FUNCTIONAL ANALYSIS
BANACH SPACES*

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Don’t worry about your difficulties in mathematics;
I can assure you that mine are still greater!
(Albert Einstein, physicist).

In mathematics, our role is more of servant than of master.
(Charles Hermite, mathematician).
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CHAPTER 1

BANACH SPACES

1. Linear normed spaces

(Cfr. [1] Hilbert Spaces, § 1.2) We remember that \((X, \| \cdot \|)\) is a normed space if:

\[ N_1 \] \( \| x \| \geq 0, \forall x \in X \)
\[ N_2 \] \( \| x \| = 0 \iff x = 0 \)
\[ N_3 \] \( \| \alpha x \| = |\alpha| \| x \|, \forall x \in X, \forall \alpha \in K = \mathbb{R}, \mathbb{C} \)
\[ N_4 \] \( \| x + y \| \leq \| x \| + \| y \|, \forall x, y \in X \)

and that \( d(x, y) = \| x - y \| \) is called metric induced by the norm.

We also remember that the norm is a continuous map and that \( N_4 \) implies

\[ |\| y \| - \| x \| | \leq \| y - x \|. \]

A normed space is a vector space with a metric defined by a norm which is a
generalization of the length of a vector in the plane and in the space.

A Banach space is a normed space that is a complete metric space.

A normed space has a completion (unique) which is a Banach space.

In a Banach space we can define and use the series.

(Cfr. [1] Hilbert Spaces, § 3.1) A map of a normed space \( X \) in a normed space \( Y \) is called an operator; if \( Y = \mathbb{R} o Y = \mathbb{C} \) it is called functional. Linear and bounded operators and linear and bounded functionals are particularly important. In fact a linear operator is continuous if and only if it’s bounded (cfr. [1] Hilbert Space, Theorem 1.5, § 3.1).

We remember that the vector space \( X = C[0, 2\pi] \) normed by:

\[ \| x \| = \sup_{t \in [0, 2\pi]} |x(t)| \]

is a Banach space, but not a Hilbert space (cfr. [1] Hilbert Space, Example 2.1, § 1.2).

**Lemma 1.1** (Translation-invariant). A metric \( d \) over a vector space is induced by a norm \( \iff \) it satisfies the following properties:

a) \( d(x + a, y + a) = d(x, y) \)

b) \( d(\alpha x, \alpha y) = |\alpha| d(x, y) \)
Proof. \( \Rightarrow: \) 

\[
d(x + a, y + a) = \|x + a - (y + a)\| = \|x - y\| = d(x, y),
\]

\[
d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha|\|x - y\| = |\alpha|d(x, y).
\]

\( \Leftarrow: \) We define the norm as \( \|x\| = d(x, 0). \) \( N_1 \) and \( N_2 \) are obviously verified. \( N_3 \) follows from b). While \( N_4 \) is a consequence of a) (united with the triangle inequality), in fact it follows from

\[
\|x + y\| = d(x + y, 0) \leq d(x, 0) + d(x + y, x) = d(x, 0) + d(y, 0) = \|x\| + \|y\|.
\]

Every normed space is metric, but generally the convers is false, as the following one shows:

**Example 1.1.** Space \( S \) - It consists of the set of all the sequences of complex numbers and the metric \( d \) is defined, for each \( x = (\xi_j) \) and \( y = (\eta_j) \) by:

\[
d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}.
\]

Let’s see that \( d \) verifies the triangle inequality.

Let \( f : \mathbb{R}_0^+ \to \mathbb{R} \) be defined by:

\[
f(t) = t \frac{t}{1 + t}
\]

since \( f \) is monotone creasing hence:

\[
f'(t) = \frac{1}{(1 + t)^2} > 0.
\]

Therefore:

\[
|a + b| \leq |a| + |b| \implies f(|a + b|) \leq f(|a| + |b|).
\]

Given \( t = |a + b| \), it results:

\[
\frac{|a + b|}{1 + |a + b|} \leq \frac{|a| + |b|}{1 + |a + b|} = \frac{|a|}{1 + |a| + |b|} + \frac{|b|}{1 + |a| + |b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|}.
\]

Therefore, letting \( a = \xi_j - \zeta_j \) and \( b = \zeta_j - \eta_j \), \( z = (\zeta_j) \), we have:

\[
\frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}
\]

Then multiplying both sides by \( \frac{1}{2^j} \) and adding up from 1 to \( \infty \) we obtain the triangle inequality.

Check by exercise that \( d \) is a metric and see that it can’t be induced by a norm (In fact it doesn’t prove the previous lemma, the only hypothesis b) isn’t verified).
1. LINEAR NORMED SPACES

DEF. 1.1. A SET $A$ of a vector space $X$ is called CONVEX if $\forall x, y \in A$ it has:

$$M = \{ z \in X : z = \alpha x + (1 - \alpha) y : 0 \leq \alpha \leq 1 \} \subset A.$$  

$M$ is called CLOSED SEGMENT with BORDER POINTS $x$ and $y$ and every other POINT is called INNER.

THEOREM 1.1. The closed balls $B(a, r) = \{ x \in X : \| a - x \| \leq r \}$ of a normed space $X$ are convex.

Proof. For every $x, y \in B(a, r)$ we have that $(0 \leq \alpha \leq 1)$:

$$\|a - \alpha x - (1 - \alpha) y\| = \|\alpha a + (1 - \alpha) a - \alpha x + (1 - \alpha) y\| \leq \alpha \|a - x\| + (1 - \alpha) \|a - y\| \leq \alpha r + (1 - \alpha) r = r.$$  

REMARK 1.1. A subspace of a vector space is obviously convex.

THEOREM 1.2 (Subspace of a Banach space). A linear manifold $Y$ of a Banach space $X$ is complete $\iff Y$ is a subspace.

Proof. It’s banal because the completeness is equivalent to the closure. $\square$

Shortly, we remember that $(x_n)_n$ converges to $x$ on normed space $X$ therefore

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$  

DEF. 1.2. A NORM $\| \cdot \|$ of a vector space $X$ is called EQUIVALENT TO A NORM $\| \cdot \|_0$ of $X$ if there exist two positive numbers $a$ and $b$ such that $\forall x \in X$ it results:

$$a \|x\|_0 \leq \|x\| \leq b \|x\|_0$$

REMARK 1.2. The concept of equivalent norms is motivated by the fact that two equivalent norms define the same topology on $X$.

This follows from (1) and by the fact that every nonempty open set of a metric space is an union of open balls.

For exercise, the details and the proof that the Cauchy sequence in $(X, \| \cdot \|)$ and $(X, \| \cdot \|_0)$ are the same.

REMARK 1.3. Generally, lacking the scalar product in Banach spaces it isn’t possible to define orthogonal vectors; moreover, if a vector is unitary in a given norm, generally, it isn’t unitary in an equivalent norm and then a base is whatever system of linearly independent vectors that generates the space.

LEMMA 1.2 (Linear combinations lemma). Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space $X$. Then there exists a number $k > 0$ such that for every choice of scalars $\alpha_1, \ldots, \alpha_n$ it results:

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq k(|\alpha_1| + \ldots + |\alpha_n|).$$
1. BANACH SPACES

Proof. Let \( s = |\alpha_1| + \ldots + |\alpha_n| \). If \( s = 0 \) then \( \alpha_j = 0 \) for \( j = 1, \ldots, n \) therefore (2) is held for every \( k > 0 \).

If \( s > 0 \) then (2) is equivalent to the inequality that we obtain dividing the (2) by \( s \) and letting \( \beta_j = \frac{\alpha_j}{s}, j = 1, \ldots, n \), we have:

\[
\|\beta_1 x_1 + \ldots + \beta_n x_n\| \geq k.
\]

Therefore it’s sufficient to prove the existence of \( k > 0 \) such that (3) is satisfied for every \( n \)-ple of scalars \( \beta_1, \ldots, \beta_n \) with \( \sum_{j=1}^{n} |\beta_j| = 1 \).

Suppose that this is false. Then there exists a sequence \((y_m)_m\) of vectors:

\[
y_m = \beta_1^{(m)} x_1 + \ldots + \beta_n^{(m)} x_n, \quad \sum_{j=1}^{n} |\beta_j^{(m)}| = 1
\]

such that:

\[
\|y_m\| \to 0 \quad \text{for} \quad m \to \infty.
\]

Since \( \sum_{j=1}^{n} |\beta_j^{(m)}| = 1 \) it follows that \( |\beta_j^{(m)}| \leq 1 \) for \( j = 1, \ldots, n \). Hence for every fixed \( j \), the sequence \((\beta_j^{(m)})_m = (\beta_j^{(1)}, \beta_j^{(2)}, \ldots)\) is bounded. Therefore, from Bolzano-Weierstrass theorem \((\beta_1^{(m)})_m\) has a convergent subsequence, let say, to \( \beta_1 \). Denote by \((y_{1,m})_m\) the subsequence corresponding to \((y_m)_m\). Analogously, the sequence \((y_{1,m})_m\) has a subsequence \((y_{2,m})_m\), hence the corresponding subsequence of scalars \((\beta_2^{(m)})_m\) converges, let say, to \( \beta_2 \).

Proceeding in this way at step \( n \)th we obtain a subsequence \((y_{n,m})_m = (y_{n,1}, y_{n,2}, \ldots)\) of \((y_m)_m\) whose terms are of the form:

\[
y_{n,m} = \sum_{j=1}^{n} \gamma_j^{(m)} x_j, \quad \sum_{j=1}^{n} |\gamma_j^{(m)}| = 1, \quad \text{where} \quad \gamma_j^{(m)} \to \beta_j \quad \text{for} \quad m \to \infty.
\]

Therefore for \( m \to \infty \) it follows:

\[
y_{n,m} \to y = \sum_{j=1}^{n} \beta_j x_j \quad \text{where} \quad \sum_{j=1}^{n} |\beta_j| = 1
\]

so that not all \( \beta_j \) can be null.

Since \( \{x_1, \ldots, x_n\} \) is a linearly independent set then we must have \( y \neq 0 \). But \( y_{n,m} \to y \implies \|y_{n,m}\| \to \|y\| \). Since \( \|y_m\| \to 0 \) and \((y_{n,m})\) is its subsequence we must have \( \|y_{n,m}\| \to 0 \). Hence \( \|y\| = 0 \) so that \( y = 0 \). This is in contrast with \( y \neq 0 \).

**Theorem 1.3 (Completeness theorem).** Every finite dimensional linear manifold \( Y \) of a normed space \( X \) is complete.

**In particular:** Every finite dimensional normed space is complete.
Proof. Let \((y_m)_m\) be a Cauchy sequence in \(Y\). Let \(\dim Y = n\) and let \(\{e_1, \ldots, e_n\}\) be a base of \(Y\). Then \(y_m = \alpha_1^{(m)} e_1 + \ldots + \alpha_n^{(m)} e_n\). Since \((y_m)_m\) is a Cauchy sequence:

\[
\forall \varepsilon > 0 \exists N = N(\varepsilon) \text{ such that } \forall m, j > N \implies \|y_m - y_j\| < \varepsilon.
\]

From Lemma 1.2 there exists \(k > 0\) such that:

\[
\varepsilon > \|y_m - y_j\| = \left\| \sum_{i=1}^{n} [\alpha_i^{(m)} - \alpha_i^{(j)}] e_i \right\| \geq k \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(j)}| \text{ for } m, j > N.
\]

Dividing by \(k\) it results:

\[
|\alpha_i^{(m)} - \alpha_i^{(j)}| \leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(j)}| < \frac{\varepsilon}{k}, m, j > N.
\]

This proves that \((\alpha_i^{(m)})_m, i = 1, \ldots, n\), is Cauchy sequence in \(\mathbb{R}\) (or \(\mathbb{C}\)) and so it converges, let say, to \(\alpha_i\).

Let \(y = \alpha_1 e_1 + \ldots + \alpha_n e_n\). Obviously \(y \in Y\), moreover:

\[
\|y_m - y\| = \left\| \sum_{i=1}^{n} [\alpha_i^{(m)} - \alpha_i] e_i \right\| \leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i| \|e_i\|.
\]

But \(\alpha_i^{(m)} \to \alpha_i\). Hence \(\|y_m - y\| \to 0\) so \(y_m \to y\).

**Theorem 1.4 (Closure theorem).** Every linear manifold \(Y\) of a normed space \(X\), that is finite dimensional, is a subspace.

Proof. It follows from the Theorem 1.3 and from the necessary part of the Theorem 1.2.

**Theorem 1.5 (Equivalent norms theorem).** Let \(X\) be a finite dimensional vector space. Then every norm \(\|\cdot\|\) is equivalent to every other norm \(\|\cdot\|_0\).

Proof. Let \(\dim X = n\) and let \(\{e_1, \ldots, e_n\}\) be a base for \(X\). Then for every \(x \in X\) we can write in a single way

\[
x = \alpha_1 e_1 + \ldots + \alpha_n e_n.
\]

From Lemma 1.2 there exists \(k > 0\) such that:

\[
\|x\| > k(|\alpha_1| + \ldots + |\alpha_n|).
\]

From the inequality:

\[
\|x\|_0 \leq \sum_{j=1}^{n} |\alpha_j| \|e_j\|_0 \leq M \sum_{j=1}^{n} |\alpha_j|
\]
it follows that:
\[ a \|x\|_0 \leq \|x\| \quad \text{where} \quad a = \frac{k}{M} > 0. \]

The proof is completed changing the role between \( \|\cdot\| \) and \( \|\cdot\|_0 \).

