ASYMPTOTICS FOR THE MULTIPlicITIES IN THE 
COCHARACTERS OF SOME PI-ALGEBRAS

FRANCESCA BENANTI, ANTONIO GIAMBRUNO, AND IRINA SVIRIDOVA

Abstract. We consider associative PI-algebras over a field of characteristic zero. We study the asymptotic behavior of the sequence of multiplicities of the cocharacters for some significant classes of algebras. We also give a characterization of finitely generated algebras for which this behavior is linear or quadratic.

1. Introduction

Let $F$ be a field of characteristic zero and $F(X)$ the free associative algebra over $F$ of countable rank with set of generators $X = \{x_1, x_2, \ldots\}$. If $A$ is an associative algebra over $F$, we denote by $\text{Id}(A) \subseteq F(X)$ the $T$-ideal of all polynomial identities of $A$. It is well known that in characteristic zero every $T$-ideal is completely determined by its multilinear elements. Hence, if $V_n$ is the space of all multilinear polynomials of degree $n$ in $x_1, \ldots, x_n$, we study the sequence of spaces $V_n \cap \text{Id}(A)$, $n = 1, 2, \ldots$.

A useful approach to this study is through the representation theory of the symmetric group $S_n$. In fact, there is a natural action of $S_n$ on $V_n$ leaving $V_n \cap \text{Id}(A)$ invariant: if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in V_n$ then one defines $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This in turn makes $V_n(A) = V_n/V_n \cap \text{Id}(A)$ an $S_n$-module.

The $S_n$-character of $V_n(A)$, denoted $\chi_n(A)$, is called the $n$-th cocharacter of $A$ or of $\text{Id}(A)$. By complete reducibility $\chi_n(A)$ decomposes into irreducibles and let $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, where $\chi_\lambda$ is the irreducible $S_n$-character associated to the partition $\lambda$ of $n$ and $m_\lambda$ is the corresponding multiplicity. Through the sequence of cocharacters $\{\chi_n(A)\}_{n \geq 1}$ one can attach to $A$ three numerical sequences. The first, called the sequence of codimensions, is given by

$$c_n(A) = \chi_n(A)(1) = \dim_F V_n/V_n \cap \text{Id}(A),$$

$n = 1, 2, \ldots$ The second sequence is

$$m_n(A) = \max_{\lambda \vdash n} m_\lambda,$$

$n = 1, 2, \ldots$ and we call it the sequence of multiplicities. The third, called the sequence of colengths, is

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda.$$

1991 Mathematics Subject Classification. 2000 Mathematics Subject Classification. Primary 16R10, 16P90.

Key words and phrases. polynomial identities, multiplicities, codimensions, growth.

The first and the second authors were partially supported by MURST of Italy.

The third author was partially supported by the scientific program "The Universities of Russia".
We are interested in the asymptotic behaviour of these three sequences and in their interrelations.

About the sequence of codimensions it was first proved by Regev in [24] that if $A$ is a PI-algebra, then $c_n(A)$ is exponentially bounded. The sequence $c_n(A)$ or even its asymptotic behaviour has only been computed for some special algebras [6, 9, 18, 19, 23, 25]; it turns out that it is in general a very hard problem to determine the precise asymptotic behaviour of such sequence.

It was recently proved in [10, 11] that for a PI-algebra $A$, 

$$\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}$$

exists and is an integer; $\exp(A)$ is called the PI-exponent of the algebra $A$ and it has been computed for some classes of algebra [2, 7, 12, 25].

About the other sequences, the most important result is due to Berele and Regev who proved in [1] that for a PI-algebras $A$, the sequence $\{l_n(A)\}_{n \geq 1}$ is polynomial bounded. Hence $\{m_n(A)\}_{n \geq 1}$ is also polynomial bounded. Notice that if $A$ is a nilpotent algebra then for $n$ large, $V_n(A)$ is the zero module. We then define for any non nilpotent PI-algebra $A$

$$
\text{mlt}(A) = \limsup_{n \to \infty} \log_n m_n(A)
$$

and

$$
\text{col}(A) = \limsup_{n \to \infty} \log_n l_n(A).
$$

Note that for any numerical sequence $p(n)$, if $b \cdot n^k \leq p(n) \leq a \cdot n^k$ for some constants $a, b > 0$, then $\limsup_{n \to \infty} \log_n (p(n)) = k$. Hence $\text{mlt}(A)$ and $\text{col}(A)$ actually capture the polynomial behaviour of the sequences $\{m_n(A)\}_{n \geq 1}$ and $\{l_n(A)\}_{n \geq 1}$ respectively. In this paper we find the precise value of $\text{mlt}(A)$ for some significant minimal algebras $A$ of small PI-exponent. Also we characterize finitely generated algebras (or better the corresponding T-ideals of identities) for which $\text{mlt}(A) = 1$ and we find precise relations among $\text{mlt}(A)$ and $\exp(A)$.

