + P_Mg) + (P_M^\perp f + P_M^\perp g), \ P_Mf, P_Mg \in M, \ P_M^\perp f, P_M^\perp g \in M^\perp
\]
\[
\lambda f = \lambda P_Mf + \lambda P_M^\perp f, \ \lambda P_Mf \in M, \ \lambda P_M^\perp f \in M^\perp,
\]

and from the uniqueness (cfr. Corollary 2.1) of this decomposition we conclude that \(P_M\) is linear. From (7) it results:
\[
\|f\|^2 = \|P_Mf\|^2 + \|P_M^\perp f\|^2 \geq \|P_Mf\|^2.
\]

So \(P_M\) is bounded and \(\|P_M\| \leq 1\).

For \(M = \{0\}\) obviously \(\|P_M\| = 0\).

Else for every \(f \in M \setminus \{0\}\) we have \(P_Mf = f\) and so \(\|P_M\| = 1\).

\footnote{In these passages we make use of the linearity.}
Example 1.3. Let $D$ be the linear manifold of all finite sequences of $l_2$ and let $B : D \to l_2$ so defined:

$$B(\alpha_k)_{k=1}^\infty = (k\alpha_k)_{k=1}^\infty.$$ 

Obviously $B$ is linear. Indicating by $\{e_k\}_{k=1}^\infty$ the standard base of $l_2$ we have:

$$\|Be_k\| = k\|e_k\| = k \text{ and } \lim_{k \to \infty} \|Be_k\| = \infty.$$ 

So $B$ is boundless.

Def. 1.4. A LINEAR APPLICATION $A : D \to H'$ is CONTINUOUS if for every $f_0 \in D$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$\|Af_0 - Af\| < \varepsilon \text{ for every } f \in D \text{ with } \|f - f_0\| < \delta.$$ 

Theorem 1.3. A (linear) application $A : D \to H'$ is continuous $\iff$ for every sequence $(f_n)_{n=1}^\infty \subset D$ convergent to any limit $f_0 \in D$ it results $Af_0 = A(\lim_{n \to \infty} f_n) = \lim_{n \to \infty} Af_n$.

Proof. $\implies$: We fix $f_0 \in D$ and $\varepsilon > 0$. We take $\delta > 0$ as in the Def. 1.4 and we suppose:

$$\|f_0 - f_n\| < \delta \text{ for every } n \geq n(\delta).$$

Then $\|Af_0 - Af_n\| < \varepsilon$ for every $n \geq n(\delta)$. Therefore $Af_0 = \lim_{n \to \infty} Af_n$.

$\impliedby$: If $A$ isn’t continuous then for any $f_0 \in D$ and any $\varepsilon > 0$ there exists a sequence $(f_n)_{n=1}^\infty$ such that $\|f_0 - f_n\| < \frac{1}{n}$ and $\|Af_0 - Af_n\| \geq \varepsilon$ for every $n \geq 1$. Therefore $\lim_{n \to \infty} f_n = f_0$ but $(Af_n)_{n=1}^\infty$ doesn’t converge to $Af_0$ and this is absurd. □

We note that, in the Def. 1.4, in the Theorem 1.3 the linearity of $A$ isn’t used. We work with linear applications, because we have interested to the properties of these applications.

Theorem 1.4. If a linear application $A : D \to H'$ is continuous in a $f_0 \in D \implies$ it is continuous in $D$.

Proof. Anyhow the application is continuous in $f_0$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$\|Af_0 - Af\| < \varepsilon \text{ for every } f \in D \text{ with } \|f - f_0\| < \delta.$$ 

Let $g$ be an other point of $D$. If $\|f - g\| < \delta$ we have $\|(f + f_0 - g) - f_0\| < \delta$ so $\|Af_0 - A(f + f_0 - g)\| < \varepsilon$, that is from the linearity of $A$, $\|Ag - Af\| < \varepsilon$. □

Theorem 1.5. A linear application $A : D \to H'$ is bounded $\iff$ it is continuous.
3. BOUNDED AND LINEAR OPERATORS

Proof. \( \implies \): If \( \|A\| = 0 \) the assertion is banal. We suppose so \( \|A\| > 0 \). We fix \( f_0 \in D \) and \( \varepsilon > 0 \). Then for every \( f \in D \) with \( \|f_0 - f\| < \frac{\varepsilon}{\|A\|} \) we have

\[
\|A f_0 - A f\| \leq \|A\| \|f_0 - f\| < \varepsilon.
\]

\( \iff \): If \( A \) is boundless there exists a sequence \( (f_n)_{n=1}^{\infty} \subset D \) of unity vectors (cfr. Remark 1.2) such that \( \|A f_n\| \geq n^2 \) for every \( n \geq 1 \). So the sequence \( (f_n)_{n=1}^{\infty} \) converges to 0 but \( \|A f_n\| \geq n \) for every \( n \geq 1 \). From the Theorem 1.3 the application \( A \) can’t be continuous and this is absurd. \( \square \)

**Theorem 1.6.** A bounded and linear application \( A : D \to H' \) can be extended univocally to a bounded and linear application:

\[
\overline{A} : \overline{D} \to H', \text{ con } \|\overline{A}\| = \|A\|.
\]

Proof. \( \overline{D} \) is a subspace. For the first thing we assure that if there exists a such extension \( \overline{A} \), it must be unique.

In fact if \( f \in \overline{D} \) is assigned, there exists a sequence \( (f_n)_{n=1}^{\infty} \subset D \) such that \( f = \lim_{n \to \infty} f_n \). Since \( \overline{A} \) must be continuous (cfr. Theorem 1.5) this implies:

\[
(8) \quad \overline{A} f = \lim_{n \to \infty} \overline{A} f_n = \lim_{n \to \infty} A f_n.
\]

So we will use (8) as definition of \( \overline{A} : \overline{D} \to H' \).

Prove that the definition is well placed and that doesn’t depend from the particular sequence convergent to \( f \in \overline{D} \); that is bounded and linear and \( \|\overline{A}\| = \|A\| \). \( \square \)

We will suppose that let \( H \) be a Hilbert space on \( \mathbb{C} \) and we will take as norm on \( \mathbb{C} \) the absolute value.

**Def. 1.5.** A linear application \( \phi : H \to \mathbb{C} \) is said LINEAR FUNCTIONAL ON \( H \).

**Example 1.4.** Let \( h \) be a vector of \( H \). For every \( f \in H \) we define \( \phi f = \langle f, h \rangle \in \mathbb{C} \). Prove that \( \phi \) is a bounded and linear functional on \( H \) and \( \|\phi\| = \|h\| \).

**Theorem 1.7 (Riesz representation theorem).** If \( \phi \) is a bounded and linear functional on \( H \), then there exists one unique vector \( h \in H \) such that \( \phi f = \langle f, h \rangle \) for every \( f \in H \) and so \( \|\phi\| = \|h\| \).

Proof. Let \( M = \{g \in H : \phi g = 0\} \). Prove that \( M \) is a subspace of \( H \). If \( M = H \), then the vector 0 \( \in H \) verifies the thesis of the theorem.