**Lemma 1.3** (Compactness lemma). A compact subset \( M \) of a metric space is bounded and closed.

**Proof.** For exercise.

**Remark 1.4.** The converse of this lemma is on the whole false. In fact let \( (e_n)_n \subset l_2 \) where \( e_n = (\delta_{n,j})_{j=1}^\infty \). This sequence is bounded since \( \|e_n\| = 1 \) for \( n \geq 1 \). Its terms constitute a set of points that is closed because it hasn’t accumulation points. For the same reason this set isn’t compact.

**Theorem 1.6.** In a finite dimensional normed space every subset \( M \subset X \) is compact \( \iff \) it’s bounded and closed.

**Proof.** The necessary part follows from Lemma 1.3.

It remains to prove the sufficient part. Let \( \dim X = n \) and let \( \{e_1, \ldots, e_n\} \) be a base for \( X \). Let \( M \) bounded and closed. Moreover let \( (x_m)_m \subset M \). Then:

\[ x_m = \alpha_1^{(m)} e_1 + \ldots + \alpha_n^{(m)} e_n. \]

Since \( M \) is bounded it follows: \( \|x_m\| \leq N \) for every \( m \) and from Lemma 1.2 there exists \( k > 0 \) such that:

\[ N \geq \|x_m\| = \|\sum_{j=1}^n \alpha_j^{(m)} e_j\| \geq k \sum_{j=1}^n |\alpha_j^{(m)}|. \]

Hence the sequence \( (\alpha_j^{(m)})_m \) is bounded; therefore it has an accumulation point \( \alpha_j, 1 \leq j \leq n \). Proceeding analogously to the proof of Lemma 1.2 it concludes that \( (x_m)_m \) has a subsequence \( (z_m)_m \) which converges to \( z = \sum_{j=1}^n \alpha_j e_j \). Since \( M \) is closed it follows \( z \in M \). \( \square \)

**Lemma 1.4** (Riesz lemma). Let \( Y, Z \) be two linear manifolds of a normed space \( X \) and suppose that \( Y \) is a subspace properly contained in \( Z \). Then for every \( \alpha \in ]0, 1[ \) there exists \( z \in Z \) such that \( \|z\| = 1 \) and \( \|z - y\| \geq \alpha \ \forall \ y \in Y. \)

**Proof.** Let \( v \in Z - Y \) and let \( a = \inf_{y \in Y} \|v - y\| \). Clearly \( a > 0 \) since \( Y \) is closed. Let \( \alpha \in ]0, 1[ \). In correspondence there exists \( y_0 \in Y \) such that:

\[ a \leq \|v - y_0\| \leq \frac{a}{\alpha}. \]

Let \( z = c(v - y_0) \), where \( c = \frac{1}{\|v - y_0\|} \). Then \( \|z\| = 1 \) and for every \( y \in Y \) it results:
\[ \| z - y \| = \| c(v - y_0) - y \| = c\| v - y_0 - c^{-1}y \| = c\| v - y_1 \|, \text{ where } y_1 = y_0 + c^{-1}y. \]

Therefore \( y_1 \in Y \) and so \( \| v - y_1 \| \geq a \). Using (4) we obtain:

\[ \| z - y \| = c\| v - y_1 \| \geq ca = \frac{a}{\| v - y_0 \|} \geq \frac{a\alpha}{a} = \alpha. \]

\[ \Box \]

**Theorem 1.7.** If a normed space \( X \) has the property that the closed unitary ball \( M = \{ x \in X : \| x \| \leq 1 \} \) is compact, then \( X \) has finite dimension.

Proof. Suppose that \( M \) is compact but \( \dim X = \infty \). Choose \( x_1 \in M \) with \( \| x_1 \| = 1 \). Thus \( x_1 \) generates a linear manifold \( X_1 \) of \( X \) that is closed (cfr. Theorem 1.4), and it is a proper subspace of \( X \) since \( \dim X = \infty \). From Lemma 1.4 there exists a \( x_2 \in X \) with \( \| x_2 \| = 1 \) such that \( \| x_2 - x_1 \| \geq \alpha = \frac{1}{2} \). The elements \( x_1, x_2 \) generate a two-dimensional subspace \( X_2 \) of \( X \). From Lemma 1.4 there exists a \( x_3 \in X \) with \( \| x_3 \| = 1 \) such that for every \( x \in X_2 \) we have \( \| x_3 - x \| \geq \frac{1}{2} \). In particular we have:

\[ \| x_3 - x_1 \| \geq \frac{1}{2}, \quad \| x_3 - x_2 \| \geq \frac{1}{2}. \]

Proceeding by induction we build a sequence \( (x_n)_n \subset M \) such that \( \| x_m - x_n \| \geq \frac{1}{2} \) for \( m \neq n \). Obviously \( (x_n)_n \) can’t have any convergent subsequences and this contradicts the compacteness of \( M \). \[ \Box \]

N.S.C. 1.1. For a normed space \( X \) to be finite dimensional is that the closed unitary ball \( M = \{ x \in X : \| x \| \leq 1 \} \) is compact.

Proof. The necessary parts follows from the Theorem 1.6 \((\iff)\) because \( M \) is clearly bounded and closed. The sufficient part is in the Theorem 1.7. \[ \Box \]

**Def. 1.3.** A metric space \( X \) is called **LOCALLY COMPACT** if each point of \( X \) has a compact neighbourhood.

**Theorem 1.8.**

\[ \mathbb{R}^n, n \geq 1, \text{ is locally compact.} \]

\[ \mathbb{C}^n, n \geq 1, \text{ is locally compact.} \]

**2. Linear operators**

By the definition of linear operator we refer to the one given in Hilbert spaces (cfr. [1] Hilbert Spaces, § 3.1).
Example 2.1 (Derivative). Let $X$ be the vector space of all polynomials on $[a, b]$. Define $T : X \to X$ letting:

$$Tx(t) = x'(t).$$

Prove that $T$ is linear and surjective.

Example 2.2 (Integration). Let $T : C[a, b] \to C[a, b]$ be defined by:

$$Tx(t) = \int_a^t x(\tau)d\tau, \forall t \in [a, b].$$

Prove that $T$ is linear.

Example 2.3 (Multiplication for $t$). Let $T : C[a, b] \to C[a, b]$ be defined by:

$$Tx(t) = tx(t).$$

Prove that $T$ is linear.

We indicate with $D_T$, $R_T$, $N_T$ respectively the domain, the codomain (or rank) and the null space (or kernel) of the operator $T$.

Theorem 2.1 (Rank-Nullity Theorem). Let $T$ be a linear operator on a vector space. Then:

a) The rank $R_T$ is a linear manifold.

b) If $\dim D_T = n < \infty \implies \dim R_T \leq n$.

c) The null space $N_T$ is a linear manifold.

Proof. a): Let $y_1, y_2 \in R_T$ and let $\alpha, \beta$ be two scalars. Then $y_1 = Tx_1$ and $y_2 = Tx_2$ for any $x_1, x_2 \in D_T$ and $\alpha x_1 + \beta x_2 \in D_T$ since $D_T$ is linear manifold. From the linearity of $T$ it follows:

$$T(\alpha x_1 + \beta x_2) = \alpha y_1 + \beta y_2.$$ 

Therefore $\alpha y_1 + \beta y_2 \in R_T$.

b): Let $y_1, \ldots, y_{n+1}$ be $n + 1$ elements of $R_T$. Then $y_1 = Tx_1, \ldots, y_{n+1} = Tx_{n+1}$ for any $x_1, \ldots, x_{n+1} \in D_T$. Since $\dim D_T = n$, the set $\{x_1, \ldots, x_{n+1}\}$ must be linearly dependent. Therefore:

$$\alpha_1 x_1 + \ldots + \alpha_{n+1} x_{n+1} = 0$$

for opportune scalars $\alpha_1, \ldots, \alpha_{n+1}$ not all null.

Since $T$ is linear it follows $T0 = 0$ and we have:

$$T(\alpha_1 x_1 + \ldots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \ldots + \alpha_{n+1} y_{n+1} = 0.$$
This proves that the set \( \{ y_1, \ldots, y_{n+1} \} \) is linearly dependent. Whence it follows that \( R_T \) hasn’t subsets of \( n + 1 \) elements which are linearly independent. Therefore \( \text{dim } R_T \leq n \).

c): Let \( x_1, x_2 \in N_T \) and let \( \alpha, \beta \) be two scalars. Then \( Tx_1 = Tx_2 = 0 \). Therefore:

\[
T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = 0
\]

and hence \( \alpha x_1 + \beta x_2 \in N_T \).

Remark 2.1. Remark that from b) of the previous theorem it follows that the linear operators preserve the linear dependence.

Remark 2.2. (cfr. [1] Hilbert Space, § 3.2) The following theorems still hold for the normed spaces: the set \( B \) of all operators on \( X \) is a normed algebra (the completeness of \( X \Rightarrow \) the completeness of \( B \)), the inverse operator theorem.

Define as in Hilbert space the linear operators which are bounded and continuous (cfr. [1] Hilbert Space, § 3.1).

Example 2.4 (Derivative). Let \( X \) be a normed space of all polynomials on \( I = [0, 1] \) with norm:

\[
\|x\| = \max \{|x(t)| : t \in I\}.
\]

A derivative operator \( T : X \to X \) is defined by:

\[
Tx(t) = x'(t).
\]

This operator is linear but not bounded. In fact let \( x_n(t) = t^n, n \in \mathbb{N} \). Then \( \|x_n\| = 1 \) and \( Tx_n(t) = x'_n(t) = nt^{n-1} \). Therefore:

\[
\|Tx_n\| = n \quad \text{and} \quad \frac{\|Tx_n\|}{\|x_n\|} = n.
\]

Since \( n \in \mathbb{N} \) is arbitrary, this proves that \( T \) isn’t bounded.

Example 2.5 (Integral). We can define an integral operator \( T : C[0, 1] \to C[0, 1] \) letting \( y = Tx \) where:

\[
y(t) = \int_0^t K(t, \tau)x(\tau)d\tau
\]

where \( K \) is an assigned function, called the kernel of \( T \), that we suppose it’s continuous on the closed square \( I \times I, I = [0, 1] \). This operator is linear.

Prove that this operator is bounded.

Since \( K \) is continuous on \( I \times I \) it follows that \( K \) is bounded, and hence:

\[
|K(t, \tau)| \leq K_0, \forall (t, \tau) \in I \times I.
\]

Moreover \( x(t) \leq \max \{|x(t)| : t \in I\} = \|x\| \).
So it results:

\[ \|y\| = \|Tx\| = \max_{t \in I} \left| \int_0^t K(t, \tau)x(\tau)d\tau \right| \leq \max_{t \in I} \int_0^t |K(t, \tau)||x(\tau)|d\tau \leq K_0\|x\| \]

Therefore \( T \) is bounded.

**Example 2.6 (Matrix).** Remember that a matrix \( A = [a_{jk}] \) having \( m \) rows and \( n \) columns defines an operator \( T : \mathbb{C}^n \to \mathbb{C}^m \) which is bounded and linear \( (\|T\| \leq (\sum_{j=1}^m \sum_{k=1}^n |a_{jk}|^2)^{\frac{1}{2}}) \).

**Theorem 2.2 (Finite dimension).** If \( X \) is a normed space and \( \dim X < \infty \), then every linear operator is bounded.

**Proof.** Let \( \dim X = n \) and let \( \{e_1, \ldots, e_n\} \) be a base for \( X \). Let \( x = \sum_{i=1}^n \xi e_i \) and let \( T : X \to X \) be a linear operator. Since \( T \) is linear, it results:

\[ \|Tx\| = \left\| \sum_{i=1}^n \xi_i Te_i \right\| \leq \sum_{i=1}^n |\xi_i||Te_i| \leq \max_k\|Te_k\| \sum_{i=1}^n |\xi_i|. \]

By virtue of linear combinations lemma we obtain:

\[ \sum_{i=1}^n |\xi_i| \leq \frac{1}{k} \| \sum_{i=1}^n \xi_i e_i \| = \frac{1}{k} \|x\|. \]

Therefore:

\[ \|Tx\| \leq M\|x\| \quad \text{where} \quad M = \frac{1}{k} \max_k\|Te_k\|. \]

**Remark 2.3.** Remember that a linear operator in a normed space is continuous if and only if is bounded (cfr. [1] Hilbert Space, Theorem 1.5, § 3.1).

**Remark 2.4.** If a linear operator is continuous in a point \( x_0 \in D_T \) it is continuous in \( 0 \) because the continuity in \( x_0 \) tells that \( \forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall x \in D_T \) with \( \|x - x_0\| < \delta_\varepsilon \implies \|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon \) therefore letting \( y = x - x_0 \) we have \( \forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall y \in D_T \) with \( \|y\| < \delta_\varepsilon \implies \|Ty\| < \varepsilon \).