2. Preliminaries

Throughout the paper we will denote by $F$ a field of characteristic zero. Recall that an algebra $A$ is a PI-algebra if it satisfies a non-trivial polynomial identity. If $f$ is a polynomial identity on $A$ we usually write $f \equiv 0$ in $A$. Let $Id(A) = \{f \in F(X) \mid f \equiv 0 \text{ in } A\}$ be the ideal of identities of $A$. Recall that $Id(A)$ is a T-ideal i.e., an ideal invariant under all endomorphisms of $F(X)$. If $\mathcal{V}$ is a variety of associative algebras, $\mathcal{V}$ determines uniquely a T-ideal $I = Id(\mathcal{V})$. Also, if $\mathcal{V}$ is generated by the algebra $A$ we write $\mathcal{V} = \text{var}(A) = \var(I)$ and $I = Id(\mathcal{V}) = Id(A)$. In case $\mathcal{V} = \text{var}(A) = \var(B)$ i.e., $Id(A) = Id(B)$, we also say that $A$ and $B$ are PI-equivalent.

Let $G$ be the infinite dimensional Grassmann algebra. Recall that $G$ is the algebra generated by a countable set $\{e_1, e_2, \ldots\}$ subject to the conditions $e_i e_j = -e_j e_i$ for all $i, j = 1, 2, \ldots$. Let $G = G_0 \oplus G_1$ be the natural $\mathbb{Z}_2$-grading on $G$ where $G_0$ and $G_1$ are the spaces generated by all monomials in the generators $e_i$’s of even and odd length, respectively.

Now, if $B = B_0 \oplus B_1$ is any $\mathbb{Z}_2$-graded algebra, then $G(B) = B_0 \otimes G_0 \oplus B_1 \otimes G_1$ is called the Grassmann envelope of $B$. The importance of such algebras is given by
a well known result of Kemer ([17, Theorem 2.3]) which states that for any proper variety $V$, there exists a finite dimensional superalgebra $B$ such that $V = \text{var}(G(B))$.

Let $V_n = \text{Span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ be the space of multilinear polynomials in $x_1, \ldots, x_n$. If $A$ is a PI-algebra, let $c_n(A) = \dim_F V_n / V_n \cap \text{Id}(A)$ be the $n$-th codimension of $A$. As it was mentioned in the introduction, then the PI-exponent of $A$ is defined as $\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}$. In [10, 11] it was proved that $\exp(A)$ always exists and is a nonnegative integer; in that paper the authors also gave a constructive way for computing the exponent as follows: let $B$ be a finite dimensional $\mathbb{Z}_2$-graded algebra such that $\text{var}(A) = \text{var}(G(B))$ and suppose that $F$ is algebraically closed. Let $B = B_1 \oplus \cdots \oplus B_k + J$ be the Wedderburn-Malcev decomposition of the algebra $B$ where $B_1, \ldots, B_k$ are simple subalgebras and $J$ is the Jacobson radical of $B$ (see [4, Theorem 72.19]). It is also well known (see for instance [17]) that in such decomposition we may take $B_1 \ldots B_k$ to be stable under the $\mathbb{Z}_2$-grading. Then

$$\exp(A) = \max_{i_1, \ldots, i_\ell} \dim_F(B_{i_1} \oplus \cdots \oplus B_{i_\ell})$$

where $B_{i_1}, \ldots, B_{i_\ell}$ satisfy the condition $B_{i_1}JB_{i_2}J \cdots JB_{i_\ell} \neq 0$.

We remark that the codimensions of a PI-algebra do not change if we extend the base field (see for instance [10, Remark 1]).

As an example, that we shall use in the next section, we now compute the exponent of the algebra $UT_2(G_0, G) = \begin{pmatrix} G_0 & G \\ 0 & G \end{pmatrix}$. Notice that $UT_2(G_0, G)$ is the Grassmann envelope of the algebra $\begin{pmatrix} F & F \oplus tF \\ 0 & F \oplus tF \end{pmatrix}$, where $t^2 = 1$ with $\mathbb{Z}_2$-grading

$$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \begin{pmatrix} 0 & tF \\ 0 & tF \end{pmatrix}.$$  

As we remarked above we may assume $F$ to be algebraically closed. Hence, since $J = (F \oplus tF)e_{12}$ and $Fe_{11}(F \oplus tF)e_{22} \neq 0$, we get that $\exp(UT_2(G_0, G)) = 3$.

3. Growth of the multiplicities of some PI-algebras

In this section we shall compute $\text{mlt}(A)$ for some significant algebras.