If \( M \neq H \), then there must exist one unity vector \( e \perp M \) (cfr. Corollary 2.2). Therefore \( \phi e \neq 0 \) (\( e \notin M \)) and \( h = \overline{\phi e} e \perp M \). For every vector \( f \in H \) it results:

\[
f = \left( f - \frac{\phi f}{|\phi e|^2} h \right) + \frac{\phi f}{|\phi e|^2} h.
\]
From:
\[ \phi \left( f - \frac{\phi f}{|\phi e|^2} h \right) = \phi f - \frac{\phi f}{|\phi e|^2} \phi e = 0 \]
we have:
\[ f_1 = f - \frac{\phi f}{|\phi e|^2} h = P_M f \in M, \quad f_2 = \frac{\phi f}{|\phi e|^2} h = P_{M^\perp} f \in M^\perp \]
and
\[ \langle f, h \rangle = \langle f_1, h \rangle + \langle f_2, h \rangle = 0 + \phi f. \]
Prove that \( \|\phi\| = \|h\| \) and that the vector \( h \) is unique. \( \square \)

2. Linear operators

**Def. 2.1.** An application \( A : D \rightarrow H, \) \( D \subset H, \) is said OPERATOR IN \( H. \)
An application \( A : H \rightarrow H \) is said OPERATOR ON \( H. \)

We will be interested to the linear operators. Examples of bounded operators are given by the automorphisms and by the projections on subspaces. A special automorphism is the identity operator, \( I, \) defined by \( If = f, \forall f \in H. \) A special projection is the zero operator, \( 0. \)

**Example 2.1.** Let \( A : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be a linear operator and let \( \{e_k\}_{k=1}^n \) be a base of \( \mathbb{C}^n. \) The action of \( A \) on the vectors \( e_k \) defines the behaviour of \( A \) on the whole \( \mathbb{C}^n. \) In fact we suppose
\[ Ae_h = \sum_{k=1}^n \alpha_{k,h} e_k, 1 \leq h \leq n \left( \alpha_{k,h} = \langle Ae_h, e_k \rangle \right). \]
So for every vector \( \sum_{h=1}^n \beta_h e_h \in H, \) we obtain:
\[ A \left( \sum_{h=1}^n \beta_h e_h \right) = \sum_{h=1}^n \beta_h Ae_h = \sum_{h=1}^n \beta_h \left( \sum_{k=1}^n \alpha_{k,h} e_k \right) = \sum_{k=1}^n \left( \sum_{h=1}^n \alpha_{k,h} \beta_h \right) e_k. \]
The behaviour of \( A \) on \( \mathbb{C}^n \) is so described by the matrix:
\[ A' = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \cdots & \cdots & \cdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{pmatrix}. \]
The converse, if a matrix \( A' \) is assigned as in (10), then the corresponding operator \( A \) defined on \( \mathbb{C}^n \) by (9) is bounded and linear. In fact from:
\[ \|A(\sum_{h=1}^{n} \beta_h e_h)\|^2 = \sum_{k=1}^{n} \left( \sum_{h=1}^{n} \alpha_{k,h} \beta_h \right)^2 \leq \left( \sum_{k=1}^{n} \sum_{h=1}^{n} |\alpha_{k,h}|^2 \sum_{h=1}^{n} |\beta_h|^2 \right) = \]

\[ = \left( \sum_{k=1}^{n} \sum_{h=1}^{n} |\alpha_{k,h}|^2 \right) \| \sum_{h=1}^{n} \beta_h e_h \|^2 \]

it follows that:

(11) \[ \|A\| \leq \left( \sum_{k=1}^{n} \sum_{h=1}^{n} |\alpha_{k,h}|^2 \right)^{\frac{1}{2}}. \]

Prove that \( A \) is linear.

**Theorem 2.1.** Let \( A \) and \( B \) two linear operators in \( H \) with respectively linear manifolds \( D_A \) and \( D_B \). Then the applications \( A + B \), \( \lambda A \), and \( AB \) defined by:

\[(A + B)f = Af + Bf, \quad \forall \ f \in D_A \cap D_B\]

\[(\lambda A)f = \lambda (Af), \quad \forall \ f \in D_A\]

\[(AB)f = A(Bf), \quad \forall \ f \in D_{AB} = \{f \in D_B : Bf \in D_A\}\]

are linear operators in \( H \) in the corresponding linear manifolds.

Proof. For exercise. \( \square \)

**Def. 2.2.** Two OPERATORS \( A \) and \( B \) in \( H \) with respectively domain \( D_A \) and \( D_B \) are EQUAL if \( D_A = D_B \) and \( Af = Bf, \quad \forall \ f \in D_A = D_B \).

**Remark 2.1.** Thus from the example 2.1 it generally follows \( AB \neq BA \); in fact \( D_{AB} \) can be different from \( D_{BA} \).

**Corollary 2.1.** With the operations defined in the Theorem 2.1 the set of all linear operators on \( H \) is linear space verifying the following properties:

a) \( (AB)C = A(BC) \)

b) \( A(B + C) = AB + AC, \ (A + B)C = AC + BC \)

c) \( (\alpha A)B = A(\alpha B) = \alpha (AB) \)

d) \( IA = AI = A \)

e) \( 0A = A0 = 0 \)

Proof. For exercise. \( \square \)
**Theorem 2.2.** Let $A$ and $B$ be two operators on bounded and linear $H$. Then $A + B$, $\lambda A$ and $AB$ verify the following inequalities:

\[(12) \quad ||A + B|| \leq ||A|| + ||B|| \]

\[(13) \quad ||\lambda A|| = |\lambda||A|| \]

\[(14) \quad ||AB|| \leq ||A|| ||B|| \]

**Proof.**

\[||A + B|| = \sup_{||f||=1} ||Af + Bf|| \leq \sup_{||f||=1} (||Af|| + ||Bf||) \leq \sup_{||f||=1} ||Af|| + \sup_{||f||=1} ||Bf|| = ||A|| + ||B||.\]

\[||\lambda A|| = \sup_{||f||=1} ||\lambda Af|| = \sup_{||f||=1} |\lambda||Af|| = |\lambda| \sup_{||f||=1} ||Af|| = |\lambda||A||.\]

\[||(AB)f|| = ||A(Bf)|| \leq ||A|| ||Bf|| \leq ||A|| ||B|| ||f||.\]

\[\square\]

The corollary of the Theorem 2.1 and the Theorem 2.2 prove that the set $B$ of all operators on bounded and linear $H$ is a normed linear space.

Algebraically the presence of multiplication in $B$ through its structure described by the corollary of the Theorem 2.1 becomes $B$ an algebra with unity.

Topologically the inequality (13) implies that this multiplication is continuous, becoming $B$ a normed algebra.

Moreover from the completeness of $H$ it follows that $B$ is complete that is a Banach algebra.