**Theorem 2.3.** Let \( T : D_T \to Y, D_T \subset X \), let \( X \) and \( Y \) be normed spaces, a linear operator. Then if \( T \) is continuous in a point, \( T \) is continuous.

**Proof.** (or cfr. [1] Hilbert Space, Theorem 1.4, § 3.1) From Remark 2.4 it follows that \( T \) is continuous in \( 0 \). Suppose that \( T \) isn’t bounded \( \implies \forall n \in \mathbb{N} \exists x_n \in X \) with \( \|x_n\| \leq 1 \) such that \( \|Tx_n\| \geq n^2 \). From the linearity \( \|T\frac{x_n}{n}\| \geq n \) but \( \|\frac{x_n}{n}\| \to 0 \) \( (x_n \text{ is } \|x_n\| \leq 1) \) and being \( T \) continuous in \( 0 \) it follows that \( T\frac{x_n}{n} \to T0 = 0 \).

The boundedness of \( T \iff \) its continuity (Remark 2.3).
Theorem 2.4. Let $T$ be a bounded and linear operator in a normed space. Then:

a) $x_n \mapsto x$ implies $Tx_n \mapsto Tx$, $x_n, x \in D_T$

b) $N_T$ is closed.

Proof. a): By exercise (cfr. [1] Hilbert Space, Theorem 1.3 $\implies$, § 3.1).

b): $\forall x \in N_T$ there exists a sequence $(x_n)_n \subset N_T$ such that $x_n \mapsto x$. Hence from a) $Tx_n \mapsto Tx$ therefore $Tx = 0$.

Remark 2.5. Also in the normed spaces let introduce the concepts of included operators and extended operators.

Theorem 2.5 (Bounded linear extension theorem). Let $T : D_T \to Y$ be a bounded and linear operator, where $D_T \subset X$ and $X$ and $Y$ are Banach spaces. Then $T$ has an extension $\overline{T} : D \to Y$ where $\overline{T}$ is a bounded and linear operator and $\|\overline{T}\| = \|T\|$.


Remark 2.6. Remark that if $X$ and $Y$ are normed spaces then the linear operator $T : X \to Y$ is bounded if and only if it transforms bounded sets of $X$ into bounded sets of $Y$.

3. Linear functionals

Def. 3.1. A linear map $\phi : X \to K$ is called LINEAR FUNCTIONAL ON $X$.

Example 3.1. The norm $\|x\| : X \to \mathbb{R}$ on a normed space $(X, \|\cdot\|)$ isn’t a linear functional.

Example 3.2. Let $a \in \mathbb{R}^3$ fixed and let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by:

$$f(x) = xa = \xi_1a_1 + \xi_2a_2 + \xi_3a_3.$$ 

Then $f$ is linear and moreover it is bounded. In fact:

$$|f(x)| = |xa| \leq \|x\| \|a\|$$

so that:

$$\|f\| \leq \|a\|.$$ 

But taken $x = a$ it results:

$$\|f\| \geq \frac{|f(a)|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|.$$ 

Therefore: $\|f\| = \|a\|$. 

Example 3.3 (Integral). Let \( X = C[a, b] \) and let \( f : C[a, b] \to \mathbb{R} \) be defined by:

\[
f(x) = \int_{a}^{b} x(t)dt.
\]

\( f \) is linear. Moreover, letting \( I = [a, b] \), and remembering that the norm on \( C[a, b] \) is that one uniform, we have:

\[
|f(x)| = \left| \int_{a}^{b} x(t)dt \right| \leq (b - a) \max_{t \in I} |x(t)| = (b - a) \|x\|.
\]

Therefore \( \|f\| \leq b - a \).

Let now \( x = x_0 = 1 \) therefore \( \|x_0\| = 1 \) and

\[
\|f\| \geq \frac{|f(x_0)|}{\|x_0\|} = \frac{|f(x_0)|}{1} = \int_{a}^{b} dt = b - a.
\]

Hence: \( \|f\| = b - a \).

Example 3.4. Another important functional on \( C[a, b] \) is obtained by choosing \( t_0 \in I = [a, b] \) and letting \( f_1(x) = x(t_0), \forall x \in C[a, b] \). \( f_1 \) is linear. Moreover:

\[
|f_1(x)| = |x(t_0)| \leq \|x\| \quad \text{and hence} \quad \|f_1\| \leq 1.
\]

Taken \( x_0 = 1 \) it results \( \|x_0\| = 1 \) and \( \|f_1\| \geq |f_1(x_0)| = 1 \).

Therefore: \( \|f_1\| = 1 \).

The set of all linear functionals defined on a vector space \( X \) give rise to a vector space, denoted by \( X^* \), called ALGEBRIC DUAL SPACE OF \( X \). Its algebraic operations of vector space are defined in a natural way as follows:

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{Sum}
\]

\[
(\alpha f)(x) = \alpha f(x) \quad \text{Scalar product} \times \text{vector}.
\]

We can consider the algebraic dual \( (X^*)^* \) of \( X^* \) whose elements are the linear functionals on \( X^* \). We denote \( (X^*)^* \) with \( X^{**} \) which is called THE ALGEBRIC DUAL SPACE (OR BIDUAL) OF \( X \).

We consider the following important relation between \( X \) and \( X^{**} \).

We can obtain a \( g \in X^{**} \), that is a linear functional on \( X^* \) by choosing a \( x \in X \) and letting for every \( f \in X^* \):

\[
g(f) = g_x(f) = f(x).
\]

Then \( g_x \) is linear, being:

\[
g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha g_x(f_1) + \beta g_x(f_2).
\]

Hence \( g_x \in X^{**} \), from definition of \( X^{**} \).

For every \( x \in X \) there exists the correspondent \( g_x \in X^{**} \). Thus define:
4. Linear operators and functionals on finite dimensional spaces

Let $T : X \to Y$ be a linear operator and $X$ and $Y$ two finite dimensional vector spaces over the same field $K$. Let $B_X = \{e_1, \ldots, e_n\}$ and $B_Y = \{b_1, \ldots, b_m\}$ be two bases for $X$ and $Y$ respectively.

For every $x \in X$ it has a unique representation:

(5) \[ x = \xi_1 e_1 + \ldots + \xi_n e_n. \]

Since $T$ linear it results:

(6) \[ y = Tx = T(\sum_{k=1}^{n} \xi_k e_k) = \sum_{k=1}^{n} \xi_k T e_k. \]

Since the representation (5) is unique we can say that:

$T$ is univocally determinate if the images $y_k = T e_k$ of $n$ vectors of the base $B_X$ are given.

Since $y$ and $y_k = T e_k$ are in $Y$, they have a unique representation:

(7) \[ y = \sum_{j=1}^{m} \eta_j b_j \]

(8) \[ T e_k = \sum_{j=1}^{m} \alpha_{j,k} b_j. \]

From (6) it follows:

\[ y = \sum_{j=1}^{m} \eta_j b_j = \sum_{k=1}^{n} \xi_k T e_k = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{m} \alpha_{j,k} b_j = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{j,k} \xi_k) b_j. \]
Since the $b_j$, $j = 1, \ldots, m$, are linearly independent it follows:

$$\eta_j = \sum_{k=1}^{n} \alpha_{j,k} \xi_k, \quad j = 1, \ldots, m.$$  \hfill (9)

The coefficients in (9) form a matrix $T_{B_X B_Y} = [\alpha_{jk}]$ of $m$ rows and $n$ columns that is univocally determinate by the operator $T$ if the elements of the bases $B_X$ and $B_Y$ are given in a fixed order.

So we say that the matrix $T_{B_X B_Y}$ represents the operator $T$ with respect to those bases.

Introducing the column vector $\bar{x} = (\xi_k)$ and $\bar{y} = (\eta_j)$ we can write

$$\bar{y} = T_{B_X B_Y} \bar{x}.$$  \hfill (9')

Analogously (8) can be written:

$$Te = T_{B_X B_Y} b,$$  \hfill (8')

where $Te$ is the column vector $(Te_k)$ and $b$ is the column vector $(b_j)$.

The converse, we already know that every matrix of $m$ rows and $n$ columns determines a linear operator represented with respect to the bases of $X$ and $Y$.

Now, let dim $X = n$, let $\{e_1, \ldots, e_n\}$ be a base for $X$ and $f : X \to K$ linear functional on $X$. Then for every $x = \sum_{j=1}^{n} \xi_j e_j \in X$ we have:

$$f(x) = f(\sum_{j=1}^{n} \xi_j e_j) = \sum_{j=1}^{n} \xi_j f(e_j) = \sum_{j=1}^{n} \xi_j \alpha_j$$  \hfill (10)

where

$$\alpha_j = f(e_j), \quad j = 1, \ldots, n$$  \hfill (11)

and $f$ is univocally determinate by its values $\alpha_j$.

The converse, every $n$th $\alpha_1, \ldots, \alpha_n$ of scalars determine a linear functional on $X$. In particular if we take the $n$th: $(\delta_{jk})_{k=1}^{n}$, $j = 1, \ldots, n$, they determine $n$ functionals $f_1, \ldots, f_n$, such that:

$$f_k(e_j) = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$  \hfill (12)

$\{f_1, \ldots, f_n\}$ is called the DUAL BASE of $\{e_1, \ldots, e_n\}$ for $X$.

This is justified by the following one:
4. LINEAR OPERATORS AND FUNCTIONALS ON FINITE DIMENSIONAL SPACES

**Theorem 4.1 (Dimension of $X^*$ theorem).** Let $X$ be a vector space with $\dim X = n$ and let $E = \{e_1, \ldots, e_n\}$ be a base for $X$. Then $F = \{f_1, \ldots, f_n\}$ given by (12) is a base for the algebraic dual $X^*$ of $X$.

Hence $\dim X^* = \dim X = n$.

Proof. $F$ is linearly independent since:

\[
\sum_{k=1}^{n} \beta_k f_k(x) = 0 \quad \text{with} \quad x = e_j \quad \text{gives} \quad \sum_{k=1}^{n} \beta_k f_k(e_j) = \sum_{k=1}^{n} \beta_k \delta_{jk} = \beta_j = 0
\]

therefore all coefficients $\beta_k$ of (13) are zero.

Let $f \in X^*$. Let as in (11) $f(e_j) = \alpha_j$; then from (10) we have:

\[
f(x) = \sum_{j=1}^{n} \xi_j \alpha_j, \quad \forall \ x \in X.
\]

On the other hand from (12) it results:

\[
f_j(x) = f_j(\xi_1 e_1 + \ldots + \xi_n e_n) = \xi_j.
\]

Therefore:

\[
f(x) = \sum_{j=1}^{n} \alpha_j f_j(x).
\]

\[\square\]

**Lemma 4.1 (Zero vector lemma).** Let $X$ be a finite dimensional vector space.

If $x_0 \in X$ is such that $f(x_0) = 0$, $\forall \ f \in X^* \implies x_0 = 0$.

Proof. Let $x_0 = \alpha_{10}e_1 + \ldots + \alpha_{n0}e_n$. We use the dual base of $\{e_1, \ldots, e_n\}$ for $X$.

$\forall \ k = 1, \ldots, n \ f_k(x_0) = f_k(\alpha_{10}e_1 + \ldots + \alpha_{k0}e_k + \ldots + \alpha_{n0}e_n) = \alpha_{10}f_k(e_1) + \ldots + \alpha_{k0}f_k(e_k) + \ldots + \alpha_{n0}f_k(e_n) = 0$, and hence $\alpha_{k0} = 0$. Then $x_0 = 0$ \[\square\]

**Lemma 4.2.** Let $X$ and $Y$ vector spaces over $K$ and let $T: D_T \to Y$ be a linear operator with $D_T \subset X$ and $R_T \subset Y$. Then:

a) $T^{-1}: R_T \to D_T$ exists $\iff Tx = 0 \implies x = 0$

b) If there exists $T^{-1}$, it’s a linear operator

c) If $\dim D_T = n < \infty$ and there exists $T^{-1}$ then $\dim R_T = \dim D_T$.

Proof. a): If there exists $T^{-1}$, $T$ must be a bijection of $D_T \to R_T$ and therefore being $T0 = 0$ 0 is the unique element which has as image 0. The converse, if 0 is the unique element which has as image 0, it holds $\forall x, y \in D_T$ with $x \neq y \implies Tx \neq Ty$ because if it had been $Tx = Ty$, it would have followed $Tx - Ty = T(x - y) = 0$ with $x - y \neq 0$. Then $T$ is injective and surjective ($T: D_T \to R_T$) and hence invertible.
b): \( T^{-1} : R_D \to R_T, \forall z_0, z_1 \in R_D \) and \( \forall \alpha_0, \alpha_1 \in K \) \( \exists! x_0 \in D_T : T x_0 = z_0, \exists! x_1 \in D_T : T x_1 = z_1 \) therefore \( T^{-1}(\alpha_0 z_0 + \alpha_1 z_1) = T^{-1}(\alpha_0 T x_0 + \alpha_1 T x_1) = T^{-1}(T(\alpha_0 x_0) + T(\alpha_1 x_1)) = T^{-1}(\alpha_0 x_0 + \alpha_1 x_1) = \alpha_0 x_0 + \alpha_1 x_1 = \alpha_0 T^{-1} z_0 + \alpha_1 T^{-1} z_1. \)

c): Being \( T \) a linear operator from the Theorem 2.1 b) \( \dim R_T \leq \dim D_T \) and since \( T^{-1} \) is a linear operator it has \( \dim D_T \leq \dim R_T. \) It follows that \( \dim D_T = \dim R_T. \)

THEOREM 4.2 (Algebraic reflexivity theorem). A finite dimensional vector space \( X \) is algebraically reflexive.