In PI-theory an important role is played by the so-called minimal algebras of given PI-exponent (see [13] and [14]). Recall that an algebra $A$ is minimal of PI-exponent $d \geq 2$ if $\exp(A) = d$ and $\exp(B) < d$ for all algebras $B$ such that $\text{Id}(B) \geq \text{Id}(A)$.

If $V = \text{var}(A)$ is the variety generated by $A$ we shall write $\exp(V) = \exp(A)$ and, in case $A$ is minimal of PI-exponent $d$, we shall say that $V$ is minimal of exponent $d$.

A complete list of minimal algebras of given PI-exponent was determined (up to PI-equivalence) in [14] in case $A$ is a finitely generated algebra (see also [13]). Moreover in [12], by giving a characterization of algebras of PI-exponent $> 2$, the authors determined a list of algebras minimal of PI-exponent $\leq 4$. We next describe these algebras and we compute for them the function $\text{mlt}(A)$.

Let $UT_3(F)$ be the algebra of $k \times k$ upper triangular matrices over $F$. In [16] it was essentially proved that $G$ and $UT_2(F)$ generate the only two minimal varieties of exponent $2$. We shall see below that the algebras

$$UT_3(F), \quad UT_2(G_0, G) = \begin{pmatrix} G_0 & G \\ 0 & G \end{pmatrix} \quad \text{and} \quad UT_2(G, G_0) = \begin{pmatrix} G & G \\ 0 & G_0 \end{pmatrix}$$
generate the only minimal varieties of exponent 3. We shall also see that $M_2(F)$, the algebra of $2 \times 2$ matrices over $F$, and $M_{1,1}(G) = \begin{pmatrix} G_0 & G_1 \\ G_1 & G_0 \end{pmatrix}$ are also minimal of PI-exponent 4.

If $f_1, \ldots, f_n \in F(X)$, let $(f_1, \ldots, f_n)_T$ denote the $T$-ideal of $F(X)$ generated by the polynomials $f_1, \ldots, f_n$.

In the next lemma we shall need some results on the proper polynomial identities of a PI-algebra. Recall that the space of multilinear proper polynomials in $x_1, \ldots, x_n$ is the subspace $P_n$ of $V_n$ spanned by all products of Lie commutators of length $\geq 2$. Then one has in a natural way the notion of proper polynomial identity for an algebra $A$ (we refer to [5] for the basic properties of the proper identities). Let $Id^n(A)$ be the space of proper identities of the algebra $A$. The $S_n$-action on $V_n$ induces a structure of $S_n$-module on $P_n$ and on $P_n(A) = P_n/P_n \cap Id^n(A)$. Then one considers in a natural way the $S_n$-character of $P_n(A)$, denoted $\psi_n(A)$, which is called the $n$-th proper cocharacter of $A$. A result of Drensky [6, Theorem 2.6] gives the precise relation between the ordinary cocharacters and the proper cocharacters of any PI-algebra $A$: if $\psi_p(A) = \sum_{\nu \vdash p} m_{\nu} \chi_{\nu}$, where $\chi_{\nu}$ is the irreducible $S_n$-character associated to the partition $\nu \vdash n$, then

$$\chi_n(A) = \sum_{p=0}^{n} \psi_p(A) \otimes \chi_{(n-p)} = \sum_{p=0}^{n} \sum_{k=0}^{p} m_{\lambda k} \chi_{\lambda k} \otimes \chi_{(n-p)}$$

and the tensor product $\chi_{\lambda} \otimes \chi_{(n-p)}$ is computed according to Young’s rule (see [15, Theorem 2.8.2]). We now apply this result in the following

**Lemma 3.1.** $\mlt(M_{1,1}(G)) = 1$.

**Proof.** It is well known (see [17, pag. 24]) that $M_{1,1}(G)$ is PI-equivalent to $G \otimes G$. Popov in [22] proved that $Id(G \otimes G) = \langle [x_1,x_2,x_3,x_4,x_5],[x_1,x_2]^2,x_1 \rangle_T$ and he also described the $S_n$-module structure of the proper multilinear polynomial identities of $G \otimes G$. In particular, if $\psi_n(G \otimes G)$ denotes the $n$-th proper cocharacter of this algebra, then he proved the following decomposition [22, Theorem 7.1]

$$\psi_n(G \otimes G) = \sum_{\lambda = (r,1^p) \vdash n} \chi_{\lambda} + \sum_{\lambda = (r,2^q,1^t) \vdash n} \chi_{\lambda}.$$ 