**Def. 2.3.** A (real or complex) linear space $U$ is said a (real or complex) **algebra** if for every $(a,b) \in U \times U$ there exists one unique element $ab \in U$, said product of $a$ and $b$, verifying the following conditions:

a) associative rule: $(ab)c = a(bc)$  

b) distributive rule: $a(b + c) = ab + ac; (a + b)c = ac + bc$  

c) $(aa)b = a(ab) = \alpha(ab)$.  

The element $e \in U$ is said unity if:

d) $ea = ae = a$  

The algebra $U$ is said commutative if:

e) commutative rule: $ab = ba$.

**Remark 2.2.** If there exists one unity in $U$ from d) it’s univocally determined.

In an algebra $U$ we have: $0a = a0 = 0$.

In fact: $0a = (a - a)a = a^2 - a^2 = 0$ and from $a0 = a(a - a) = a^2 - a^2 = 0$.

**Def. 2.4.** A normed linear space $U$ which, is also an algebra, is said **normed algebra** if $\forall a, b \in U$ holds:

\[||ab|| \leq ||a|| ||b||.\]
A complete normed algebra is said BANACH ALGEBRA.

**Theorem 2.3.** Under the operations defined in Theorem 2.1 the set $\mathcal{B}$ of all bounded and linear operators on $H$ is one unitary Banach algebra.

Proof. From the corollary of the Theorem 2.1 and from the Theorem 2.2 it follows that $\mathcal{B}$ is a normed unitary algebra.

It remains to prove that $\mathcal{B}$ is complete.

Let $(A_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{B}$, that is:

$$\lim_{m,n \to \infty} \|A_m - A_n\| = 0.$$ 

For the first consequence we have:

$$\lim_{m,n \to \infty} |\|A_m\| - \|A_n\|| \leq \lim_{m,n \to \infty} \|A_m - A_n\| = 0$$

and so the sequence $(\|A_n\|)_{n=1}^\infty$ converges. From:

$$\|A_m f - A_n f\| \leq \|A_m - A_n\| \|f\|$$

it follows that $\forall f \in H$ the sequence $(A_n f)_{n=1}^\infty$ is a Cauchy sequence and so it converges. We define on $H$ an operator $A$:

$$Af = \lim_{n \to \infty} A_n f, \; \forall f \in H.$$ 

Prove that $A$ is bounded and linear.

We prove that $A$ is the limit of the sequence $(A_n)_{n=1}^\infty$ in the sense of the norm operator.

In fact for every $\varepsilon > 0$ we can choose $n(\varepsilon) > 0$ such that:

$$\|A_m - A_n\| < \varepsilon, \; \forall \; m \geq n(\varepsilon), \; \forall \; n \geq n(\varepsilon).$$

Thus $\forall f \in H$ we have:

$$\|A_m f - A_n f\| \leq \|A_m - A_n\| \|f\| < \varepsilon \|f\|, \; \forall \; m \geq n(\varepsilon), \; \forall \; n \geq n(\varepsilon)$$

and, taking the limit as $m \to \infty$

$$\|Af - A_n f\| \leq \varepsilon \|f\|, \; \forall \; n \geq n(\varepsilon)$$

$$\|A - A_n\| \leq \varepsilon, \; \forall \; n \geq n(\varepsilon).$$

Hence $A = \lim_{n \to \infty} A_n$ and in particular $\|A\| = \lim_{n \to \infty} \|A_n\|$. \qed
Theorem 2.4. If \((A_n)_{n=1}^{\infty}\) and \((B_n)_{n=1}^{\infty}\) are two convergent sequences in \(B\) and \(A = \lim_{n \to \infty} A_n, B = \lim_{n \to \infty} B_n,\) then:

\[
\lim_{n \to \infty} A_n B_n = AB(= \lim_{n \to \infty} A_n \lim_{n \to \infty} B_n).
\]

Proof. \(\|A_n B_n - AB\| = \|A_n B_n - A_n B + A_n B - AB\| \leq \|A_n (B_n - B)\| + \|(A_n - A)B\| \leq \|A_n\| \|B_n - B\| + \|A_n - A\| \|B\|.\)

We note that the assertion of the Theorem 2.4 remains true if \(B\) is substituted with any normed algebra.

Definition 2.5. A BOUNDED AND LINEAR OPERATOR \(A : H \to H\) is said INVERTIBLE if there exists a bounded and linear operator \(A^{-1} : H \to H,\) said inverse of \(A,\) such that:

\[
AA^{-1} = A^{-1} A = I
\]

Remark 2.3. An automorphism of \(H\) is invertible. But the converse is false, that is, it isn’t said that an invertible operator preserves the inner product and so it isn’t said that it is an automorphism.

Theorem 2.5. An invertible bounded and linear operator is a bijection. The inverse is unique.

Proof. For exercise. \(\square\)

We suppose that \(H\) is a separable Hilbert space and that \(\{e_k\}_{k=1}^{\infty}\) is a base for \(H\) and \(A : H \to H\) let be a bounded and linear operator on \(H.\) Then, as in the dimensional finite case, the action of \(A\) on \(H\) is always determined by the action of \(A\) on the elements of the assigned base.

In fact from \(Ae_h = \sum_{k=1}^{\infty} \alpha_{k,h} e_k, h \geq 1,\) from the continuity of \(A,\) for every vector \(f = \sum_{h=1}^{\infty} \beta_h e_h \in H\) we have

\[
\langle Af, e_k \rangle = \left( \sum_{h=1}^{\infty} \beta_h A e_h, e_k \right) = \sum_{h=1}^{\infty} \langle \beta_h A e_h, e_k \rangle = \sum_{h=1}^{\infty} \alpha_{k,h} \beta_h
\]

\[
Af = \sum_{k=1}^{\infty} \langle Af, e_k \rangle e_k = \sum_{k=1}^{\infty} \sum_{h=1}^{\infty} \alpha_{k,h} \beta_h e_k.
\]

The action of \(A\) on \(H\) can described by the infinite matrix:

\[
A' = \begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \ldots \\
\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \ldots \\
\alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \ldots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Note that \( \alpha_{k,h} = \langle Ae_h, e_k \rangle \), \( k \geq 1 \), \( h \geq 1 \) and \( \sum_{k=1}^{\infty} |\alpha_{k,h}|^2 = \|Ae_h\|^2 \leq \|A\|^2 < \infty \), \( h \geq 1 \).

The following examples of bounded and linear operators are very important for the spectral analysis.

**Example 2.2.** The linear operator \( A : l_2 \to l_2 \) defined by:

\[
Aa = A(\alpha_k)_{k=1}^{\infty} = (\alpha_{k-1})_{k=1}^{\infty}
\]

or \( A(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots) \) is said on the right translation operator in \( l_2 \).

Obviously \( \langle Aa, Ab \rangle = \sum_{k=1}^{\infty} \alpha_{k-1} \beta_{k-1} = \langle a, b \rangle \).

Thus \( A \) is an isometry of \( l_2 \) in itself.

In fact \( A \) transforms \( l_2 \) on a proper subspace that is the set \( M \) of all sequences to summable square having the first term equal to zero. So \( A \) is certainly noninvertible.