Proof. The canonical map \( C : X \to X^{**} \) is linear. \( C x_0 = 0 \implies (C x_0)(f) = g_{x_0}(f) = f(x_0) = 0, \forall f \in X^*. \) This implies \( x_0 = 0 \) from Lemma 4.1. Therefore from Lemma 4.2 there exists \( C^{-1} : R_C \to X \) and \( \dim R_C = \dim X. \)

From the Theorem 4.1 we have \( \dim X^{**} = \dim X^* = \dim X. \)

Hence: \( \dim R_C = \dim X^{**}. \)

But then \( R_C = X^{**} \) because from the Theorem 2.1 \( R_C \) is a vector subspace of \( X^{**} \) and a proper subspace of \( X^{**} \), it has minor dimension of \( \dim X^{**}. \)

5. Normed spaces of operators - Dual space

Let \( X \) and \( Y \) be two normed spaces over \( K \). Indicate with \( B(X, Y) \) a set of all bounded and linear operators \( T : X \to Y \). \( B(X, Y) \) is obviously a vector space with respect to the usual operations \( T_1 + T_2 \) and \( \alpha T. \)

THEOREM 5.1. \( B(X, Y) \) is a normed space with respect to the norm defined by:

\[
\|T\| = \sup_{x \in X, \|x\| \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in X, \|x\| = 1} \|Tx\|
\]

Proof. For exercise.

THEOREM 5.2. If \( Y \) is a Banach space then \( B(X, Y) \) is a Banach space.

Proof. Let \( (T_n)_n \subset B(X, Y) \) be a Cauchy sequence. For every \( \varepsilon > 0 \) there exists \( N = N(\varepsilon) \) such that:

\[
\|T_n - T_m\| < \varepsilon \text{ for every } m, n > N.
\]

For every \( x \in X \) and for every \( m, n > N \) it results:

\[
\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\| < \varepsilon \|x\|.
\]

For every fixed \( x \) and a given \( \varepsilon \) we choose in (15) \( \varepsilon = \varepsilon_x \) in a way that \( \varepsilon_x \|x\| < \varepsilon. \) Then from (15) it follows:
\[ \| T_n x - T_m x \| < \varepsilon \]

hence \((T_n x)_n\) is Cauchy sequence in \(Y\). Since \(Y\) is a Banach space it follows that \((T_n x)_n\) converges to any \(y \in Y\) that depends from \(x\). Therefore we obtain an operator \(T : X \to Y\) defined by:

\[
T x = y.
\]

The operator \(T\) is linear since

\[
\lim_{n \to \infty} T_n (\alpha x + \beta z) = \lim_{n \to \infty} (\alpha T_n x + \beta T_n z) = \alpha \lim_{n \to \infty} T_n x + \beta \lim_{n \to \infty} T_n z.
\]

From the continuity of the norm we obtain from (15) for every \(n > N\) and for every \(x \in X\)

\[
\| T_n x - T x \| = \| T_n x - \lim_{m \to \infty} T_m x \| = \lim_{m \to \infty} \| T_n x - T_m x \| \leq \varepsilon \| x \|.
\]

Therefore \((T_n - T)\) is a bounded and linear operator; hence

\[
T = T_n - (T_n - T)
\]

is bounded, whence \(T \in B(X, Y)\). From (16) it follows that

\[
\| T_n - T \| \leq \varepsilon \quad \text{for} \quad n > N
\]

and hence \(\| T_n - T \| \to 0\) that is \(T_n \to T\). \(\square\)

**Def. 5.1.** Let \(X\) be a normed space. Indicate with \(B(X, K)\) the set of all the bounded and linear functionals on \(X\). Then \(B(X, K)\) is a normed space with respect to the norm defined by:

\[
\| f \| = \sup_{x \in X, \| x \| \neq 0} \frac{| f(x) |}{\| x \|} = \sup_{x \in X, \| x \| = 1} | f(x) |
\]

\(B(X, K)\) is called **dual space** of \(X\) and we’ll denote it with \(X'\).

From the Theorem 5.2 we can enunciate the following one:

**Corollary 5.1.** The dual space \(X'\) of a normed space \(X\) is a Banach space.

**Example 5.1 (The dual of \(\mathbb{R}^n\) is \(\mathbb{R}^n\)).** We have \(\mathbb{R}^n' = \mathbb{R}^{n*}\). For every \(f \in \mathbb{R}^{n*}\) we have:

\[
f(x) = \sum_{k=1}^{n} \xi_k \alpha_k, \quad \alpha_k = f(e_k).
\]

From the Cauchy-Schwarz-Buniakowski inequality we have:
\[ |f(x)| = |\sum_{k=1}^{n} \xi_k \alpha_k| \leq \left( \sum_{k=1}^{n} \xi_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} \alpha_k^2 \right)^{\frac{1}{2}} = \|x\| \left( \sum_{k=1}^{n} \alpha_k^2 \right)^{\frac{1}{2}}. \]

Therefore:

\[ \|f\| \leq \left( \sum_{k=1}^{n} \alpha_k^2 \right)^{\frac{1}{2}}. \]

Taking \( x = (\alpha_1, \ldots, \alpha_n) \) we have that:

\[ \|f\| = \left( \sum_{k=1}^{n} \alpha_k^2 \right)^{\frac{1}{2}}. \]

This proves that the norm of \( \|f\| \) is the euclidean norm \( \varrho \):

\[ \|f\| = \|a\| \]

where \( a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \).

Thus the map of \( \mathbb{R}^n \) over \( \mathbb{R}^n \) defined by:

\[ f \mapsto a = (\alpha_k), \quad \alpha_k = f(e_k) \]

preserves the norm and since it’s linear and bijective it follows that it’s an isomorphism\(^1\).

**Example 5.2 (The spaces \( l^p \)).** Let \( p \geq 1 \) be a real number. Every element of \( l^p \) is a sequence:

\[ x = (\xi_i) = (\xi_1, \xi_2, \ldots) \]

of numbers such that:

\[ |\xi_1|^p + |\xi_2|^p + \ldots \text{ converges}. \]

So:

\[ \|x\| = \left( \sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}} \text{ is a norm.} \]

In fact, from the Hölder, Cauchy-Schwarz-Minkowski inequalities it follows the triangle inequality.

\( l^p \) is separable because the set \( M \) of all sequences \( y = (\eta_1, \ldots, \eta_n, 0, 0, \ldots) \) where \( n \) is any integer number and \( \eta_i \) are rational (real o rational complex) is countable and

\(^1\text{cfr. [1] Hilbert Space, Def. 4.3, § 2.4 and Corollary 3.2, § 3.3.} \)
wherever dense in $l^p$. The proof is analogously to that one of $l^2$ (cfr. Hilbert Spaces, Theorem 3.5, § 1.3).

Analogously still for what we have proved for $l^2$ it proves that $l^p$ is complete (cfr. [1] Hilbert Spaces, Theorem 3.3, § 1.3).

**Example 5.3 (The space $l^\infty$).** It’s clear that the set $l^\infty$ of all bounded sequences $x = (\xi_n)$ is a vector space.

The function:

$$\|x\| = \sup_{n \in \mathbb{N}} |\xi_n|, \forall x \in l^\infty$$

is a norm.

In fact for every $n \in \mathbb{N}$ it results:

$$|\xi_n + \eta_n| \leq |\xi_n| + |\eta_n| \leq \|x\| + \|y\|.$$ 

Therefore:

$$\|x + y\| \leq \|x\| + \|y\|.$$ 

Prove the remaining axioms of norm by exercise.

Analogously for what it has done for $l^p$, $1 \leq p < \infty$, it proves that $l^\infty$ is a Banach space.

But $l^\infty$ isn’t separable. In fact let $y = (\eta_1, \eta_2, \ldots)$ be a sequence constitute of elements which are or zero or one. Then $y \in l^\infty$. To $y \in l^\infty$ we associate the number $\hat{y}$ whose binary representation is:

$$\frac{\eta_1}{2} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + \ldots$$

But the interval $[0, 1]$ isn’t countable and every $\hat{y} \in [0, 1]$ has a binary representation and two different $\hat{y}$ have different binary representation. Hence there exists a set not countable of sequences of zero and one. The norm of $l^\infty$ tells that two of these sequences which aren’t equal must have distance 1. If we take the balls with centre each of these sequences and radius for example, $\frac{1}{3}$, they are pairwise disjoint and not countable. If $M$ is a dense set in $l^\infty$, every ball so done it must at least meet $M$ at a point. Hence $M$ can’t be countable and therefore $l^\infty$ isn’t separable.

**Example 5.4 (The dual of $l^1$ is $l^\infty$).** In fact: $\{e_k\}_{k=1}^\infty = \{((\delta_{kj})_{j=1}^\infty)_{k=1}^\infty\}$ is a base for $l^1$.

For every $x \in l^1$ we have a unique representation:

$$x = \sum_{k=1}^\infty \xi_k e_k.$$ 

(18)

Let $f \in l^1'$. Since $f$ is bounded and linear, it follows:
(19) \[ f(x) = \sum_{k=1}^{\infty} \xi_k \alpha_k, \quad \alpha_k = f(e_k) \]

where the \( \alpha_k \) are univocally determinated by \( f \). Still: \( \|e_k\| = 1 \) and:

(20) \[ |\alpha_k| = |f(e_k)| \leq \|f\| \|e_k\| = \|f\|, \quad \sup_k |\alpha_k| \leq \|f\|. \]

Hence \( (\alpha_k)_k \in l^\infty \).

On the other hand for every \( b = (\beta_k) \in l^\infty \) we can obtain a corresponding bounded and linear functional \( g \) on \( l^1 \). In fact we can define \( g \) on \( l^1 \) by:

\[ g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \quad \text{where} \quad x = (\xi_k) \in l^1. \]

From:

\[ |g(x)| \leq \sum_{k=1}^{\infty} |\xi_k \beta_k| \leq \sup_j |\beta_j| \sum_{k=1}^{\infty} |\xi_k| = \|x\| \sup_j |\beta_j| \]

it follows that \( g \) is bounded. But \( g \) is obviously linear and hence \( g \in l^1' \).

Now we prove that the norm of \( f \) is the norm of \( l^\infty \).

From (19) we have:

\[ |f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \alpha_k \right| \leq \sup_j |\alpha_j| \sum_{k=1}^{\infty} |\xi_k| = \|x\| \sup_j |\alpha_j|. \]

Whence:

\[ \|f\| \leq \sup_j |\alpha_j|. \]

From this one and (20) it follows

(21) \[ \|f\| = \sup_j |\alpha_j| \]

that is the norm of \( l^\infty \). Therefore (21) can be written:

\[ \|f\| = \|a\| \quad \text{where} \quad a = (\alpha_j) \in l^\infty. \]

We have so proved that the linear bijection of \( f \in l^1' \) on \( l^\infty \) defined by \( f \mapsto a \) is an isomorphism.
Example 5.5 (The dual of $l^p$, $1 < p < +\infty$, is $l^q$ where $q$ is the conjugate of $p$, that is $\frac{1}{p} + \frac{1}{q} = 1$). In fact: $\{e_k\}_{k=1}^{\infty}$, where $e_k = (\delta_{kj})_{j=1}^{\infty}$ is a base for $l^p$. Then every $x \in l^p$ has a unique representation:

$$x = \sum_{k=1}^{\infty} \xi_k e_k. $$

Let $f \in l^q'$. Since $f$ is bounded and linear it follows:

$$f(x) = \sum_{k=1}^{\infty} \xi_k \alpha_k, \quad \alpha_k = f(e_k). $$

Let $q$ the conjugate of $p$ and let $x_n = (\xi^{(n)}_k)$ such that:

$$\xi^{(n)}_k = \begin{cases} \frac{|\alpha_k|^q}{\alpha_k} & \text{if } k \leq n \text{ and } \alpha_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \alpha_k = 0 \end{cases} $$

Replacing it in (23) we obtain:

$$f(x_n) = \sum_{k=1}^{\infty} \xi^{(n)}_k \alpha_k = \sum_{k=1}^{n} |\alpha_k|^q. $$

Using (24) and by the fact that $(q-1)p = q$, we have:

$$f(x_n) \leq \|f\|\|x_n\| = \|f\| \left( \sum_{k=1}^{\infty} |\xi^{(n)}_k|^p \right)^{\frac{1}{p}} = \|f\| \left( \sum_{k=1}^{n} |\alpha_k|^{(q-1)p} \right)^{\frac{1}{p}} = \|f\| \left( \sum_{k=1}^{n} |\alpha_k|^q \right)^{\frac{1}{p}}. $$

Therefore:

$$f(x_n) = \sum_{k=1}^{n} |\alpha_k|^q \leq \|f\| \left( \sum_{k=1}^{n} |\alpha_k|^q \right)^{\frac{1}{p}}. $$

Since $1 - \frac{1}{p} = \frac{1}{q}$ we have:

$$\left( \sum_{k=1}^{n} |\alpha_k|^q \right)^{1-\frac{1}{p}} = \left( \sum_{k=1}^{n} |\alpha_k|^q \right)^{\frac{1}{q}} \leq \|f\|. $$

From the arbitrariness of $n$ it follows that:
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(25) \[ \left( \sum_{k=1}^{\infty} |\alpha_k|^q \right)^{\frac{1}{q}} \leq \|f\| . \]

hence \((\alpha_k) \in l^q\).