Now, by Drensky’s result mentioned above and by applying Young’s rule, we obtain that in the decomposition of the $n$-th cocharacter of $A$ only partitions of the type $\lambda = (r,1^p), \lambda = (r,2^q,1^t), \lambda = (r,s,1^t), \lambda = (r,s,2^q,1^t)$ appear. Moreover for each such partition $\lambda$, $\mu_{\lambda} \leq t(n+1)$, for some constant $t \in \mathbb{Z}^+$. In particular, if $\lambda_0 = (2^{[n/2]},[2^{\sharp},1^{n-2\sharp}])$, where $[a]$ denotes the integral part of $a$, then we have that $m_{\lambda_0} = t([n/2] + 1)$. Therefore it turns out that

$$\frac{n}{2} t \leq m_{\lambda_0} \leq m_n(M_{1,1}(G)) \leq t(n+1) \leq 2tn$$

and, so, $\mlt(M_{1,1}(G)) = \lim_{n \to \infty} \sup \log_n m_n(M_{1,1}(G)) = 1$. \qed

In the next lemmas we shall use a result of Berele and Regev which allows to compute the $n$-th cocharacter of a product of T-ideals. The result is the following
Theorem 3.2 ([3], Theorem 1.1). Let $A, A_1, A_2$ be PI algebras such that $Id(A) = Id(A_1)Id(A_2)$. Then
\[ \chi_n(A) = \chi_n(A_1) + \chi_n(A_2) + \chi(1) \otimes \sum_{j=0}^{n-1} \chi_j(A_1) \otimes \chi_{n-j-1}(A_2) - \sum_{j=0}^{n} \chi_j(A_1) \otimes \chi_{n-j}(A_2). \]

Lemma 3.3. \(\mlt(UT_2(G_0, G)) = \mlt(UT_2(G, G_0)) = 1\).

Proof. It is easy to check that \(UT_2(G_0, G)\) satisfies the identity \([x_1, x_2][x_3, x_4, x_5] = 0\). Hence \(\var(UT_2(G_0, G)) \subseteq \var([x_1, x_2][x_3, x_4, x_5])\). Now, in [26] it was proved that the identity \([x_1, x_2][x_3, x_4, x_5] = 0\) defines a minimal variety of exponent 3. As it was noticed in the previous section, \(\exp(UT_2(G_0, G)) = 3\), hence we obtain that \(\var(UT_2(G_0, G)) = \var([x_1, x_2][x_3, x_4, x_5])\).

By applying Theorem 3.2, we obtain that
\[ \chi_n(UT_2(G_0, G)) = \sum_{\lambda=(s,1)^+ \vdash n} m_{\lambda} \chi_{\lambda}, \]
where \(m_{(0)} = 1\) and \(m_{\lambda} = q_1(r - s + q_2) \leq q_1(n + q_2) \leq 2q_1n\), for some constants \(q_1, q_2 \in \mathbb{Z}^+\). In particular, for \(\lambda_0 = ([n^2], [n], 1^{n-3})\), we obtain \(m_{\lambda_0} = q_1([n^2] + q_2)\). As in the proof of Lemma 3.1, we obtain \(\mlt(UT_2(G_0, G)) = 1\). Similarly one can prove that \(\var(UT_2(G, G_0)) = \var([x_1, x_2][x_3, x_4, x_5])\) and, as above, it follows that \(\mlt(UT_2(G, G_0)) = 1\). \(\square\)

In the proof of the previous lemma we saw that \(UT_2(G_0, G)\) and \(UT_2(G, G_0)\) are minimal algebras of exponent 3. But then, by [12, Theorem 1], it follows that \(UT_3(F)\), \(UT_2(G_0, G)\) and \(UT_2(G, G_0)\) are, up to PI-equivalence, the only minimal algebras of PI-exponent 3.

We next deal with the case \(A = M_2(F)\) and \(A = UT_k(F)\). Notice first that from the definition of the exponent, \(\exp(M_k(F)) = k^2\) and \(\exp(UT_k(F)) = k\), for all \(k \geq 1\). Moreover, by [7] both \(M_k(F)\) and \(UT_k(F)\) are minimal algebras.

We next compute the functions \(\mlt(A)\) for \(M_2(F)\) and \(UT_k(F)\), \(k \geq 1\).

Lemma 3.4. \(\mlt(M_2(F)) = 3\).