Describing the behaviour of \( A \) in term of the standard base \( \{e_k\}_{k=1}^{\infty} \) of \( l_2 \) we obtain:

\[
Ae_h = e_{h+1}, \ h \geq 1.
\]

Thus the matrix \( A' \) associated to the operator \( A \) is in the form:

\[
A' = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
& & & \cdots \cdots \cdots
\end{pmatrix}
\]

**Example 2.3.** Let \( H \) be a separable Hilbert space with base \( \{e_k\}_{k=1}^{\infty} \) and let \( A' \) be an infinite matrix in the form (14) with the additional property that:

\[
\alpha^2 = \sum_{k=1}^{\infty} \sum_{h=1}^{\infty} |\alpha_{k,h}|^2 < \infty, \ \alpha \geq 0.
\]

In the example 2.2 this condition isn’t verified from the matrix \( A' \).

For every vector \( f = \sum_{h=1}^{\infty} \beta_h e_h \in H \) we have from the Cauchy inequality in \( l_2 \)

\[
\sum_{k=1}^{\infty} \sum_{h=1}^{\infty} |\alpha_{k,h} \beta_h|^2 \leq \sum_{k=1}^{\infty} (\sum_{h=1}^{\infty} |\alpha_{k,h}|^2) \sum_{h=1}^{\infty} |\beta_h|^2 = \alpha^2 \|f\|^2.
\]

The series \( \sum_{k=1}^{\infty} (\sum_{h=1}^{\infty} \alpha_{k,h} \beta_h) e_k \) converges so in \( H \).

It’s therefore possible to define an operator \( A : H \to H \) so:

\[
A(\sum_{h=1}^{\infty} \beta_h e_h) = \sum_{k=1}^{\infty} (\sum_{h=1}^{\infty} \alpha_{k,h} \beta_h) e_k, \ for \ f = \sum_{h=1}^{\infty} \beta_h e_h \in H.
\]

The operator \( A \) is obviously linear.
3. BILINEAR FORMS

From (15) it follows that $A$ is bounded and moreover that:

$$\|A\| \leq \alpha = \left( \sum_{k=1}^{\infty} \sum_{h=1}^{\infty} |\alpha_{k,h}|^2 \right)^{\frac{1}{2}}.$$ 

3. Bilinear forms

**Def. 3.1.** A function $\varphi : H \times H \to K$ is said **BILINEAR FORM** (or **BILINEAR FUNCTIONAL**) on $H$ if $\forall (f, g) \in H \times H, \forall \alpha \in K$ we have:

$$\varphi(f_1 + f_2, g) = \varphi(f_1, g) + \varphi(f_2, g)$$

$$\varphi(\alpha f, g) = \alpha \varphi(f, g)$$

$$\varphi(f, g_1 + g_2) = \varphi(f, g_1) + \varphi(f, g_2)$$

$$\varphi(f, \alpha g) = \bar{\alpha} \varphi(f, g)$$

The bilinear form $\varphi$ is said **SYMMETRIC** if:

$$\varphi(g, f) = \overline{\varphi(f, g)}, \forall (f, g) \in H \times H.$$ 

The bilinear form $\varphi$ is said **BOUNDED** if there exists a real number $k \geq 0$ such that:

$$|\varphi(f, g)| \leq k \|f\| \|g\|, \forall (f, g) \in H \times H.$$ 

If $\varphi$ is bounded, then the nonnegative number:

$$\|\varphi\| = \sup_{f \neq 0, g \neq 0} \frac{|\varphi(f, g)|}{\|f\| \|g\|}$$

is said the **NORM** of $\varphi$.

**Def. 3.2.** If $\varphi$ is a bilinear form on $H$, then the function $\hat{\varphi}$ on $H$ defined by:

$$\hat{\varphi}(f) = \varphi(f, f)$$

is said **ASSOCIATED QUADRATIC FORM** $\varphi$.

The quadratic form $\hat{\varphi}$ is said **REAL** if $\hat{\varphi}(f) \in \mathbb{R}, \forall f \in H$.

The quadratic form $\hat{\varphi}$ is said **BOUNDED** if there exists a real $M \geq 0$ such that:

$$|\hat{\varphi}(f)| \leq M \|f\|^2, \forall f \in H.$$ 

If $\hat{\varphi}$ is bounded, then the nonnegative number:
is said NORM of $\hat{\varphi}$.

**Remark 3.1.** For a bilinear form $\varphi$ and for a quadratic form $\hat{\varphi}$ on $H$ we have respectively:

$$
\|\varphi\| = \sup_{f \neq 0, g \neq 0} \frac{|\varphi(f, g)|}{\|f\| \|g\|} = \sup_{\|f\| = \|g\| = 1} |\varphi(f, g)|
$$

$$
\|\hat{\varphi}\| = \sup_{f \neq 0} \frac{|\hat{\varphi}(f)|}{\|f\|^2} = \sup_{\|f\| = 1} |\hat{\varphi}(f)|
$$

So:

$\varphi$ is bounded $\iff$ the terms of the first chain of equalities are finite.

$\hat{\varphi}$ is bounded $\iff$ the terms of the second chain of equalities are finite.

Anyone bilinear form $\varphi$ determines a quadratic form $\hat{\varphi}$ associated to it.

But not only $\hat{\varphi}$ is determined by $\varphi$, but it’s possible to get back from $\hat{\varphi}$ the original bilinear form $\varphi$.

**Theorem 3.1 (Polar identity).**

$$
\varphi(f, g) = \frac{1}{4}[\hat{\varphi}(f + g) - \hat{\varphi}(f - g) + i\hat{\varphi}(f + ig) - i\hat{\varphi}(f - ig)]
$$

Proof. In fact:

$$
\hat{\varphi}(f + g) = \varphi(f + g, f + g) = \varphi(f, f) + \varphi(f, g) + \varphi(g, f) + \varphi(g, g)
$$

$$
\hat{\varphi}(f - g) = \varphi(f - g, f - g) = \varphi(f, f) - \varphi(f, g) - \varphi(g, f) + \varphi(g, g).
$$

Subtracting the second equality from the first, we have:

$$
(16) \quad \hat{\varphi}(f + g) - \hat{\varphi}(f - g) = 2\varphi(f, g) + 2\varphi(g, f).
$$

Substituting $g$ with $ig$ we obtain:

$$
(17) \quad \hat{\varphi}(f + ig) - \hat{\varphi}(f - ig) = -2i\varphi(f, g) + 2i\varphi(g, f).
$$

Multiplying (17) for $i$ and summing to (16):

$$
\hat{\varphi}(f + g) - \hat{\varphi}(f - g) + i\hat{\varphi}(f + ig) - i\hat{\varphi}(f - ig) = 4\varphi(f, g).
$$

□

**Corollary 3.1.** If $\varphi$ and $\phi$ are bilinear form on $H$ and if $\hat{\varphi} = \hat{\phi}$, then $\varphi = \phi$. 
Corollary 3.2. A linear operator \( A \) on \( H \) is isometric \( \iff \|Af\| = \|f\|, \forall f \in H. \)

Theorem 3.2. \( \varphi \) is symmetric \( \iff \hat{\varphi} \) is real.