Conversely to every \(b = (\beta_k) \in l^q\) corresponds a bounded and linear functional \(g \in l^{p'}\) defining:

\[ g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k, \quad \text{where } x = (\xi_k) \in l^p. \]

Then \(g\) is linear and its boundedness follows from the Hölder inequality. Hence \(g \in l^{p'}\).

From (23) and from the Hölder inequality, we have:

\[ |f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \alpha_k \right| \leq \left( \sum_{k=1}^{\infty} |\xi_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{\infty} |\alpha_k|^q \right)^{\frac{1}{q}} = \|x\| \left( \sum_{k=1}^{\infty} |\alpha_k|^q \right)^{\frac{1}{q}} \]

Therefore:

\[ \|f\| \leq \left( \sum_{k=1}^{\infty} |\alpha_k|^q \right)^{\frac{1}{q}} \]

and from (25) it finally follows:

(26) \[ \|f\| = \left( \sum_{k=1}^{\infty} |\alpha_k|^q \right)^{\frac{1}{q}} . \]

Therefore we can write:

\[ \|f\| = \|a\|_q, \quad \text{where } a = (\alpha_k) \in l^q \text{ and } \alpha_k = f(e_k). \]

Hence the map of \(l^{p'}\) over \(l^q\) defined by \(f \mapsto a\) is linear and bijective, and from (26) it preserves the norm, hence it’s an isomorphism.

6. Hahn-Banach theorem

**Def. 6.1.** A **SUB-LINEAR functional** is a functional \(p : X \to \mathbb{R}, \text{ with } X \text{ vector space, which is sub-additive, that is:} \)

(S.A.) \[ p(x + y) \leq p(x) + p(y), \quad \forall \ x, y \in X \]
and positively homogeneous, that is:

\[ p(\alpha x) = \alpha p(x), \quad \forall \alpha \in \mathbb{R}^+_0, \quad \forall x \in X. \]

**Remark 6.1.** Note that the norm is a sub-linear functional.

**Theorem 6.1 (Hahn-Banach Theorem).** Let \( X \) be a real vector space and let \( p \) be a sub-linear functional on \( X \). Let \( f : Z \to \mathbb{R} \) be a linear functional defined on subspace \( Z \) of \( X \), such that:

\[ f(x) \leq p(x), \quad \forall x \in Z. \]

Then \( f \) admits a linear extension \( \overline{f} : X \to \mathbb{R} \) such that:

\[ \overline{f}(x) \leq p(x), \quad \forall x \in X. \]

That is \( \overline{f} \) is a linear functional on \( X \) that verifies (27\*) and such that \( \overline{f}(x) = f(x), \quad \forall x \in Z \).

Proof. Let divide the proof in three parts.

a) Let \( E \) be the set of all linear extensions \( g \) of \( f \), such that \( g(x) \leq p(x), \quad \forall x \in D_g \). Clearly \( E \neq \emptyset \) since \( f \in E \). On \( E \) we can define a partial order\(^1\):

\[ g \preceq h \iff h \text{ is an extension of } g, \text{ that is:} \]

\[ g \preceq h \iff D_h \supset D_g \text{ and } h(x) = g(x), \quad \forall x \in D_g. \]

For every totally ordered set (or chain)\(^2\) \( C \subset E \) we define:

\[ \forall g \in C, \quad \hat{g}(x) = g(x) \text{ if } x \in D_g. \]

\( \hat{g} \) is a linear functional with domain:

\[ D_{\hat{g}} = \bigcup_{g \in C} D_g \]

which is a vector space, being \( C \) a chain. \( \hat{g} \) is well defined, in fact if \( x \in D_{g_1} \cap D_{g_2}, \)
\( g_1, g_2 \in C \), then \( g_1(x) = g_2(x) \) because \( C \) is a chain hence \( g_1 \preceq g_2 \) or \( g_2 \preceq g_1 \). Clearly \( g \preceq \hat{g} \) for every \( g \in C \). Thus \( \hat{g} \) is an upper bound of \( C \). Since \( C \subset E \) is arbitrary

\(^1\)They say that the dyad constituted by a set \( E \) and by an order relation \( \preceq \) on it is a partially ordered set if it verifies the reflexive, transitive and antisymmetric property.

\(^2\)They say that a subset \( C \subset E \) is totally ordered (or chain) if for every dyad \( g, h \in C \) they have one of the relations \( g \preceq h \) or \( h \preceq g \).
from Zorn lemma\(^1\) there exists in \(E\) a maximal element \(\mathcal{F}\). From definition of \(E\), \(\mathcal{F}\) is a linear extension of \(f\) such that:

\[(28)\quad \mathcal{F}(x) \leq p(x), \quad \forall \ x \in D_{\mathcal{F}}.\]

b) Suppose that there exists \(y_1 \in X - D_{\mathcal{F}}\) and consider the subspace \(Y_1\) of \(X\) generated by \(D_{\mathcal{F}}\) and \(y_1\) (Note that \(y_1 \neq 0\)). Hence every \(x \in Y_1\) can be written:

\[x = y + \alpha y_1, \quad y \in D_{\mathcal{F}}.\]

This representation is unique, in fact:

\[y + \alpha y_1 = \bar{y} + \beta y_1, \quad \text{with} \quad \bar{y} \in D_{\mathcal{F}} \implies y - \bar{y} = (\beta - \alpha)y_1 \quad \text{where} \quad y - \bar{y} \in D_{\mathcal{F}}\]

while \(y_1 \not\in D_{\mathcal{F}}\) so that necessarily the unique solution is \(^2\):

\[y - \bar{y} = 0 \quad \text{and} \quad \beta - \alpha = 0.\]

A functional \(g_1\) on \(Y_1\) is defined by:

\[(29)\quad g_1(y + \alpha y_1) = \mathcal{F}(y) + \alpha c\]

where \(c\) is a real number. \(g_1\) is obviously linear. Moreover for \(\alpha = 0\) we have:

\[g_1(y) = \mathcal{F}(y).\]

Hence \(g_1\) is a proper extension of \(\mathcal{F}\), that is an extension such that \(D_{\mathcal{F}}\) is a proper subset of \(D_{g_1}\).

If we prove that \(g_1 \in E\), letting see that:

\[(30)\quad g_1(x) \leq p(x), \quad \forall \ x \in D_{g_1}\]

we contradict the maximality of \(\mathcal{F}\), so that \(D_{\mathcal{F}} = X\).

c) Finally we must prove that \(g_1\) with an opportune \(c\) in \(\mathbb{R}\) verifies (30).

Let \(y, z \in D_{\mathcal{F}}\). From (28) and (S.A.) we obtain:

\[\mathcal{F}(y) - \mathcal{F}(z) = \mathcal{F}(y - z) \leq p(y - z) = p(y + y_1 - y_1 - z) \leq p(y + y_1) + p(-y_1 - z).\]

Whence we have:

\(^1\)ZORN LEMMA: If \(E\) is a partially ordered set and every chain \(C\) of \(E\) is superiorily bounded then \(E\) has a maximal element.

\(^2\)\(D_{\mathcal{F}}\) is a subspace. But since \(y_1 \not\in D_{\mathcal{F}}\), it follows \(\beta - \alpha = 0\) otherwise \(y_1 = (\beta - \alpha)^{-1}(\beta - \alpha)y_1 \in D_{\mathcal{F}}\).

From \(\beta - \alpha = 0\) it follows \(y - \bar{y} = 0\).
(31) \[-p(y_1 - z) - \overline{f}(z) \leq p(y + y_1) - \overline{f}(y), \text{ where } y_1 \text{ is fixed.}\]

Since in (31), \(y\) doesn’t appear on the left and \(z\) doesn’t appear on the right, the inequality continues to subsist if we passe to left to sup for \(z \in D_f\) that we’ll call \(m_0\) and to right to inf for \(y \in D_f\) that we’ll call \(m_1\). Then:

\[m_0 \leq m_1 \text{ and for } c \text{ such that } m_0 \leq c \leq m_1\]

from (31) we have:

(32) \[-p(y_1 - z) - \overline{f}(z) \leq c, \forall z \in D_f\]

(33) \[c \leq p(y + y_1) - \overline{f}(y), \forall y \in D_f.\]

Let \(\alpha < 0\). From (32) with \(z = \alpha^{-1}y\), we have:

\[-p(y_1 - \alpha^{-1}y) - \overline{f}(\alpha^{-1}y) \leq c,\]

multiplying for \(\alpha > 0\), we obtain:

\[\alpha p(y_1 - \alpha^{-1}y) + \overline{f}(y) \leq -\alpha c.\]

From this last one and from (29) for \(x = y + \alpha y_1\), we have:

\[g_1(x) = \overline{f}(y) + \alpha c \leq -\alpha p(-y_1 - \alpha^{-1}y) = p(\alpha y_1 + y) = p(x).\]

For \(\alpha = 0\) it results \(x \in D_f\) and there’s nothing to prove.

For \(\alpha > 0\) we reason analogously working on (33).

**Theorem 6.2 (Generalized Hahn-Banach theorem).** Let \(X\) be a vector space over the complex (or over the real) and let \(p : X \to \mathbb{R}\) be sub-additive and such that:

\[(M.O.) \quad p(\alpha x) = |\alpha|p(x).\]

Let \(f : Z \to C \ (f : Z \to \mathbb{R})\) be a linear functional on a subspace \(Z\) of \(X\), such that:

\[(34) \quad |f(x)| \leq p(x), \forall x \in Z.\]

Then \(f\) has a linear extension \(\overline{f}\) on \(X\) such that:

\[(34^*) \quad |\overline{f}(x)| \leq p(x), \forall x \in X.\]
Proof. a) $X$ vector space over $\mathbb{R}$. From (34) it follows:

$$f(x) \leq p(x), \forall x \in Z.$$ 

Then from the Theorem 6.1 there exists a linear extension $\overline{f}$ over $X$ such that:

(35) 

$$\overline{f}(x) \leq p(x), \forall x \in X.$$ 

From (35) and from (M.O.) we obtain:

$$-\overline{f}(x) = \overline{f}(-x) \leq p(-x) = p(x)$$

that is:

$$\overline{f}(x) \geq -p(x).$$

From this one and from (35) it follows (34*).

b) $X$ vector space over $\mathbb{C}$. Since $f$ has complex values we can write $f(x) = f_1(x) + if_2(x), \forall x \in Z$, where $f_1$ and $f_2$ have real values. Denoted by $X_r$ and $Z_r$ the real vector spaces that we obtain restricting the multiplication to the real scalars, since $f$ is linear on $Z$, $f_1$ and $f_2$ are linear functionals on $Z_r$. Therefore being:

$$f_1(x) \leq |f(x)| \leq p(x), \forall x \in Z_r$$

from the Theorem 6.1 there exists a linear extension $\overline{f}_1$, of $f_1$ on $X_r$ such that:

(36) 

$$\overline{f}_1(x) \leq p(x), \forall x \in X_r.$$ 

Analogously we reason about $f_2$. Since:

$$f = f_1 + if_2 \text{ we have } \forall x \in Z :$$

$$i[f_1(x) + if_2(x)] = if(x) = f(ix) = f_1(ix) + if_2(ix).$$

Therefore, from the equality of the complex numbers:

(37) 

$$f_2(x) = -f_1(ix), \forall x \in Z.$$ 

Hence if letting:

(38) 

$$\overline{f}(x) = \overline{f}_1(x) - i\overline{f}_1(ix), \forall x \in X.$$ 

from (37) it follows that:

$$\overline{f}(x) = f_1(x) - if_1(ix) = f_1(x) + if_2(x) = f(x), \forall x \in Z.$$ 

This proves that $\overline{f}$ is an extension of $f$ over $X$. 
6. HAHN-BANACH THEOREM

We must still prove that:

i) \( \overline{f} \) is a linear functional on \( X \)

ii) \( \overline{f} \) verifies \((34^*)\) on \( X \).

i): Since \( \overline{f}_1 \) is linear on \( X \), and since \((38)\) holds, we have:

\[
\overline{f}((a + ib)x) = \overline{f}_1(ax + ibx) - i\overline{f}_1(ix) - i[a\overline{f}_1(ix) - b\overline{f}_1(x)] = (a + ib)\overline{f}_1(x) - i\overline{f}_1(ix) = (a + ib)\overline{f}(x).
\]

The first property of the linearity is immediate.

ii): Since \( p(x) \geq 0^1 \), for every \( x \in X \) such that \( \overline{f}(x) = 0 \), \((34^*)\) holds.