Proof. The multiplicities in the cocharacter sequence of \(M_2(F)\) were computed in [6, 9, 23]. An inspection of these multiplicities shows that for all \(n\), \(\max_{\lambda \vdash n} m_{\lambda} = m_{\lambda} \) for some \(\lambda = (\lambda_1, \ldots, \lambda_4)\) with \(\lambda_4 > 0\). Since by [5, Theorem 12.6.5] when \(\lambda_4 > 0\), \(m_{\lambda} = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)\) it turns out that \(m_{\lambda} \leq (n + 1)^3 \leq 8n^3\). Let now \(\lambda = n\) be the partition \(\lambda = (4\left\lfloor \frac{n}{4} \right\rfloor, 3\left\lfloor \frac{n}{3} \right\rfloor, 2\left\lfloor \frac{n}{2} \right\rfloor, n - 9\left\lfloor \frac{n}{9} \right\rfloor)\). Then \(m_{\lambda} = (\left\lfloor \frac{n}{4} \right\rfloor + 1)^3\) and this implies
\[ \frac{n^3}{10^3} \leq m_{\lambda} \leq 8n^3. \]

Hence \(\mlt(M_2(F)) = 3\). \(\square\)

Lemma 3.5. \(\mlt(UT_k(F)) = \binom{k}{2}\).
$\lambda$ and $m$

It is well known (see [20]) that the $T$-ideal of identities of $UT_k(F)$ is generated by the polynomial $[x_1, x_2] \cdots [x_{2^k-1}, x_{2^k}]$, then $Id(UT_k(F)) = Id(UT_{k-1}(F)) Id(F)$. In particular, for $k = 2$, $Id(UT_2(F)) = Id(F)Id(F)$. Then, by applying Theorem 3.2,

$$\chi_n(UT_2(F)) = \chi_n(F) + \chi_1(F) \otimes (\sum_{j=0}^{n-1} \chi_j(F) \otimes \chi_{n-j-1}(F)) - \sum_{j=0}^{n} \chi_j(F) \otimes \chi_{n-j}(F).$$

In this decomposition the irreducible character corresponding to partition $\lambda = (n)$ appear

$$\chi(n) = 2\chi(n) + \chi_1(1) \otimes \sum_{j=0}^{n-1} \chi(j) \otimes \chi(n-j-1) - \sum_{j=0}^{n} \chi(j) \otimes \chi(n-j).$$

Then its multiplicity is $m(n) = 2 + n - n - 1 = 1$.

The irreducible character corresponding to $\lambda = (\lambda_1, \lambda_2) \vdash n$ appear

$$\chi_\lambda = \chi(1) \otimes \sum_{j=\lambda_2-1}^{\lambda_1} \chi(j) \otimes \chi(n-j-1) + \sum_{j=\lambda_2}^{\lambda_1-1} \chi(j) \otimes \chi(n-j-1) - \sum_{j=\lambda_2}^{\lambda_1} \chi(j) \otimes \chi(n-j).$$

Then $m_{(\lambda_1, \lambda_2)} = [\lambda_1 - (\lambda_2 - 1) + 1] + [\lambda_1 - \lambda_2 + 1] - [\lambda_1 - \lambda_2 + 1] = \lambda_1 - \lambda_2 + 1$.

The irreducible characters corresponding to $\lambda = (\lambda_1, \lambda_2, \lambda_3) \vdash n$ appear only if $\lambda_3 = 1$ and in this case

$$\chi_\lambda = \chi(1) \otimes \sum_{j=\lambda_2}^{\lambda_1} \chi(j) \otimes \chi(n-j),$$

then $m_{(\lambda_1, \lambda_2, 1)} = \lambda_1 - \lambda_2 + 1$.

Moreover $m_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} = 0$ since $\dim_F(UT_2) = 3$.

Similarly it is possible to calculate $\chi_n(UT_k(F))$ for $k > 2$ and then to prove that the characters corresponding to partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ have multiplicity larger than those of any other shape and for any such $\lambda$

$$m_\lambda = q \prod_{i,j=1}^{k} (\lambda_i - \lambda_j + q_{ij}),$$

for some constants $q \in \mathbb{Q}^+$, $q_{ij} \in \mathbb{N}$. Hence for any $\lambda \vdash n$ and for large enough $n$,

$$m_\lambda \leq q \prod_{i,j=1}^{k} (n + q_{ij}) \leq q^{2d}n^d,$$

where $d = \frac{k(k-1)}{2} = \binom{k}{2}$. Let $d' = \frac{k(k+1)}{2}$. If we now consider the partition $\lambda' = (k\lfloor \frac{n}{d} \rfloor, (k-1)\lfloor \frac{n}{d} \rfloor, \ldots, n-(d'-1)\lfloor \frac{n}{d} \rfloor)$ we have that

$$m_{\lambda'} = q \prod_{i,j=1}^{k} (\lfloor \frac{n}{d'} \rfloor + q_{ij}) \geq \overline{q}n^d,$$

with $\overline{q} \in \mathbb{Q}^+$. Hence

$$\overline{q}n^d \leq m_{\lambda'} \leq m_n(A) \leq q^{2d}n^d$$

and $m(UT_k(F)) = d = \binom{k}{2}$. \hfill \box
We now state and prove a remark that will be used throughout the paper.