Proof. If \( \varphi \) is symmetric, then \( \hat{\varphi}(f) = \varphi(f, f) = \overline{\varphi(f, f)} = \overline{\hat{\varphi}(f)}. \)

The converse, we suppose that \( \hat{\varphi} \) is real.

From the polar identity and from the following one:

\[
\hat{\varphi}(f) = \hat{\varphi}(-f) = \hat{\varphi}(if)
\]

we deduct:

\[
\varphi(g, f) = \frac{1}{4}[\hat{\varphi}(g + f) - \hat{\varphi}(g - f) + i\hat{\varphi}(g + if) - i\hat{\varphi}(g - if)] = \\
= \frac{1}{4}[\hat{\varphi}(f + g) - \hat{\varphi}(f - g) + i\hat{\varphi}(f - ig) - i\hat{\varphi}(f + ig)] = \frac{1}{4} \varphi(f, g).
\]

\[\square\]

Theorem 3.3. \( \varphi \) is bounded \( \iff \hat{\varphi} \) is bounded.

If \( \varphi \) is bounded, then:

\[
\|\hat{\varphi} \| \leq \|\varphi\| \leq 2\|\hat{\varphi}\|
\]

Proof. We suppose bounded \( \varphi \). Then:

\[
\sup_{\|f\|=1} |\hat{\varphi}(f)| = \sup_{\|f\|=1} |\varphi(f, f)| \leq \sup_{\|f\|=\|g\|=1} |\varphi(f, g)| = \|\varphi\|.
\]

Therefore: \( \|\hat{\varphi}\| \leq \|\varphi\|. \)

The converse, we suppose bounded \( \hat{\varphi} \). Then it follows, for every \( f, g \in H \) with \( \|f\| = \|g\| = 1 \):

\[
|\varphi(f, g)| \leq \frac{1}{4}\|\hat{\varphi}\|(\|f + g\|^2 + \|f - g\|^2 + \|f + ig\|^2 + \|f - ig\|^2) = \\
= \frac{1}{4}\|\hat{\varphi}\|2(\|f\|^2 + \|g\|^2 + \|f\|^2 + \|g\|^2) = \|\hat{\varphi}\|(\|f\|^2 + \|g\|^2).
\]

Therefore: \( \|\varphi\| \leq 2\|\hat{\varphi}\|. \)

\[\square\]

Theorem 3.4. If \( \varphi \) is bounded and symmetric, then \( \|\varphi\| = \|\hat{\varphi}\|. \)

Proof. From the Theorem 3.2 it results real \( \hat{\varphi} \) and for the Theorem 3.3 we have:

\( \|\hat{\varphi}\| \leq \|\varphi\|. \) Therefore we prove that:

\( |\varphi(f, g)| \leq \|\hat{\varphi}\| \) for every \( f, g \in H \) with \( \|f\| = \|g\| = 1 \).

Suppose that \( \varphi(f, g) = \rho e^{i\alpha} \). We denote by \( f' \) unitary vector \( e^{-i\alpha}f \); then

\[
|\varphi(f, g)| = \rho = \varphi(e^{-i\alpha}f, g) = \varphi(f', g) = \frac{1}{4} [\hat{\varphi}(f' + g) - \hat{\varphi}(f' - g)].
\]

\[\text{Here we use the reality of } \hat{\varphi}.\]
since $\varphi(f', g)$ is real it follows that the purely imaginary terms in the polar identity it must be zero. Therefore:

$$|\varphi(f, g)| \leq \frac{1}{4}\|\hat{\varphi}\|((\|f' + g\|^2 + \|f' - g\|^2) = \frac{1}{2}\|\hat{\varphi}\|((\|f'\|^2 + \|g\|^2) = \|\hat{\varphi}\|.$$ 

□

**Theorem 3.5.** a) Let $A : H \to H$ a bounded and linear operator. Then the function $\varphi : H \times H \to K$ defined by:

$$\varphi(f, g) = \langle f, Ag \rangle$$

is a bounded bilinear on $H$ and $\|\varphi\| = \|A\|$.

b) The converse, let $\varphi : H \times H \to K$ be a bounded bilinear form. Then there exists one unique bounded and linear operator $A : H \to H$, such that:

$$\varphi(f, g) = \langle f, Ag \rangle, \ \forall \ (f, g) \in H \times H.$$ 

**Corollary 3.3.** a) Let $A : H \to H$ be a bounded and linear operator. Then the function $\phi : H \times H \to K$ defined by:

$$\phi(f, g) = \langle Af, g \rangle$$

is a bounded bilinear form on $H$ and $\|\phi\| = \|A\|$.

b) The converse, let $\phi : H \times H \to K$ be a bounded bilinear form. Then there exists one unique bounded and linear operator $A : H \to H$, such that:

$$\phi(f, g) = \langle Af, g \rangle, \ \forall \ (f, g) \in H \times H.$$ 

**Corollary 3.4.** If $A : H \to H$ is a bounded and linear operator, then:

$$\|A\| = \sup_{\|f\| = \|g\| = 1} |\langle f, Ag \rangle| = \sup_{\|f\| = \|g\| = 1} |\langle Af, g \rangle|.$$ 

4. Added operators

Not all the bounded and linear operators $A : H \to H$ have an inverse, but they have always an operator $A^* : H \to H$, such that: $\langle Af, g \rangle = \langle f, A^*g \rangle$.

In fact:

**Theorem 4.1.** Let $A : H \to H$ be a bounded and linear operator. Then there exists one unique bounded and linear operator $A^* : H \to H$ said the added of $A$, such that:

$$\langle Af, g \rangle = \langle f, A^*g \rangle, \ \forall \ f \in H, \ \forall \ g \in H.$$
Moreover: $\|A\| = \|A^*\|$. 

Proof. From the Corollary 3.3 the equation $\varphi(f, g) = \langle Af, g \rangle$ defines a bounded bilinear form $\varphi$. From the Theorem 3.5 b) there exists one unique bounded and linear operator $A^* : H \to H$, such that:

$$\varphi(f, g) = \langle f, A^*g \rangle, \quad \forall f \in H, \forall g \in H.$$ 

Moreover, from the same theorems:

$$\|A\| = \|\varphi\| = \|A^*\|.$$ 

\[\square\]

Def. 4.1. It’s called ADDED OPERATOR of a bounded and linear operator $A : H \to H$, that bounded and linear operator $A^* : H \to H$, such that:

$$\langle Af, g \rangle = \langle f, A^*g \rangle, \quad \forall f \in H, \forall g \in H. \tag{18}$$

Example 4.1. Let $H = C^n$ and let $\{e_k\}_{k=1}^n$ be a base of $C^n$. Let $A : C^n \to C^n$ be a linear operator given by the matrix $A' = (\alpha_{k,h})_{k,h=1}^n$.