Thus let \( x \in X \) such that \( \overline{f}(x) \neq 0 \). Therefore:

\[
\overline{f}(x) = |\overline{f}(x)|e^{i\theta}
\]

hence

\[
|\overline{f}(x)| = \overline{f}(x)e^{-i\theta} = |\overline{f}(e^{-i\theta}x)|.
\]

Thus from (M.O.) it follows:

\[
|\overline{f}(x)| = \overline{f}(e^{-i\theta}x) = 2\overline{f}_1(e^{-i\theta}x) \leq p(e^{-i\theta}x) = p(x).
\]

\( \square \)

**Theorem 6.3** (Hahn-Banach Theorem for normed spaces). Let \( f \) be a linear functional which is bounded on a subspace \( Z \) of a normed space \( X \). Then there exists a bounded and linear functional \( \overline{f} \) on \( X \) that is an extension of \( f \) to \( X \) and such that:

\[
(39) \quad \|\overline{f}\|_X = \|f\|_Z
\]

where \( \|\overline{f}\|_X = \sup_{x \in X, \|x\|=1} |\overline{f}(x)| \) and \( \|f\|_Z = \sup_{x \in Z, \|x\|=1} |f(x)| \) (and \( \|f\|_Z = 0 \) in the banal case \( Z = \{0\} \)).

Proof. If \( Z = \{0\} \), the \( f = 0 \) and the extension is \( \overline{f} = 0 \).

Thus let \( Z \neq \{0\} \). For every \( x \in Z \) we have:

\[
|f(x)| \leq \|f\|_Z \|x\|.
\]

But this is the relation \((34)\) of the Theorem 6.2 where:

\[
(40) \quad p(x) = \|f\|_Z \|x\|, \forall x \in X.
\]

\( ^1 \)

\( ^2 \)

because it’s real.
Prove by exercise that \( p \) is sub-additive and verifies the (M.O.) of Theorem 6.2. Therefore, in virtue by the Theorem 6.2, there exists a linear functional \( \overline{f} \) on \( X \) that is an extension of \( f \) and such that:

\[
|\overline{f}(x)| \leq p(x) = \|f\|_Z \|x\|, \forall x \in X.
\]

Hence:

\[
\|\overline{f}\|_X \leq \|f\|_Z.
\]

Since, from its same definition, on an extension the norm can’t be decreasing we have:

\[
\|\overline{f}\|_X \geq \|f\|_Z
\]

and the (39) of the Theorem is so proved. \( \square \)

REMARK 6.2. Let \( Z \) be a subspace of a Hilbert space, \( H \); then, from Riesz representation theorem, we have:

\[
f(x) = \langle x, z \rangle, \forall x \in Z \text{ where } \|z\| = \|f\|.
\]

Obviously, since the inner product is defined over all \( H \), we have a linear extension \( \overline{f} \) of \( f \) on \( H \) and moreover:

\[
\|\overline{f}\| = \|z\| = \|f\|.
\]

Hence in this case the extension is immediate.

THEOREM 6.4 (Bounded and linear functionals theorem). Let \( X \) be a normed space and let \( x_0 \in X \) with \( x_0 \neq 0 \). Then there exists a bounded and linear functional \( \overline{f} \) over \( X \) such that:

\[
\|\overline{f}\| = 1 \text{ and } \overline{f}(x_0) = \|x_0\|.
\]

Proof. Let \( Z \) be the subspace of \( X \) of all elements \( x = \alpha x_0 \) for scalar \( \alpha \). On \( Z \) we define a linear functional \( f \) from:

\[
f(x) = f(\alpha x_0) = \alpha \|x_0\|.
\]

(41)

\( f \) is bounded and it has norm 1 because:

\[
|f(x)| = |f(\alpha x_0)| = |\alpha| \|x_0\| = \|\alpha x_0\| = \|x\|.
\]

In virtue by the Theorem 6.3 it follows that \( f \) admits a linear extension \( \overline{f} \) over \( X \) and \( \|\overline{f}\| = \|f\| = 1 \).

From (41) it follows that:
\[ \overline{f}(x_0) = f(x_0) = \|x_0\|. \]

\[ (42) \]
\[ \|x\| = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|}. \]

Hence if \( x_0 \) is such that \( f(x_0) = 0 \), \( \forall f \in X' \implies x_0 = 0 \).

Proof. If \( x = 0 \) being \( \forall f \in X' \quad f(0) = 0 \), we obtain:

\[ \|0\| = \sup_{f \in X', f \neq 0} \frac{|f(0)|}{\|f\|} = 0. \]

From the Theorem 6.4 it results, for \( x \neq 0 \):

\[ \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} \geq \frac{\overline{f}(x)}{\|\overline{f}\|} = \|x\| \]

and from \( |f(x)| \leq \|f\| \|x\| \), we obtain:

\[ \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} \leq \|x\|. \]
**Lemma 7.1.** For every fixed $x$ of a normed space $X$ the functional $g_x$, defined by:

$$g_x(f) = f(x), \forall f \in X'$$

is a bounded and linear functional on $X'$. Hence $g_x \in X''$ and moreover it results:

$$\|g_x\| = \|x\|.$$ 

**Proof.** $g_x$ is linear; in fact:

$$g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha g_x(f_1) + \beta g_x(f_2).$$

From (43) and from Corollary 6.1 it follows:

$$\|g_x\| = \sup_{f \in X', f \neq 0} \frac{|g_x(f)|}{\|f\|} = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{\|f\|} = \|x\|. \quad \Box$$

We can consider the dual $(X')'$ of $X'$ whose elements are bounded and linear functionals on $X'$. We denote $(X')'$ with $X''$ which is called the SECOND DUAL (OR BIDUAL) SPACE OF $X$.

**Def. 7.1.** From the previous lemma, we have a correspondence from every $x \in X$ to an unique bounded and linear functional $g_x \in X''$ which is defined by (43). Therefore we can define:

$$C : X \to X'', \ x \mapsto g_x.$$ 

$C$ is called CANONICAL MAP from $X$ to $X''$.

**Lemma 7.2.** The canonical map $C$ given by (46) is an isomorphism of the normed space $X$ over the normed space $R_C$.

**Proof.** The linearity of $C$ follows from:

$$C(\alpha x + \beta y)(f) = g_{\alpha x + \beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f) =$$

$$= \alpha (Cx)(f) + \beta (Cy)(f).$$

In particular it results $g_x - g_y = g_{x-y}$. Hence from (44) we obtain:

$$\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|.\,$
This proves that $C$ is an isometry\footnote{cfr. Hilbert Spaces, Def. 4.2, § 2.4 and Corollary 3.2, § 3.3.}. Isometry implies injectivity\footnote{$x \neq y \implies 0 \neq \|x - y\| = \|g_x - g_y\| = \|g_x - g_y\|$.}.

**Def. 7.2.** The normed space $X$ is called **IMMERSED** in a normed space $Z$ if $X$ is isomorphic to a subspace of $Z$.

**Remark 7.1.** The previous lemma proves that $X$ is immersed in $X''$ and $C$ is so called **CANONICAL IMMERSION** of $X$ into $X''$.

Generally $C$ isn’t surjective.

**Def. 7.3.** A normed space is called **REFLEXIVE** if $R_C = X''$.

**Remark 7.2.** From Lemma 7.2 it follows that if $X$ is reflexive then it is isomorphic to $X''$. But R.C.James has proved that, generally, the converse is false.

**Theorem 7.1.** If a normed space $X$ is reflexive, then it’s complete, that is Banach space.

Proof. Since $X''$ is the dual of $X'$ it follows that $X''$ is Banach space from Corollary 5.1. Hence $X$ is complete from Lemma 7.2.

**Remark 7.3.** We have proved that every functional of a finite dimensional normed space $X$ is bounded. Hence $X' = X^*$ and since $X$ is algebraically reflexive (cfr. Theorem 4.2) it follows:

**Theorem 7.2.** Every finite dimensional normed space is reflexive.

**Remark 7.4.** Every space $L^p$, $1 < p < +\infty$ is reflexive. Hence every space $L^p[a, b]$, with $1 < p < +\infty$ is reflexive.

**Theorem 7.3.** Every Hilbert space is reflexive.

Proof. Let $A : H' \to H$ defined by $Af = z$ where, from the Riesz representation theorem:

$$f(x) = \langle x, z \rangle.$$  

From those we have seen $A$ is bijective, isometric and linear conjugate. $H'$ is complete and it’s a Hilbert space with respect to the inner product defined by:

$$\langle f_1, f_2 \rangle_1 = \langle Af_2, Af_1 \rangle$$

Prove it by exercise.

Let $g \in H''$. From the Riesz representation theorem it results:

$$g(f) = \langle f, f_0 \rangle_1 = \langle Af_0, Af \rangle.$$  

Remembering that:
and letting:

\[ Af_0 = x \]

we have:

\[ \langle Af_0, Af \rangle = \langle x, z \rangle = f(x). \]

Therefore: \( g(f) = f(x) \), that is \( g = Cx \) from definition of \( C \). Hence \( C \) is surjective. \( \square \)

**Lemma 7.3.** Let \( Y \) be a proper subspace of a normed space \( X \). Let \( x_0 \in X \setminus Y \) and let \( \delta \) be the distance of \( x_0 \) from \( Y \)

\[ \delta = \inf_{y \in Y} \| y - x_0 \|. \]

Then there exists \( \bar{f} \in X' \), such that

\[ \| \bar{f} \| = 1, \bar{f}(y) = 0 \quad \forall \ y \in Y, \bar{f}(x_0) = \delta. \]

Proof. Consider the subspace \( Z \subset X \) generated by \( Y \) and \( x_0 \) and define a bounded and linear functional on \( Z \) as follows:

\[ f(z) = f(y + \alpha x_0) = \alpha \delta, \ y \in Y. \]

Every \( z \in Z \) has an unique representation:

\[ z = y + \alpha x_0, \ y \in Y \]

that we have used in (49). Prove by exercise that \( f \) is linear.

Since \( Y \) is closed it follows that \( \delta > 0 \) and hence \( f \neq 0 \). Moreover for \( \alpha = 0 \) it results \( f(y) = 0, \forall \ y \in Y \) and for \( \alpha = 1 \) and \( y = 0 \) it has \( f(x_0) = \delta \).

Thus we prove that \( f \) is bounded. For \( \alpha = 0 \) it results

\[ f(z) = 0. \]

Let, so, \( \alpha \neq 0 \). Using (47) and noted that:

\[ -\alpha^{-1} y \in Y \]

we obtain:

\[ |f(z)| = |\alpha| \delta = |\alpha| \inf_y \| y - x_0 \| \leq |\alpha| \| -\alpha^{-1} y - x_0 \| = \| y + \alpha x_0 \|. \]
That is: $|f(z)| \leq \|z\|$. Hence $f$ is bounded and $\|f\| \leq 1$.
Moreover there exists a sequence $(y_n)_n \subset Y$ such that:

$$\lim_n \|y_n - x_0\| = \delta.$$

Let $z_n = y_n - x_0$. Then for $\alpha = -1$ from (49) we have

$$f(z_n) = -\delta.$$

Hence:

$$\|f\| = \sup_{z \in Z, z \neq 0} \frac{|f(z)|}{\|z\|} \geq \frac{|f(z_n)|}{\|z_n\|} \xrightarrow{n \to \infty} \delta = 1.$$

Therefore $\|f\| \geq 1$ and so $\|f\| = 1$. From the Theorem 6.3 we can extend $f$ over all $X$ without increase the norm and this concludes the proof. \qed

**Theorem 7.4.** If the dual $X'$ of a normed space $X$ is separable, then $X$ is separable.

**Proof.** If $X'$ is separable, the unitary sphere $U' = \{f : \|f\| = 1\}$ of $X'$ contains a dense countable subset $(f_n)_n$. From $f_n \in U'$ it follows that:

$$\|f_n\| = \sup_{\|x\|=1} |f_n(x)| = 1$$

and hence we can find some points $x_n \in X$ with $\|x_n\| = 1$ such that $|f_n(x_n)| \geq \frac{1}{2}$. Let $Y$ be the closure of the linear manifold generated by $(x_n)_n$. Then $Y$ is separable because the set of all linear combinations of $x_n$, with rational coefficients, is a dense and countable subset.

We want prove that $Y = X$. Suppose that $Y \neq X$. Then for Lemma 7.3 there exists $\overline{f} \in X'$ with $\|\overline{f}\| = 1$ and $\overline{f}(y) = 0 \ \forall \ y \in Y$.

Since $x_n \in Y$ it follows that: $\overline{f}(x_n) = 0 \ \forall \ n$, hence:

$$\frac{1}{2} \leq |f_n(x_n)| = |f_n(x_n) - \overline{f}(x_n)| = |(f_n - \overline{f})(x_n)| \leq \|f_n - \overline{f}\| \|x_n\|,$$

where $\|x_n\| = 1$. From $\|f_n - \overline{f}\| \geq \frac{1}{2}$ it contradicts the hypothesis that $(f_n)$ is dense in $U'$ because $\overline{f} \in U'$. \qed

**Remark 7.5.** The connectedness between the separability and the reflexivity is quite simple. The separability of $X'$ implies the separability of $X$, but generally the converse is false. But if a normed space $X$ is reflexive, $X''$ is isomorphic to $X$; in this case, it follows that the separability from $X$ implies that one of $X''$ and, from the previous Theorem, it implies that one of $X'$.