**Remark 3.6.** Let $A$ and $B$ be PI-algebras. If $B \in \text{var}(A)$ then $\exp(B) \leq \exp(A)$, $\text{mlt}(B) \leq \text{mlt}(A)$ and $\text{col}(B) \leq \text{col}(A)$.

**Proof.** Since $\text{Id}(B) \supseteq \text{Id}(A)$, it follows that for all $n \geq 1$, $V_n/V_n \cap \text{Id}(B)$ is isomorphic to a quotient module of $V_n/V_n \cap \text{Id}(A)$. Thus $c_n(B) \leq c_n(A)$ and, so, $\exp(B) \leq \exp(A)$. Moreover by complete reducibility, if $\chi_n(B) = \sum m_{\lambda} \chi_{\lambda}$ and $\chi_n(A) = \sum m_{\lambda} \chi_{\lambda}$, then we must have $m_{\lambda} \leq m'_{\lambda}$ for all $\lambda \vdash n$. Thus $\text{mlt}(B) \leq \text{mlt}(A)$ and $\text{col}(B) \leq \text{col}(A)$ follows. □

In the next corollary we combine Lemmas 3.1, 3.3, 3.4, 3.5 with a characterization of varieties of exponent $d \geq 2$ given in [12].

**Corollary 3.7.** For any PI-algebra $A$, if $\exp(A) > 2$ then $\text{mlt}(A) \geq 1$.

**Proof.** Since $\exp(A) > 2$, by [12, Theorem 1] one of the algebras $\text{UT}_2(G_0, G)$, $\text{UT}_2(G, G_0)$, $\text{UT}_3(F)$, $M_2(F)$, $M_{1,1}(G)$ lies in $\text{var}(A)$. Call $B$ such an algebra. Since by Lemmas 3.1, 3.3, 3.4, 3.5 $\text{mlt}(\text{UT}_2(G_0, G)) = \text{mlt}(\text{UT}_2(G, G_0)) = \text{mlt}(M_{1,1}(G)) = 1$ and $\text{mlt}(\text{UT}_3(F)) = \text{mlt}(M_2(F)) = 3$, by the previous remark we obtain that $\text{mlt}(A) \geq 1$. □

**Corollary 3.8.** $\text{col}(\text{UT}_2(F)) = 2$.

**Proof.** From the definition of colength and the proof of Lemma 3.5 we have

\[
\text{l}_n(\text{UT}_2(F)) = \sum_{\lambda \vdash n} m_{\lambda} = m(n) + \sum_{\lambda_1 + \lambda_2 = n} m_{\lambda_1, \lambda_2} + \sum_{\lambda_1 + \lambda_2 = n-1} m_{\lambda_1, \lambda_2, 1} = 1 + \sum_{\lambda_1 + \lambda_2 = n} (\lambda_1 - \lambda_2 + 1) + \sum_{\lambda_1 + \lambda_2 = n-1} (\lambda_1 - \lambda_2 + 1) = 1 + \sum_{\lambda_1 = n/2} (\lambda_1 - (n - \lambda_1) + 1) + \sum_{\lambda_1 = n/2} (\lambda_1 - (n - 1 - \lambda_1) + 1) = 1 + \frac{1}{4}(n + 2)^2 + \frac{1}{4}(n + 4)(n + 2) = \frac{1}{2}n^2 + \frac{5}{2}n + 4.
\]

Hence $\text{col}(\text{UT}_2(F)) = 2$. □

4. Growth of the multiplicities of finitely generated algebras

It is well known (see for instance [10, Remark 1] for its proof) that if $A$ is a PI-algebra over a field $F$ and $K \supseteq F$ is an extension field then $\text{Id}(A) \otimes_F K = \text{Id}(A \otimes_F K)$. Hence we shall assume from now on that $F$ is an algebraically closed field of characteristic zero. Also throughout this section we shall assume that $A$ is a finitely generated PI-algebra over $F$,

\[
\chi_n(A) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda}
\]

is its $n$-th cocharacter and

\[
\text{l}_n(A) = \sum_{\lambda \vdash n} m_{\lambda}
\]

its colength.
Recently in [8] Drensky and Kassabov have obtained a characterization of a finitely generated algebra satisfying a nonmatrix identity i.e., an identity not satisfied by $2 \times 2$ matrices over $F$. Here we state a special case of their result which will be essential in the proof of the next theorem. Recall that if $\lambda \vdash n$ we write $\lambda = (\lambda_1, \ldots, \lambda_s)$.

**Theorem 4.1.** [8] Let $A$ be a finitely generated algebra satisfying a nonmatrix polynomial identity. Then the multiplicities $m_\lambda$, in the $n$-th cocharacter of $A$, are bounded by a linear function of $n$ if and only if there exists a positive integer $q$ such that $m_\lambda = 0$ whenever $\lambda_3 > q$.