If $b = \sum_{k=1}^n \beta_k e_k, c = \sum_{k=1}^n \gamma_k e_k \in C^n$ then:

$$\langle Ab, c \rangle = \left\langle \sum_{k=1}^n \sum_{h=1}^n \alpha_{k,h} \beta_h e_k, \sum_{k=1}^n \gamma_k e_k \right\rangle = \sum_{k=1}^n \sum_{h=1}^n \beta_k \alpha_{k,h} \beta_h.$$ 

We can imagine to have obtained this number multiplying the matrix $A'$ on the right for the column vector of components $\beta_k$ and on the left for the row vector of components $\gamma_k$. Let $A'' = (\alpha_{k,h}^*)_{k,h=1}^n$ be the matrix of elements $\alpha_{k,h}^* = \overline{\alpha_{h,k}}$ ($A''$ is the complex conjugate transposed of $A'$ and it’s said added matrix of $A'$). Then:

$$\langle b, A\gamma c \rangle = \left\langle \sum_{k=1}^n \beta_k e_k, \sum_{k=1}^n \sum_{h=1}^n \alpha_{k,h}^* \gamma_h e_k \right\rangle = \sum_{k=1}^n \sum_{h=1}^n \beta_k \overline{\alpha_{k,h}} \gamma_h =$$

$$= \sum_{k=1}^n \sum_{h=1}^n \overline{\gamma_h} \alpha_{h,k} \beta_k = \langle Ab, c \rangle.$$ 

The operator $A^*$ corresponding to the matrix $A''$ is so the added of $A$ and the added operators correspond to the added matrixes.

Example 4.2. Let $A : l_2 \to l_2$ be the right traslation operator defined by

$$A(\alpha_1, \alpha_2, \alpha_3, \ldots) = (0, \alpha_1, \alpha_2, \ldots).$$

It follows that the added operator $A^* : l_2 \to l_2$ is the left traslation operator of $l_2$, given by: $A^*(\beta_1, \beta_2, \beta_3, \ldots) = (\beta_2, \beta_3, \beta_4, \ldots).$ 

In fact for $a = (\alpha_k)_{k=1}^\infty$ and $b = (\beta_k)_{k=1}^\infty$ di $l_2$, we have:
\[ \langle Aa, b \rangle = \sum_{k=1}^{\infty} \alpha_k \overline{\beta}_{k+1} = \langle a, A^*b \rangle. \]

With respect to the standard base \( \{e_k\}_{k=1}^{\infty} \) of \( l_2 \), the action of \( A^* \) is described by:

\[ A^* e_k = e_{k-1}, k \geq 2, A^* e_1 = 0. \]

The corresponding matrix \( A^* \) is:

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We remark that:

\[ (19) \quad A^*A = I \neq AA^* \]

and that \( AA^* \) always nulls the first term of every sequence of \( l_2 \). While \( A \) is isometric, \( A^* \) isn’t certainly isometric, being \( A^*e_1 = 0 \).

Moreover: \( \|A\| = \|A^*\| = 1 \)

**Theorem 4.2.** Let \( A, B : H \to H \) be two bounded and linear operators. Then:

- a) \( A^{**} = A \)
- b) \( (\lambda A)^* = \overline{\lambda} A^* \)
- c) \( (AB)^* = B^*A^* \)
- d) \( (A + B)^* = A^* + B^* \)
- e) If \( A \) is invertible, also \( A^* \) is invertible and: \( (A^*)^{-1} = (A^{-1})^* \).

Proof. For exercise. \( \square \)

**Theorem 4.3.** If \( A : H \to H \) is bounded and linear, then:

\[ \|A^*A\| = \|A\|^2 = \|AA^*\|. \]

Proof. From \( \|Af\|^2 = \langle Af, Af \rangle = \langle f, A^*Af \rangle \leq \|A^*A\|\|f\|^2 \), we have: \( \|Af\| \leq \|A^*A\|^\frac{1}{2}\|f\| \), thus \( \|A\|^2 \leq \|A^*A\| \).
Moreover \( \|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2 \). \( \square \)

**Def. 4.2.** A bounded and linear operator \( A : H \to H \) is said:

- **Autoadded (or Hermitian)** if \( A^* = A \)
- **Unitary** if \( A^* = A^{-1} \)
- **Normal** if \( AA^* = A^*A \).

**Remark 4.1.** Obviously the bounded, linear and added operators, as those ones unitary are normal.
Remark 4.2. The right translation operator $A : l_2 \to l_2$ (cfr. example 4.2) isn’t normal from (19). For the same reason the left translation operator isn’t normal, too.

Remark 4.3. The operator $2iI$ of any Hilbert space is normal, but it isn’t neither autoadded, nor unitary. In fact we have: $(2iI)^* = -2iI$ and $(2iI)(2iI)^* = (2iI)^2(2iI) = 4I$.

Remark 4.4. From what we have seen in the example 4.1 it follows that every linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ is autoadded if and only if, with respect to a given base, the matrix $A'$ is autoadded (or hermitian) in the usual sense, that is $\alpha_{k,h} = \overline{\alpha_{h,k}}$. Moreover $A$ is unitary if and only if the corresponding matrix $A'$ is unitary in the traditional sense, that is, if and only if the added matrix coincides with the inverse matrix of $A'$ (a real unitary matrix is usually said orthogonal).

Now, we give some characterizations for the bounded and linear operators which are autoadded, unitary and normal.

At first, we remark that from the definition 4.2, it follows that a bounded and linear operator $A : H \to H$ is:

- autoadded $\iff \langle Af, g \rangle = \langle f, Ag \rangle, \forall f, g \in H$
- unitary $\iff A$ is invertible and: $(\ast) \langle Af, g \rangle = \langle f, A^{-1}g \rangle, \forall f, g \in H$.

**Theorem 4.4.** A bounded and linear operator on $H$ is unitary $\iff$ it is an automorphism.

Proof. $\Rightarrow$: If $A \in \mathcal{B}$ is unitary then since $\mathcal{B}$ is a Banach algebra it follows: $\langle Af, Ag \rangle = \langle f, A^{-1}Ag \rangle = \langle f, g \rangle$ and so $A$ is an automorphism.

$\Leftarrow$: For exercise. $\square$

Remark 4.5. If $A$ is an injective linear operator verifying $(\ast)$ then $A$ is an automorphism and so it’s bounded.

**Theorem 4.5.** Let $A : H \to H$ be a bounded and linear operator. The following affirmations are equivalent:

a) $A$ is autoadded
b) The bilinear form $\varphi$ defined by $\varphi(f, g) = \langle Af, g \rangle$ is symmetric
c) The quadratic form $\hat{\varphi}$ defined by $\hat{\varphi}(f) = \langle Af, f \rangle$ is real.