Therefore we assert that:

A separable normed space $X$, with not separable dual $X'$, isn’t reflexive.
Example 7.1. $l^1$ isn’t reflexive because $l^1$ is separable, but its dual $l^\infty$ isn’t separable.

8. Category and uniform boundedness theorem

Def. 8.1. A subset $M$ of a metric space $X$ is called:

a) RARE (or NOT EVERYWHERE DENSE) in $X$ if $M$ hasn’t internal points.

b) MEAGRE (or of 1st category) in $X$ if $M$ is countable union of rare sets in $X$.

c) NOT MEAGRE (or of 2nd category) in $X$ if $M$ isn’t meagre in $X$.

Theorem 8.1 (Baire category theorem). Let $X \neq \emptyset$ be a complete metric space. Then $X$ is not meagre in itself.

Hence if $X \neq \emptyset$ is complete and:

$$X = \bigcup_{k=1}^{\infty} A_k, A_k \text{ closed}$$

then at least an $A_k$ contains a nonempty open subset. Proof. Suppose that the complete metric space $X \neq \emptyset$ let be meagre in itself. Then:

$$X = \bigcup_{k=1}^{\infty} M_k \text{ where } M_k \text{ are rare in } X.$$

For hypothesis $M_1$ is rare in $X$ so by definition $\overline{M_1}$ doesn’t contain nonempty open subsets, but $X$ contains them. Therefore $\overline{M_1} \neq X$. Hence $X \setminus \overline{M_1} = \overline{M_1}^C$ isn’t empty and it’s open. Choose a point $p_1$ of $\overline{M_1}^C$ and an open ball with center $P_1$, such that:

$$B_1 = B(p_1, \varepsilon_1) \subset \overline{M_1}^C, \quad \varepsilon_1 < \frac{1}{2}.$$

For hypothesis $M_2$ is rare in $X$, so $\overline{M_2}$ doesn’t contain nonempty open subsets. Hence it doesn’t contain the open ball $B(p_1, \frac{1}{2} \varepsilon_1)$. This implies that:

$$\overline{M_2}^C \cap B(p_1, \frac{1}{2} \varepsilon_1)$$

is nonempty and open, so, we choose an open ball in this set such that:

$$B_2 = B(p_2, \varepsilon_2) \subset \overline{M_2}^C \cap B(p_1, \frac{1}{2} \varepsilon_1), \quad \varepsilon_2 < \frac{1}{2} \varepsilon_1.$$

By induction we obtain a sequence of open balls:
$B_k = B(p_k, \varepsilon_k)$, $\varepsilon_k < \frac{1}{2} \varepsilon_{k-1} < 2^{-k}$

such that:

$B_k \cap M_k = \emptyset$ and $B_{k+1} \subset B(p_k, \frac{1}{2} \varepsilon_k) \subset B_k$, $k = 1, 2, \ldots$

Since $\varepsilon_k < 2^{-k}$ the sequence $(p_k)_k$ of centers is Cauchy sequence and it converges to any $p \in X$ because $X$ is complete.

Hence for every $m$ and for $n > m$ we have:

$B_n \subset B(p_m, \frac{1}{2} \varepsilon_m)$

so

$d(p_m, p) \leq d(p_m, p_n) + d(p_n, p) < \frac{1}{2} \varepsilon_m + d(p_n, p) \rightarrow \frac{1}{2} \varepsilon_m$

for $n \to \infty$.

Thus $p \in B_m$ for every $m$. Since $B_m \subset \overline{M_m}$ it has: $p \not\in M_m$ for every $m$. Hence:

$p \not\in \bigcup_{k=1}^{\infty} M_k = X$

and this is a contradiction.

\[ \square \]

**Remark 8.1.** Note that, generally, the converse of Baire theorem isn’t true. An example of incomplete normed space which isn’t meagre in itself has been given by N.Bourbaki in 1955.

**Theorem 8.2 (Uniform boundedness theorem).** Let $(T_n)_n$ a sequence of bounded and linear operators $T_n : X \to Y$, where $X$ is a Banach space and $Y$ is a normed space such that:

$(\|T_n x\|)$ is bounded for every $x$, that is:

(50) \[ \|T_n x\| \leq c_x, \quad n = 1, 2, \ldots \quad \text{where } c_x \in \mathbb{R}^+. \]

Then the sequence $(\|T_n\|)_n$ is bounded, that is:

(51) \[ \|T_n\| \leq c, \quad n = 1, 2, \ldots \]

**Proof.** For every $k \in \mathbb{N}$ let $A_k \subset X$ be the set of points $x \in X$ such that $\|T_n x\| \leq k$ for every $n$.

$A_k$ is closed. In fact for every $x \in \overline{A_k}$ there exists a sequence $(x_j)$ in $A_k$ convergent to $x$. This means that for every fixed $n$ we have:
\[\|T_n x_j\| \leq k\]
and hence
\[\|T_n x\| \leq k\]
because \(T_n\) is continuous and the norm is continuous, too. Therefore \(x \in A_k\).
For (50) every \(x \in X\) it belongs to any \(A_k\). Hence \(X = \bigcup_{k=1}^{\infty} A_k\). Since \(X\) is complete the Baire Theorem 8.1 implies that at least an \(A_k\) contains an open ball:

\[B_0 = B(x_0, r) \subset A_{k_0}.\]

Let \(x \in X\) with \(x \neq 0\). Letting:

\[z = x_0 + \gamma x, \quad \gamma = \frac{r}{2\|x\|}.\]

Then \(\|z - x_0\| = \frac{r}{2} < r\) that is \(z \in B_0\).
From (52) and from the definition of \(A_{k_0}\) it results:
\[\|T_n z\| \leq k_0, \text{ for every } n.\]
Moreover:
\[\|T_n x_0\| \leq k_0\]
because \(x_0 \in B_0\).
From (53) we obtain:
\[x = \frac{1}{\gamma}(z - x_0).\]
Then, for every \(n\), we have:
\[\|T_n x\| = \frac{1}{\gamma}\|T_n(z - x_0)\| \leq \frac{1}{\gamma}(\|T_n z\| - \|T_n x_0\|) \leq \frac{4}{r}\|x\| k_0.\]
Thus, for every \(n\), we have:
\[\|T_n\| = \sup_{\|x\|=1} \|T_n x\| \leq \frac{4}{r} k_0\]
that is (51).

\[\text{Theorem 8.3 (Application: Polynomials space). The normed space } X \text{ of all polynomials with norm defined by:}\]

\[\|x\| = \max_{j} |\alpha_j|, \quad (\alpha_0, \alpha_1, \ldots \text{ coefficients of } x)\]
isn’t complete.

Proof. Let build a sequence of bounded and linear operators that satisfies (50) but not (51) so that $X$ can’t be complete.

We can write a polynomial $x \neq 0$ of degree $N_x$ in the form:

$$x(t) = \sum_{j=0}^{\infty} \alpha_j t^j \quad (\alpha_j = 0 \text{ for } j > N_x).$$

As sequence of operators on $X$ we choose the sequence of the functionals $T_n = f_n$ def. by:

$$T_n 0 = f_n(0) = 0, \quad T_n x = f_n(x) = \alpha_0 + \ldots + \alpha_{n-1}.$$

$f_n$ is linear (Prove it by exercise).

$f_n$ is bounded since from (54) we have:

$$|\alpha_j| \leq \|x\|$$

so that:

$$|f_n(x)| \leq n\|x\|.$$

Moreover for every fixed $x \in X$ the sequence $(|f_n(x)|)_n$ satisfies (50) since a polynomial $x$ of degree $N_x$ has $N_x + 1$ coefficients at plus; so from (55) we have:

$$|f_n(x)| \leq (N_x + 1) \max_j |\alpha_j| = c_x$$

and this is a relation of the form (50).

Now, we prove that $(f_n)_n$ doesn’t satisfy (51), that is there doesn’t exist $c$ such that $\|T_n\| = \|f_n\| \leq c$, for every $n$.

For $f_n$ we choose the polynomial:

$$x(t) = 1 + t + \ldots + t^n.$$

Then $\|x\| = 1$ for (54) and

$$f_n(x) = 1 + \ldots + 1 = n = n\|x\|.$$

So:

$$\|f_n\| \geq \frac{|f_n(x)|}{\|x\|} = n$$

Therefore $(\|f_n\|)_n$ isn’t bounded. $\square$
9. Weak and strong convergence

**Def. 9.1.** A sequence \((x_n)_n\) of a normed space \(X\) is called **STRONGLY CONVERGENT** (or **CONVERGENT TO THE NORM**) if there exists a \(x \in X\) such that:

\[
\lim_{n \to \infty} \|x_n - x\| = 0
\]

that is written: \(\lim_{n \to \infty} x_n = x\), or simply \(x_n \to x\).

\(x\) is called the **STRONG LIMIT** of \((x_n)_n\).

**Def. 9.2.** A sequence \((x_n)_n\) of a normed space \(X\) is called **WEAKLY CONVERGENT** if there exists a \(x \in X\) such that every \(f \in X'\) results:

\[
\lim_{n \to \infty} f(x_n) = f(x).
\]

We write \(x_n \rightharpoonup x\).

\(x\) is called the **WEAK LIMIT** of \((x_n)_n\).

**Remark 9.1.** The weak convergence consists in the convergence of the numbers \(a_n = f(x_n)\) for every \(f \in X'\).

**Lemma 9.1 (Weak convergence lemma).** Let \((x_n)_n\) be a weakly convergent sequence in a normed space \(X\), \(x_n \rightharpoonup x\). Then:

a) The weak limit \(x\) of \((x_n)_n\) is unique.

b) Every subset of \((x_n)_n\) converges weakly to \(x\).

c) The sequence \((\|x_n\|)_n\) is bounded.

Proof. a) Suppose that \(x_n \rightharpoonup x\) and \(x_n \rightharpoonup y\). Then \(f(x_n) \to f(x)\) and \(f(x_n) \to f(y)\). Since \((f(x_n))_n\) is a sequence of numbers its limit is unique. Hence \(f(x) = f(x)\).

But for every \(f \in X'\), we have:

\[f(x) - f(y) = f(x - y) = 0.\]

This, by the Norm and null-vector corollary (cfr. Corollary 6.1), implies \(x - y = 0\). Therefore we have the assertion.

b) It follows by the fact that \((f(x_n))_n\) is a convergent sequence of numbers, so that every subsequence of \((f(x_n))_n\) converges and it has the same limit of the sequence.

c) Since \((f(x_n))_n\) is a convergent sequence of numbers, it is bounded that is \(|f(x_n)| \leq c_f\) for every \(n\), where \(c_f\) is a constant dependent from \(f\) but not from \(n\). Using the canonical map \(C : X \to X''\) we define \(g_n \in X''\) by:

\[g_n(f) = f(x_n)\] for every \(f \in X'\).

Then for every \(n\) we have:
Thus the sequence $(|g_n(f)|)_n$ is bounded for every $f \in X'$. Since $X'$ is complete from the Corollary 5.1, from the Theorem 8.2 it follows that $(\|g_n\|)_n$ is bounded. Now, from Lemma 7.1 we have:

$$\|g_n\| = \|x_n\|$$

and the assertion is proved. \(\square\)

**Theorem 9.1 (Strong and weak convergence theorem).** Let $(x_n)_n$ be a sequence of a normed space $X$. Then:

a) The strong convergence implies the weak convergence with the same limit.

b) The converse of a) is generally false.

c) If $\dim X < \infty$, then the weak convergence implies the strong one.

Proof. a) For definition $x_n \to x$ means $\|x_n - x\| \to 0$.

Hence for every $f \in X'$ we have:

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\| \|x_n - x\| \to 0$$

Therefore $x_n \rightharpoonup x$.

b) Let $(e_n)_n$ be an orthonormal sequences of a Hilbert space $H$. Indeed every $f \in H'$ has a Riesz representation (cfr. [1] Hilbert Spaces, Theorem 1.7, § 3.1):

$$f(x) = \langle x, z \rangle$$

Hence $f(e_n) = \langle e_n, z \rangle$.

Now, from the Bessel inequality (cfr. [1] Hilbert Spaces, Theorem 1.8, § 1.1) it results:

$$\sum_{n=1}^{\infty} |\langle e_n, z \rangle|^2 \leq \|z\|^2.$$ 

Therefore the series converges and so the general term must go to zero for $n \to \infty$. This implies that

$$f(e_n) = \langle e_n, z \rangle \to 0.$$ 

Since $f \in H'$ was arbitrary we see that $e_n \to 0$.

However $(e_n)_n$ can’t strongly converge to zero because:

$$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n \rangle = 2 \quad m \neq n$$

c) Suppose that $x_n \rightharpoonup x$ and $\dim X = k$. Let $\{e_1, \ldots, e_k\}$ be a base for $X$ and:

$$x_n = \alpha_1^{(n)} e_1 + \ldots + \alpha_k^{(n)} e_k \quad ; \quad x = \alpha_1 e_1 + \ldots + \alpha_k e_k.$$
For hypothesis $f(x_n) \to f(x)$ for every $f \in X'$.