For $\lambda \vdash n$ let $h(\lambda)$ be the number of non-zero parts of $\lambda$. Hence $h(\lambda)$ is the height of the Young diagram corresponding to $\lambda$. Also, if $n \geq m$ and $\lambda \vdash n, \mu \vdash m$, we write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i$. Recall that the group algebra $FS_\lambda$ decomposes into the direct sum of its minimal two-sided ideals $FS_\lambda = \bigoplus_{\lambda \vdash n} I_\lambda$ where $I_\lambda$ is the ideal corresponding to the partition $\lambda$.

Another result needed in the next theorem is the following lemma essentially proved in [10]

**Lemma 4.2.** Let $A$ be a finitely generated PI-algebra and let $\exp(A) = d$. Then there exists a constant $q \geq 0$ such that

$$
\chi_n(A) = \sum_{\lambda \vdash n \atop h(\lambda) \leq r} m_\lambda \chi_\lambda.
$$

**Proof.** By a theorem of Kemer (see [17, Theorem 2.3]) there exists a finite dimensional algebra $B$ such that $Id(A) = Id(B)$. Hence, we may assume that $A$ is a finite dimensional algebra over the algebraically closed field $F$. Let $\dim_F A = r$. It is well known (see for instance [10, Lemma 1]) that

$$
\chi_n(A) = \sum_{\lambda \vdash n \atop h(\lambda) \leq r} m_\lambda \chi_\lambda
$$

i.e., if $\chi_\lambda$ participates in $\chi_n(A)$ with non-zero multiplicity then the diagram of $\lambda$ lies in a strip of height $r$. Now, by [11, Corollary 1], since $\exp(A) = d$, there exists an integer $k \geq 0$ such that $\bigoplus_{\lambda \vdash n} I_\lambda \subseteq Id(A)$. This says that in (4.1), $m_\lambda = 0$ whenever $\lambda \geq (k^d)$. Also notice that by the characterization of the exponent, $d \leq r$ (see [10]). By combining the above results, we obtain that $m_\lambda = 0$ whenever $\lambda_{d+1} \geq k$. It follows that if we set $(r-d)k = q$, then $m_\lambda = 0$ whenever $n - (\lambda_1 + \cdots + \lambda_d) \leq q$. 

**Theorem 4.3.** For a finitely generated PI-algebra $A$ the following are equivalent

1. $\mlt(A) \leq 1$
2. $\exp(A) \leq 2$
3. $UT_3(F)$, $M_2(F) \not\in \var(A)$.

**Proof.** Since by Lemma 3.4 and Lemma 3.5, $\mlt(M_2(F)) = \mlt(UT_3(F)) = 3$, it is clear that if $\mlt(A) \leq 1$ then $UT_3(F)$, $M_2(F) \not\in \var(A)$.

Suppose that $UT_3(F)$ and $M_2(F)$ do not belong to the variety generated by $A$. As we remarked above, we may assume that $A$ is a finite dimensional algebra over $F$ and $F$ is algebraically closed. Let $A = A_1 \oplus \cdots \oplus A_t + J$ be the Wedderburn-Malcev
decomposition of $A$, where $J = J(A)$ is the Jacobson radical of $A$ and $A_i \cong M_{n_i}(F)$ is a simple subalgebra of $A$ for $i = 1, \ldots, t$.

If there exists $j$ such that $A_j \cong M_{n_j}(F)$ with $n_j \geq 2$, then $M_2(F) \subseteq M_{n_j}(F) \subseteq A$ and we obtain a contradiction. Hence $A_i \cong F$ for all $i = 1, \ldots, t$. Suppose that there exist $A_i$, $A_j$, $A_m$ distinct such that $A_i A_j A_m \neq 0$. Then we can choose $j_1, j_2 \in J$ such that $1_j, 1_j 1_j 1_j \neq 0$, where $1_j, 1_j 1_j, 1_j 1_j$ are the unit elements of $A_i$, $A_j$, $A_m$, respectively. Now set $u_{11} = 1_j, u_{22} = 1_j, u_{33} = 1_j, u_{12} = 1_j 1_j, u_{23} = 1_j 1_j 1_j$ and $u_{13} = 1_j 1_j 1_j 1_j$. If $B$ is the subalgebra of $A$ generated by these elements, it is clear that $B \cong UT_2(F)$; thus $UT_2(F) \in \text{var}(A)$, contrary to the assumption. Hence $A = F_1 \oplus \cdots \oplus F_t + J$ and $F_i J F_i J F_m = 0$ for all $i, l, m$ distinct. By the characterization of the exponent given in [11], it follows that $\exp(A) \geq 2$. Notice that since $J$ is a nilpotent ideal, say $J^k = 0$, and for all $a, b \in A$, $[a, b] \in J$, it follows that $A$ satisfies the identity $[x_1, x_2] \cdots [x_{2k}, x_{2k}] \equiv 0$. Hence $A$ satisfies a nonmatrix identity and $\exp(A) \geq 2$; but then, from Lemma 4.2 and Theorem 4.1 it follows that $\text{mlt}(A) \leq 1$.