Proof. For exercise. $\square$

**Corollary 4.1.** If $A : H \to H$ is an autoadded, bounded and linear operator then:

$$\|A\| = \sup_{\|f\|=1} |\langle Af, f \rangle|.$$

Proof. Defining $\varphi(f, g) = \langle Af, g \rangle$ as in the proof of the Theorem 4.1 we obtain from the Corollary 3.3 a) and from the Theorema 3.4:
\[ \|A\| = \|\hat{\varphi}\| = \|\hat{\phi}\| = \sup_{\|f\| = 1} |\hat{\phi}(f)| = \sup_{\|f\| = 1} |\langle Af, f \rangle|. \]

**Theorem 4.6.** A bounded and linear operator \( A : H \to H \) is normal \( \iff \|Af\| = \|A^*f\|, \forall \ f \in H. \)

**Proof.** We define the bilinear forms \( \varphi \) and \( \phi \) on \( H \) by:

\[ \varphi(f, g) = \langle A^*Af, g \rangle, \phi(f, g) = \langle AA^*f, g \rangle. \]

We obtain:

\[ \hat{\varphi}(f) = \langle A^*Af, f \rangle = \langle Af, Af \rangle = \|Af\|^2, \]
\[ \hat{\phi}(f) = \langle AA^*f, f \rangle = \langle A^*f, A^*f \rangle = \|A^*f\|^2. \]

From the Corollary 3.1 we have that \( \hat{\varphi} = \hat{\phi} \iff \varphi = \phi \) that is \( \iff \langle A^*Af, g \rangle = \langle AA^*f, g \rangle, \forall \ f, g \in H \) and this one \( \iff A^*A = AA^*. \)

**Theorem 4.7.** Let \( A : H \to H \) be a bounded and linear operator. Then there exist two autoadded, bounded and linear operators \( B, C : H \to H, \) such that:

\[ A = B + iC \]

\( B \) and \( C \) are determined in one unique way.

The operator \( A \) is normal \( \iff BC = CB. \)

**Proof.** We suppose that there exist two autoadded, bounded and linear operators \( B \) and \( C \) on \( H, \) as in the theorem. Then:

\[ A = B + iC \quad , \quad A + A^* = 2B \]
\[ A^* = B^* - iC^* = B - iC \quad , \quad A - A^* = 2iC \]

So we necessarily give the formulas:

\[ B = \frac{1}{2}(A + A^*) \quad , \quad C = \frac{1}{2i}(A - A^*) \]

The converse, for every bounded and linear operator on \( H, \) the bounded and linear operators, defined by (21), are autoadded and they verify (20).

If \( A \) is normal, then we have:

\[ BC = \frac{1}{4i}(A^2 + A^*A - AA^* - A^{*2}) = \frac{1}{4i}(A^2 - A^{*2}) = CB. \]

The converse, from \( BC = CB \) it follows:

\[ AA^* = (B + iC)(B - iC) = (B - iC)(B + iC) = A^*A. \]
5. Projection operators

**Definition 5.1.** To every subspace \( M \subset H \) it corresponds a projection operator \( P = P_M : H \to M \) that we will briefly call the **PROJECTION ONTO** \( M \) (cfr. Theorem 2.6, Chapter 2).

**Theorem 5.1.** If \( P \) is a projection onto a subspace \( M \), then:

\[
M = \{ f : Pf = f \} = \{ f : \|Pf\| = \|f\| \} = \{ Pg : g \in H \}.
\]

Proof. It’s already known that \( Pg \in M, \forall g \in H \), and \( Pf = f, \forall f \in M \) and so that \( M \subset \{ f : Pf = f \} \subset \{ Pg : g \in H \} = M \) moreover it’s \( M \subset \{ f : Pf = f \} \subset \{ f : \|Pf\| = \|f\| \} \). So \( M \) is contained in all the above-mentioned sets and it coincides with the last one because from the definition \( \{ Pg : g \in H \} \) is constituted by elements of \( M \). On the other hand, for \( f \not\in M \) we have \( f = Pf + P \perp f \) (cfr. Theorem 2.9, Chapter 2) (where \( P \perp \) denotes the projection onto \( M^\perp \)) and \( P \perp f \neq 0 \).

So:

\[
\|f\|^2 = \|Pf\|^2 + \|P \perp f\|^2 > \|Pf\|^2
\]

and therefore \( M \supset \{ f : \|Pf\| = \|f\| \} \).

**Remark 5.1.** It’s reasonable to denote the set \( \{ Pg : g \in H \} \) with \( PH \). In general, if \( A \) is a linear operator in \( H \) with domain \( D_A \), then for every \( U \subset D_A \) we will write \( AU = \{ Af : f \in U \} \). In particular, the set \( AD_A \) denotes the rank of \( A \). In accordance with the Theorem 5.1, the rank of a projection coincides with the corresponding subspace.

**Example 5.1.** Let \( [c,d[ \) be a subinterval of \( ]a,b[ \subset \mathbb{R} \) and let

\[
M = \{ f \in L^2[a,b] : f(x) = 0 \text{ for almost every } x \in ]a,b[ \setminus [c,d[ \}.
\]

Then \( M \) is a subspace of \( L^2[a,b] \) that can be identified by \( L^2[c,d] \) (cfr. example 1.4 of the §1 and the example 2.1 of the §2 of the Chapter 2). If \( P \) is the projection onto \( M \), then:

\[
Pf(x) = \begin{cases} 
  f(x) & \text{for } x \in ]c,d[ \\
  0 & \text{for } x \in [a,b] \setminus [c,d[ 
\end{cases}
\]

The passage from \( f \in L^2[a,b] \) to \( Pf \) can be realized multiplying the function \( f \) for the characteristic function of \( ]c,d[ \), \( \chi_{]c,d[} \), defined by \( Pf = \chi_{]c,d[} f, \forall f \in L^2[a,b] \).

The assertion of the Theorem 5.1 can be formulated in the following way:

\[
M = \{ f \in L^2[a,b] : \chi_{]c,d[} f = f \} = \{ f \in L^2[a,b] : \int_c^d |f(x)|^2 dx = \int_a^b |f(x)|^2 dx \} =
\]

Theorem 5.2. A bounded and linear operator $P$ on $H$ is a projection $\iff P = P^* = P^2$.

Proof. We suppose that $P$ is the projection onto $M$ and $P^\perp$ that one onto $M^\perp$. Then since $Pf \in M$ we have $P^2f = P(Pf) = Pf$ and since

$$\langle Pf, P^\perp g \rangle = \langle P^\perp f, Pg \rangle = 0$$

we have

$$\langle Pf, g \rangle = \frac{1}{2} \langle Pf, Pg + P^\perp g \rangle = \langle Pf, Pg \rangle = \langle Pf + P^\perp f, Pg \rangle \frac{1}{2} = \langle f, Pg \rangle$$

and so $P^2 = P$ and $P^* = P$.

The converse, let $P = P^* = P^2$. The set $M = \{ f : Pf = f \}$ is obviously a linear manifold. Prove that $M$ is a subspace.