In particular, taking $f_1, \ldots, f_k$ defined by:

$$f_j(e_j) = 1 \quad ; \quad f_j(e_m) = 0 \quad \text{if } m \neq n$$

(I remember that $f_1, \ldots, f_k$ is the dual base of $e_1, \ldots, e_k$ (12)).

Then:

$$f_j(x_n) = \alpha_j^{(n)} \quad \text{and} \quad f_j(x) = \alpha_j.$$

Hence: $f_j(x_n) \to f(x)$ implies $\alpha_j^{(n)} \to \alpha_j$.

Therefore we obtain

$$\|x_n - x\| = \|\sum_{j=1}^{k} (\alpha_j^{(n)} - \alpha_j)e_j\| \leq \sum_{j=1}^{k} |\alpha_j^{(n)} - \alpha_j|\|e_j\| \to 0$$

for $n \to \infty$.

This proves that $x_n \to x$.  

\[\square\]

**Remark 9.2.** It’s interesting to note that there exist infinite dimensional spaces in which the strong convergence and the weak one are equivalent. An example is $l^1$ as I.Schur has proved in 1821.

**Example 9.1 (Hilbert spaces).** In a Hilbert space $H$ $x_n \to x \iff \langle x_n, z \rangle \to \langle x, z \rangle$ for every $z \in H$.

The proof follows immediately from the Riesz representation theorem (cfr. [1] Hilbert Spaces, Theorem 1.7, § 3.1).

**Def. 9.3.** A TOTAL (or FUNDAMENTAL) set in a normed space $X$ is a subset $M \subset X$ whose span is dense in $X$.

**Lemma 9.2.** In a normed space $X$.

$$x_n \to x$$

is equivalent

- $A$) The sequence $(\|x_n\|)_n$ is bounded.
- $B$) For every element $f$ of the total subset $M \subset X'$ we have $f(x_n) \to f(x)$.

Proof. ($\Longrightarrow$): It follows from Lemma 9.1 and from the definition.

($\Longleftarrow$): From A) there exists $c \in \mathbb{R}^+$ such that $\|x_n\| \leq c$ for every $n$ and $\|x\| \leq c$.

Let $f \in X'$. Since $M$ is total in $X'$ there exists a sequence $(f_j)$ of span... $M$ such that $f_j \to f$. Hence for every $\varepsilon > 0$ we can find $j$ such that:

$$\|f_j - f\| < \frac{\varepsilon}{3c}.$$  

Moreover since $f_j \in \text{span } M$, from B), there exists a $N$ such that for every $n > N$ it results:
\[ |f_j(x_n) - f(x)| < \frac{\varepsilon}{3}. \]

Therefore for every \( n > N \):

\[
|f(x_n) - f(x)| \leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)| < \\
< \|f - f_j\| \|x_n\| + \varepsilon \frac{3}{3} + \|f_j - f\| \|x\| < \frac{\varepsilon}{3c} c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} c = \varepsilon.
\]

So \( x_n \to x \).

\[ \square \]

**Theorem 9.2 (The spaces \( l^p \), \( 1 < p < \infty \)). In the space \( l^p \), \( 1 < p < \infty \).

\[ x_n \to x \]

is equivalent

\[
\begin{align*}
A) & \quad \text{The sequence } (\|x_n\|)_n \text{ is bounded}. \\
B) & \quad \text{For every fixed } j \text{ it results } \xi_j^{(n)} \to \xi_j \text{ for } n \to \infty \text{ where } x_n = (\xi_j^{(n)}) \text{ and } (\xi_j). 
\end{align*}
\]

Proof. The dual space of \( l^p \) is \( l^q \) (cfr. Example 5.5). A base for \( l^q \) is \( (e_n) \), where \( e_n = (\delta_{n,j}) \). Hence span \( (e_n) \) is dense in \( l^q \) so that from Lemma 9.2 it follows the assertion.

\[ \square \]

**Def. 9.4.** A normed space \( X \) is called **WEAKLY COMPLETE** if every Cauchy weak sequence in \( X \) weakly converges in \( X \).

### 10. Convergence of sequences of operators

**Def. 10.1.** Let \( X \) and \( Y \) be two normed spaces. A sequence \( (T_n)_n \) of operators \( T_n \in B(X,Y) \) is called:

1) **UNIFORMLY CONVERGENT** if \( (T_n) \) converges to the norm of \( B(X,Y) \), that is there exists an operator \( T : X \to Y \) such that:

\[ \|T_n - T\| \to 0, \text{ for } n \to \infty \]

2) **STRONGLY CONVERGENT** if \( (T_n) \) strongly converges in \( Y \) for every \( x \in X \), that is there exists an operator \( T : X \to Y \) such that:

\[ \|T_n x - Tx\| \to 0, \text{ for } n \to \infty, \text{ for every } x \in X \]
3) **WEAKLY CONVERGENT** if \((T_n)\) weakly converges in \(Y\) for every \(x \in X\), that is there exists an operator \(T : X \to Y\) such that:

\[ |f(T_n x) - f(T x)| \to 0, \text{ for every } x \in X, \text{ for every } f \in Y'. \]

It isn’t difficult to prove that 1) \(\implies\) 2) \(\implies\) 3); but, generally, the converse is false, as the following examples show.

**Example 10.1 (Strongly but not uniformly convergent).** In the space \(l^2\) we consider a sequence \((T_n)_n\), where \(T_n : l^2 \to l^2\) is defined by:

\[ T_n x = (0, \ldots, 0, \xi_{n+1}, \xi_{n+2}, \ldots) \]

where \(x = (\xi_1, \xi_2, \ldots) \in l^2\). This operator \(T_n\) is strongly convergent to zero since:

\[ T_n x \to 0 = 0x. \]

Moreover \((T_n)_n\) isn’t uniformly convergent since \(\|T_n - 0\| = \|T_n\| = 1\).

**Example 10.2 (Weakly but not strongly convergent).** Another sequence \((T_n)_n\) of operators \(T_n : l^2 \to l^2\) is defined by:

\[ T_n x = (0, \ldots, 0, \xi_1, \xi_2, \ldots) \]

where \(x = (\xi_1, \xi_2, \ldots) \in l^2\). This operator is bounded and linear. Clearly \((T_n)\) is weakly convergent to zero.

Every bounded and linear functional \(f\) on \(l^2\) has a Riesz representation:

\[ f(x) = \langle x, z \rangle = \sum_{j=1}^{\infty} \xi_j \overline{\zeta_j} \]

where \(z = (\zeta_j) \in l^2\).

Letting \(j = n + k\) and using the definition of \(T_n\), it results:

\[ f(T_n x) = \langle T_n x, z \rangle = \sum_{j=n+1}^{\infty} \xi_{j-n} \overline{\zeta_j} = \sum_{k=1}^{\infty} \xi_k \overline{\zeta_{n+k}} \]

From the Cauchy inequality, it results:

\[ |f(T_n x)|^2 = |\langle T_n x, z \rangle|^2 \leq \sum_{k=1}^{\infty} |\xi_k|^2 \sum_{m=n+1}^{\infty} |\zeta_m|^2. \]

The last serie is the rest of a convergent serie. Hence the part on the right of the inequality tends to 0 for \(n \to \infty\). Therefore \(f(T_n x) \to 0 = f(0x)\). Consequently \((T_n)_n\) is weakly convergent to 0.

However \((T_n)_n\) doesn’t strongly converge since for \(x = (1, 0, 0, \ldots)\) we have:

\[ \|T_m x - T_n x\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \text{ for } m \neq n. \]
The linear functionals are linear operators with rank in the field of scalars \( \mathbb{R} \) or \( \mathbb{C} \). Thus also for them we can speak about the convergences 1), 2) and 3). But for them the convergences 2) and 3) are equivalent. In fact, now, we have \( f_n(x) \in \mathbb{R} \) or \( \mathbb{C} \). Hence the convergences 2) and 3) are so considered in a finite dimensional space (1-dimensional) and the equivalence of 2) and 3) follows from the Theorem 9.1 part c).

The two remaining concepts of convergence are renamed in the following one:

**Def. 10.2.** Let \((f_n)_n\) be a sequence of bounded and linear functionals on a normed space \( X \). Then:

a) \((f_n)_n\) STRONGLY CONVERGES if there exists \( f \in X' \) such that \( \|f_n - f\| \to 0 \), and it writes \( f_n \to f \)

b) \((f_n)_n\) WEAKLY CONVERGES\(^*\) (\( \ast = \text{star} \)) if there exists \( f \in X' \) such that \( f_n(x) \to f(x) \) for every \( x \in X \), and it write \( f_n \xRightarrow{\ast} f \)

\( f \) in a) and b) is called the strong limit and the weak limit\(^*\) of \((f_n)_n\), respectively.

**Remark 10.1.** The concept of weak convergence\(^*\) is, in any way, more important than that one of the weak convergence of \((f_n)_n\) that is \( g(f_n) \to g(f) \) for every \( g \in X'' \).

Prove by exercise that the weak convergence implies the weak convergence\(^*\) (using, for example, the canonical map \( C : X \to X'', x \mapsto g_x \)) and that, if \( X \) is reflexive, holds the converse.

**Remark 10.2.** Returning to the operators \( T_n \in B(X,Y) \) we ask what we can say about the limit operator \( T : X \to Y \) in the convergences 1), 2) and 3).

If the convergence is uniform, \( T \in B(X,Y) \), otherwise \( \|T_n - T\| \) has no sense.

If the convergence is strong or weak, \( T \) is still linear, but it can be boundless, if \( X \) isn’t complete.

**Example 10.3.** The space \( X \) of all "finite" sequences \( x = (\xi_i) \in l^2 \) isn’t complete in the metric of \( l^2 \).

Define a sequence of bounded and linear operators \( T_n \) over \( X \) in the following way:

\[ T_n x = (\xi_1, 2\xi_2, \ldots, n\xi_n, \xi_{n+1}, \xi_{n+2}, \ldots), \]

so that the terms of \( T_n x \) are \( j\xi_j \) for \( j \leq n \) and \( \xi_j \) for \( j > n \).

This sequence strongly converges to a boundless linear operator \( T \) defined by:

\[ T x = (\eta_j) \text{ where } \eta_j = j\xi_j. \]

**Lemma 10.1.** Let \( T_n \in B(X,Y) \) with Banach space \( X \) and normed space \( Y \). If \((T_n)_n\) strongly converges to \( T \), then \( T \in B(X,Y) \).

Proof. The linearity of \( T \) easily follows from that one of \( T_n, n = 1, 2, \ldots \). Since \( T_n x \to Tx \) for every \( x \in X \), the sequence \((T_n x)_n\) is bounded for every \( x \). Since \( X \) is complete \((\|T_n\|)_n\) is bounded from Uniform boundedness theorem 8.2, that is:
\[ \|T_n\| \leq c \text{ for every } n. \]

From this one it follows that:

\[ \|T_n x\| \leq \|T_n\| \|x\| \leq c \|x\|. \]

Therefore:

\[ \|T_n x\| \leq c \|x\|. \]

Theorem 10.1. A sequence \((T_n)_n\) of operators \(T_n \in B(X, Y)\) with \(X\) and \(Y\) Banach spaces is strongly convergent \(\iff\)

\[ \begin{align*}
A) & \ \text{The sequence } (\|T_n\|)_n \text{ is bounded.} \\
B) & \ \text{The sequence } (T_n x)_n \text{ is a Cauchy sequence in } Y \text{ for } x \text{ of a total subset } M \text{ of } X
\end{align*} \]

Proof. \((\implies)\): If \(T_n x \to T x\) for every \(x \in X\), then A) follows from the Theorem 8.2 and B) is banal.

\((\iff)\): \(\|T_n\| \leq c\) for every \(n\). Consider \(x \in X\). Fix \(\varepsilon > 0\). Since span \(M\) is dense in \(X\) there exists \(y \in \text{span } M\) such that:

\[ \|x - y\| < \frac{\varepsilon}{3c}. \]

Since \(y \in \text{span } M\), the sequence \((T_n y)_n\) is a Cauchy sequence hence there exists \(N\) such that

\[ \|T_n y - T_m y\| < \frac{\varepsilon}{3} \text{ for } m, n > N. \]

Therefore for every \(m, n > N\) it has:

\[ \|T_n x - T_m x\| \leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| < \]

\[ < \|T_n\| \|x - y\| + \frac{\varepsilon}{3} + \|T_m\| \|x - y\| < \frac{\varepsilon c}{3c} + \frac{\varepsilon c}{3c} = \varepsilon \]

that is \((T_n x)_n\) ia a Cauchy sequence in \(Y\).

Since \(Y\) is complete, \((T_n x)_n\) converges in \(Y\). From the arbitrariness of \(x\) it follows the assertion. \(\square\)

Corollary 10.1. A sequence \((f_n)_n\) of bounded and linear functionals on a Banach space \(X\) is weakly* convergent to a bounded and linear functional if and only if:
\[
\begin{align*}
A) & \text{ The sequence } (\|f_n\|)_n \text{ is bounded.} \\
B) & \text{ The sequence } (f_n(x))_n \text{ is a Cauchy sequence for } x \text{ of a total subset } M \text{ of } X.
\end{align*}
\]
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