As a consequence of the previous theorem, Lemma 3.4 and Lemma 3.5 we get

**Corollary 4.4.** Let $A$ be a finitely generated PI-algebra. Then the following properties are equivalent

1. $\text{mlt}(A) = 1$
2. $\exp(A) = 2$
3. $UT_2(F), M_2(F) \not\in \text{var}(A)$ and $UT_2(F) \in \text{var}(A)$.

**Proof.** Observe that $\text{mlt}(A) = 0$ means that for all $n$, the multiplicities $m_\lambda$ are bounded by a constant. This is equivalent to $UT_2(F) \not\in \text{var}(A)$. Then for any finitely generated PI-algebra $A, G \not\in \text{var}(A)$. Hence by a result of Kemer, $G, UT_2(F) \not\in \text{var}(A)$ is equivalent to $\exp(A) \leq 1$. We get the desired conclusion.

Notice that in the previous corollary we actually proved that $\text{mlt}(A) = 0$, $\exp(A) = 1$ and $UT_2(F) \not\in \text{var}(A)$ are equivalent properties.

We next show that there exists no finitely generated PI-algebra such that $\text{mlt}(A) = 2$.

**Theorem 4.5.** For any finitely generated PI-algebra $A$, $\text{mlt}(A) \neq 2$.

**Proof.** Let $\text{mlt}(A) = 2$. As we remarked above we may assume that $A$ is a finite dimensional algebra and let $A = A_1 \oplus \cdots \oplus A_t + J$ be its Wedderburn-Malcev decomposition. If for some $i \geq 1$, $A_i \cong M_{n_i}(F)$ and $n_i \geq 2$, then $M_2(F) \subseteq \text{var}(A)$ and $3 = \text{mlt}(M_2(F)) \leq \text{mlt}(A) = 2$ leads to a contradiction. Hence $A = F_1 \oplus \cdots \oplus F_t + J$. If $F_i J F_i J F_m \neq 0$ then, as in the proof of Theorem 4.3, we can construct a subalgebra $B$ of $A$ isomorphic to $UT_2(F)$ and by Lemma 3.5 we get a contradiction. Consequently $M_2(F)$ and $UT_2(F)$ do not belong to the variety generated by $A$ and by Theorem 4.3, $\text{mlt}(A) \leq 1$, a contradiction.

We next study finitely generated PI-algebra for which $\text{col}(A) \leq 1$.

**Theorem 4.6.** Let $A$ be a finitely generated PI-algebra. Then $\text{col}(A) \leq 1$ if and only if $UT_2(F) \not\in \text{var}(A)$. 

Proof. If col(A) ≤ 1 then, by Remark 3.6 and Corollary 3.8, UT_2(F) ∉ var(A).
Conversely, if UT_2(F) ∉ var(A) then by [21, Theorem 1] in the decomposition of the n-th cocharacter χ_n(A) of A, m_λ ≤ q, for some constant q; also there exists a constant M such that
\[ \chi_n(A) = \sum_{\lambda \vdash n, n-\lambda_1 \leq M} m_\lambda \chi_\lambda. \]
Hence we can write
\[ l_n(A) = \sum_{\lambda \vdash n, n-\lambda_1 \leq M} m_\lambda \leq nq, \]
for some constant q and col(A) ≤ 1. □

We finish by relating the functions col(A) and exp(A) at least when they take small values.

Remark 4.7. For any finitely generated PI-algebra A we have col(A) ≤ 1 if and only if exp(A) = 1;
Proof. By Theorem 4.6, col(A) ≤ 1 if and only if UT_2(F) ∉ var(A). By [21, Theorem 1] we have that UT_2(F) ∉ var(A) if and only if exp(A) = 1 and the property is proved. □

References


Dipartimento di Matematica ed Applicazioni, Università di Palermo, via Archirafi 34, 90123 Palermo, Italy

E-mail address: fbenanti@math.unipa.it

Dipartimento di Matematica ed Applicazioni, Università di Palermo, via Archirafi 34, 90123 Palermo, Italy

E-mail address: a.giambruno@unipa.it

Department of Algebra and Geometric Computations, Faculty of Mathematics and Mechanics, Ulyanovsk State University, Ulyanovsk 4327000, Russia

E-mail address: sviridova_i@rambler.ru