Every vector $g \in H$ can be written as $g = Pg + (g - Pg)$ where $Pg \in M$ since $P(Pg) = P^2g = Pg$. To prove that $P$ is a projection onto $M$ we only must verify that $g - Pg \in M^\perp$. In fact if $h \in M$ it results:

$$\langle g - Pg, h \rangle = \langle g - Pg, Ph \rangle = \langle Pg - P^2g, h \rangle = \langle Pg - Pg, h \rangle = 0.$$ 

Remark 5.2. A linear operator $A : H \to H$ is said IDEMPOTENT if $A^2 = A$. From the Theorem 5.2 it follows that the projection operators are characterized by the fact to be autoadded and idempotent.

In particular, also the operator $0$ and that one $I$ are two projections.

While if $\mathcal{B}$ is a Banach algebra, the subset of all the projections doesn’t have an analogous algebraic and topological structure. For example, if $P$ is a different projection from $0$, then $\lambda P$ is a projection $\iff \lambda = 1$ or $\lambda = 0$ (if $\lambda = 1$ we obtain $P$ while if $\lambda = 0$ we obtain $0$, $\lambda \notin \{0, 1\}$, $P \neq 0 \implies \lambda P$ isn’t a projection because $(\lambda P)^2 = \lambda^2 P^2 = \lambda^2 P \neq \lambda P$).

Theorem 5.3. If $P$ is the projection onto $M$ and $P^\perp$ is the projection onto $M^\perp$, then $P^\perp = I - P$ and $M^\perp = \{ f : Pf = 0 \}$.

Proof. From the Theorem 2.9, Chapter 2, we have $P^\perp g = g - Pg = (I - P)g$. From the Theorem 5.1 we conclude that $M^\perp = \{ f : (I - P)f = f \} = \{ f : Pf = 0 \}$. □

Theorem 5.4. Let $P$ and $Q$ be two projections onto the subspaces $M$ and $N$ respectively. The following affirmations are equivalent:

\[1\] cfr. Theorem 2.9, Chapter 2.
a) \( PQ = QP \)

b) \( PQ \) is a projection

c) \( QP \) is a projection.

d) \( QP \) is a projection.

Proof. \( a) \Rightarrow b) \): From the Theorem 5.2 \( PQ \) is a projection if and only if \( PQ = (PQ)^* = (PQ)^2 \). \( (PQ)^* = Q^*P^* = QP = PQ. \)

\( (PQ)^2 = (PQ)(PQ) = (PQ)(QP) = P(QQ)P = PQ^2P = P(QP) = P(PQ) = P^2Q = PQ. \)

\( b) \Rightarrow a) \): \( PQ = (PQ)^* = Q^*P^* = QP. \)

Exchange the rule of \( PQ \), to obtain d).

\( \Box \)

Corollary 5.1. If \( PQ \) is a projection, then it’s the projection onto \( M \cap N \).

Proof. From the Theorem 5.1 we must prove that \( M \cap N = PQH \). Since \( PQ = QP \) from \( PQH \subset PH \subset M \) and \( PQH = QPH \subset QH \subset N \) it follows: \( PQH \subset M \cap N \).

On the other hand for \( f \in M \cap N \) we have \( f = Qf = PQf \) and so \( M \cap N \subset PQH. \)

\( \Box \)

Theorem 5.5. Let \( P \) and \( Q \) be two projections onto the subspaces \( M \) and \( N \), respectively. The following affirmations are equivalent:

a) \( M \perp N \)

b) \( PN = 0 \)

c) \( QM = 0 \)

d) \( PQ = 0 \)

e) \( QP = 0 \)

f) \( P + Q \) is a projection.

Proof. \( a) \Rightarrow b) \): If \( M \perp N \Rightarrow N \subset M^\perp \) and from the Theorem 5.3 \( \Rightarrow PN = 0. \)

\( a) \Rightarrow c) \): If \( M \perp N \Rightarrow M \subset N^\perp \) and from the Theorem 5.3 \( \Rightarrow QM = 0. \)

\( b) \Rightarrow d) \): \( (PQ)g = P(Qg) \in PN = 0. \)

\( c) \Rightarrow e) \): \( (QP)g = Q(Pg) \in QM = 0. \)

\( e) \Rightarrow a) \): \( \forall f \in M, \forall g \in N \) we have that \( \langle f, g \rangle = \langle Pf, Qg \rangle = \langle QPf, g \rangle = \langle 0, g \rangle = 0. \)

d) \Rightarrow a) \): \( \forall f \in M, \forall g \in N \) we have that \( \langle g, f \rangle = \langle Qg, Pf \rangle = \langle PQg, f \rangle = \langle 0, f \rangle = 0. \)

e) \Rightarrow f) \): \( (P + Q)(P + Q) = P^2 + PQ + QP + Q^2 = P + Q. \)

\( (P + Q)^* = P^* + Q^* = P + Q. \)

\( f) \Rightarrow d), e) \): \( (P + Q)^2 = P^2 + PQ + QP + Q^2 = P + PQ + QP + Q = P + Q, \)

from this one it follows \( PQ + QP = 0. \) \( \forall f \in H \) is \( (PQ + QP)f = PQf + QPf = 0 \) and so \( PQf = -QfP. \) But \( PQf \in M \) and \( -QfP \in N \) and since the two vectors are opposite, they belong to \( M \cap N \).

Now, we will prove that \( M \cap N = 0. \) Let \( f \in M \cap N \) then we have \( 0 = (PQ + QP)f = P(Qf) + Q(Pf) = Pf + Qf = f + f = 2f \) and so \( f = 0. \) This one implies that \( PQ = 0 \) and \( QP = 0. \)

\( \Box \)

Corollary 5.2. If \( P + Q \) is a projection, then it’s the projection onto \( M + N. \)
Proof. \( M + N \) is a subspace since \( M \perp N \). For \( f \in M + N \) we have \( f = Pf + Qf = (P + Q)f \) and conversely this equation implies \( f \in M + N \). So \( M + N = \{ f : (P + Q)f = f \} \).

**Remark 5.3.** By virtue of the equality between a), d) and e) of the previous theorem the projections \( P \) and \( Q \) are said orthogonal, \( P \perp Q \), if \( PQ = 0 \) or, equivalently, \( QP = 0 \).

If \( (P_k)_{k=1}^n \) is a sequence of mutually orthogonal projections onto the subspaces \( M_k \) respectively, then we prove by induction that \( \sum_{k=1}^n P_k \) is the projection \( \sum_{k=1}^n M_k \).

**Theorem 5.6.** Let \( P \) and \( Q \) be two projections onto the subspaces \( M \) and \( N \) respectively. The following affirmations are equivalent:

a) \( M \subset N \)

b) \( QP = P \)

c) \( PQ = P \)

d) \( Q - P \) is a projection

e) \( \langle (Q - P)f, f \rangle \geq 0, \forall f \in H \)

f) \( \|Pf\| \leq \|Qf\|, \forall f \in H \).

Proof. For exercise. \( \square \)

**Corollary 5.3.** If \( Q - P \) is a projection, then it’s the projection onto \( N - M = N \cap M^\perp \).

Proof. \( Q - P = Q(I - P) \) is the projection onto \( N \cap M^\perp \) from the Corollary 5.1 and the Theorem 5.3. \( \square \)
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